Journal of Computational Mathematics
Vol.37, No.1, 2019, 1–17.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1707-m2016-0718

UNCONDITIONALLY SUPERCLOSE ANALYSIS OF A NEW MIXED FINITE ELEMENT METHOD FOR NONLINEAR PARABOLIC EQUATIONS*

Dongyang Shi, Fengna Yan and Junjun Wang
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: shi_dy@zzu.edu.cn, yanfengnaa@163.com, wjunjun8888@163.com

Abstract

This paper develops a framework to deal with the unconditional superclose analysis of nonlinear parabolic equation. Taking the finite element pair $Q_{11}/Q_{01}\times Q_{10}$ as an example, a new mixed finite element method (FEM) is established and the τ-independent superclose results of the original variable u in H^1-norm and the flux variable $\vec{q}=-a(u)\nabla u$ in L^2-norm are deduced (τ is the temporal partition parameter). A key to our analysis is an error splitting technique, with which the time-discrete and the spatial-discrete systems are constructed, respectively. For the first system, the boundedness of the temporal errors are obtained. For the second system, the spatial superclose results are presented unconditionally, while the previous literature always only obtain the convergent estimates or require certain time step conditions. Finally, some numerical results are provided to confirm the theoretical analysis, and show the efficiency of the proposed method.

Mathematics subject classification: 65N15, 65N30
Key words: Nonlinear parabolic equation, Mixed FEM; Time-discrete and spatial-discrete systems, τ-independent superclose results.

1. Introduction

Let $\Omega\subset\mathbb{R}^2$ be a rectangle with boundary $\partial\Omega$ and $0<T<\infty$. We develop and analyze a mixed FEM to the following time-dependent nonlinear parabolic equation:

$$\begin{cases} u_t-\nabla\cdot(a(u)\nabla u)=f(X,t), & (X,t)\in\Omega\times(0,T],\\ u=0, & (X,t)\in\partial\Omega\times(0,T],\\ u(X,0)=u_0(X), & X\in\Omega,\end{cases}\tag{1.1}$$

where $X=(x,y)$, $a(u)$ and $f(X,t)$ are smooth functions. Assume that there exist constants μ, M, B such that $0<\mu\le a(u)\le M$, $|a'(u)+a''(u)|\le B$. For the nonlinear problem of (1.1), [1] constructed the linearized Galerkin FEM and derived optimal error of order $O(h^2+\tau^2)$ in L^2-norm. With the linearized Galerkin FEMs, [2] and [3] discussed three-level Galerkin method and implicit-explicit multistep FEMs, and obtained optimal order error estimates, respectively. For other nonlinear problems, numerious efforts have been devoted to the development of efficient numerical schemes, such as the nonlinear parabolic integro-differential equations [4-6], nonlinear Schrödinger equations [7-10], Navier-Stokes equations [11-13] and others [14-18].

* Received August 14, 2016 / Revised version received May 15, 2017 / Accepted July 10, 2017 / Published online August 14, 2018 /

It is known that the approximating spaces should satisfy the so-called Babuska-Brezzi condition in the usual mixed FEMs. In order to make the requirement to be satisfied easier, a mixed finite element form was established in [19] for second elliptic problems, in which the two spaces just need to fulfill a very simple inclusion relationship. Motivated by this work, the nonconforming pair $EQ_1^{rot}/Q_{10} \times Q_{01}$ was used to research a linear Sobolev equation and optimal error estimates and superclose results were received in [20]. For the linear parabolic problem, [21] deduced optimal error estimates based on the triangular nonconforming finite element pair $P_1/P_0 \times P_0$, and [22] showed the supercloseness as well as the extrapolation results with the nonconforming element pair $EQ_1^{rot}/Q_{10} \times Q_{01}$ of [20]. Note that [23,24] discussed the linear elasticity problem and the nonlinear Schrödinger equation with conforming finite element pairs, respectively.

Generally speaking, to deduce optimal error estimates of linearized Galerkin FEMs, one may use mathematical induction with an inverse inequality to bound the numerical solution in L^∞ norm, such as

$$\|U_h^n - R_h u^n\|_{L^\infty} \le Ch^{-\frac{d}{2}}\|U_h^n - R_h u^n\|_0 \le Ch^{-\frac{d}{2}}(h^{r+1} + \tau^m), \tag{1.2}$$

Here and later, U_h^n and u^n are the finite element approximation and the exact solution at time t^n, respectively, and R_h is a certain projection operator, C is a positive constant independent of τ and h. The inequality (1.2) results in the time-step restriction, and extremely time-consuming in practical computations see, e.g., [3-18,24,25]. However, it has been shown that the time restriction may not be necessary in many cases (see $[26-33]$). Not long ago, a new error analysis technique was proposed by [26] (also see [27]) for a Joule heating system with a standard Galerkin FEM, which splitted the numerical error into two parts, the spatial error and the temporal error. Then, the estiamte of (1.2) can be replaced by

$$\|U_h^n - R_h U^n\|_{L^\infty} \le Ch^{-\frac{d}{2}}\|U_h^n - R_h U^n\|_0 \le Ch^{-\frac{d}{2}}h^{r+1}, \tag{1.3}$$

where U^n is the time-discrete solution. Therefore, the bouneness of U_h^n can be deduced without any time-restriction. Consequently, [28-31] applied this idea to investigate various nonlinear problems and obtained the unconditional error estimates, respectively. But in the above studies, they only focused on the analysis of time-independent error estimates for the lineared Galerkin FEMs. Recently, [32] studied a mixed finite element scheme for the nonlinear Sobolev equation, and obtain the unconditionaliy superclose and superconvergent results by avoiding the estimate of the numerical solution in L^∞-norm. Of course, the method can't be used in this equation of (1.1). [33] derived the unconditionally superconvergent results for nonlinear parabolic equation with nonconforming EQ_1^{rot} elemrnt. In this paper, we study the linearized mixed finite element scheme for problem (1.1) with element pair $Q_{11}/Q_{01} \times Q_{10}$, and deduce the τ-independent superclose results through rigorous analysis.

The rest of the paper is organized as follows. In Section 2, the linearized time-discrete system is presented and the boundedness of the numerical solution in L^∞ norm for the original variable u and the flux variable $\vec{q} = -a(u)\nabla u$ are deduced, which will play an important role in the superclose analysis. In Section 3, we develop the new mixed finite element scheme and some notations. In Section 4, we give the linearized FEM for the spatial-discrete system and derive the corresponding superclose estimates of order $O(h^2 + \tau^2)$ unconditionally. In Section 5, some numerical results are provided to verify the theoretical analysis.