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A POSITIVE AND MONOTONE NUMERICAL SCHEME FOR VOLTERRA-RENEWAL EQUATIONS WITH SPACE FLUXES*

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Abstract

We study a numerical method for solving a system of Volterra-renewal integral equations with space fluxes, that represents the Chapman-Kolmogorov equation for a class of piecewise deterministic stochastic processes. The solution of this equation is related to the time dependent distribution function of the stochastic process and it is a non-negative and non-decreasing function of the space. Based on the Bernstein polynomials, we build up and prove a non-negative and non-decreasing numerical method to solve that equation, with quadratic convergence order in space.

Mathematics subject classification: 65R20, 45D05, 45M20.
Key words: Volterra renewal, Piecewise deterministic process, Monotone positive numerical scheme, Bernstein polynomials.

1. Introduction

In this paper we analyze a numerical method for solving the following system of Volterra-renewal integral equations with space fluxes [6]

$$u_i(x,t) = f_i(x,t) + \sum_{j=1}^{S} q_{ij} \int_0^t k_j(t-\eta) u_j(g_j(x,t,\eta),\eta) d\eta, \tag{1.1}$$

$$\text{where} \qquad f_i(x,t) = \sum_{j=1}^{S} q_{ij} \tilde{F}_j(g_j(x,t,0)) k_j(t), \tag{1.2}$$

for $i = 1, \dots S$, $t \geq 0$ and $x \in \Omega \subset \mathbb{R}$. This system of equations is part of a special form of the Chapman-Kolmogorov equation for a very wide cathegory of stochastic processes named Piecewise Deterministic Processes (PDPs) [13, 14].

Briefly, a PDP is generated from the random switching in time of deterministic motions, taken randomly from a discrete set of given functions. It can be considered as an extension of the "point processes" used in queue theory and renewal processes [27]. From the theoretical side, PDPs are known by experts working in probability calculus and operation research (e.g. see [7, 11, 16]). Within the general category of the PDPs, those characterized by a motion

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switching randomly between deterministic states driven by a semi-Markov process $\mathcal{S}(t)$, are significative. An initial work was made in [22] for Markov processes. A semi-Markov process is a discrete state and continuous time stochastic jump process where the influence of the past is erased at the epochs of jumps. These kind of stochastic processes have potentally a huge amount of applications, we quote Stochastic Hybrid Systems [10, 30] and systems driven by dichotomous noise [26]. Further applications and details of the definition of these PDPs can be found in [5,6], here we give some basic definitions in order to provide a little explanation of the meaning of the terms in Eq. (1.1).

The semi-Markov process is defined as: a discrete Markov process with S states, a stochastic transition matrix $\hat{q} := \{q_{ij}\}$, with $0 \leq q_{ij} \leq 1, \sum_{i=1}^{S} q_{ij} = 1$, jointly to a set of probability density functions $k_i(t) \geq 0, \int_0^\infty k_i(t)dt = 1$, describing the statistics of *switching time events*. The semi-Markov process $\mathcal{S}(t)$ drives the ordinary differential equation

$$dX(t)/dt = \bar{A}_{\mathcal{S}(t)}(X), \tag{1.3}$$

where the function $\bar{A}_i$, is one from a set of $\{\bar{A}_1, \ldots, \bar{A}_S\}$ given functions. The resulting motion of the state function $X(t)$ is a random sample path composed of pieces of deterministic trajectories, each of them within two switching events of the semi-Markov process.

The meaningful information of a stochastic process is provided by the marginal probability distribution functions. In this case it is defined as

$$F_i(x, y, t) := \mathbb{P}\Big(X(t) \leq x, y < Y \leq y + dy, i = \mathcal{S}(t)\Big),$$

i.e. the probability that at the time t the process $X(t)$ is in the dynamical state i, for the sojourn time y, and its value is not greater than x. Usually, a probability distribution function is computed by applying Monte Carlo methods directly to the stochastic equation model, like (1.3). This choice is motivated by the easy implementation of the method on computers, but it suffers of a notorious slow convergence rate that scales as the inverse of the square root of the number of samples, although it is robust with respect to the dimension of the spatial domain. Whenever the governing equation of the distribution function is known, it is possible to solve this one by deterministic methods [5, 12], and, if needed, Monte Carlo methods for the validation of theoretical findings (see e.g. [1, 2, 29]). Thus, we search for the probability distribution function by solving the related Chapman-Kolmogorov equation. In the case of PDP described by equation (1.3), the Chapman-Kolmogorov assumes the form of a system of hyperbolic partial differential equations with nonlocal boundary condition [5], or equivalently [6] as the system of Volterra-renewal equation (1.1)-(1.2). Eq. (1.2) is the initial condition of the problem, where $\tilde{F}_j(x)$ represent the distribution functions for the inital data of (1.3). The distribution functions $F_i(x, y, t)$ have the fundamental properties to be monotonically increasing in x and positive in y for all $t > 0$. In order to calculate them, we first solve (1.1)-(1.2), then apply the transformation

$$F_i(x, y, t) = u_i(g_i(x, t, t - y), t - y)\, e^{-\int_0^y \lambda_i(\tau)\, d\tau}, \qquad 0 < y < t, \tag{1.4}$$

where $\lambda_i(t) = k_i(t)/\int_t^\infty k_i(\tau)d\tau$ and the functions $g_j(x, t, \eta)$ represent the inverse fluxes of the solutions of the ODE (1.3). Moreover, if we are interested in the probability distribution without the dependence on the length of the sojourn time y in the state, we integrate it as follows

$$\mathcal{F}_i(x, t) = \int_0^t F_i(x, y, t)\, dy. \tag{1.5}$$

The problem formed by Eqs. (1.1), (1.2), (1.4) and (1.5) is the subject of our investigation.

Kernels of the type such as in the integral of (1.1), derive from *renewal processes*, a well investigated subject in the field of stochastic processes. In fact, Eq. (1.1) takes the form of a *renewal equation* when the dependence on x and on the flux function g are neglected. The literature on renewal type equations is wide and extensive [15, 17], and includes investigations both from an analytical and numerical point of view. Significant is the case of Lebesgue-Stieltjes integral [31], since the probability measure may have both discrete and continuous components.

The function $g_j(x,t,\eta)$ in (1.1) returns the value of the position of the process at the time η when the coordinate x at time $t > \eta$ is given for the $j-$th discrete state of the system. Here we consider $g_j(x,t,\eta)$ monotone in x [33].

The first numerical approach to Eq. (1.1) is reported in [3] where we prove a basic theorem for existence and uniqueness of the solution and propose and analyse a numerical method based on quadrature in time, and interpolation in space. In [4], we investigate the asymptotic behavior of the solution and correspondingly the asymptotic stability of the numerical method.

Here, we want to provide a different discretization scheme which, by using Bernstein polynomials for approximation in space and exploiting their positivity and monotonicity properties, is able to preserve, in the numerical solution, the monotonicity and positivity of the continuous one. We note that, with the exception of the degree 1 case, which coincides with the first degree Bernstein polynomial, Lagrangian interpolation used in [3] and [4] does not possess analogous properties and hence does not allow an analysis like the one carried out in this paper. The use of Bernstein polynomials in the numerical solution of Volterra integral equations is documented in literature, see for example [21,28,32]. However, in our case, the very special form of Eq. (1.1) and the application of Bernstein approximation to the spatial variable of the Volterra equation with flux, makes our analysis new.

The paper is organized as follows: Section 2 is devoted to the study of the analytical solution of a *single* Volterra-renewal equation (1.1). In Section 3 the discretization scheme with Bernstein polynomials is built. Its monotone preserving properties and quadratic convergence are proved. Furthermore, it is proved that these properties hold also after the application of the transformations (1.4) and (1.5). In Section 4 numerical experiments are reported for the validation of our theoretical findings. Finally, Section 5 concludes the paper with some remarks.

2. Analytical Results

Here and in the following sections we restrict our analysis to the scalar equation ($S = 1$),

$$u(x,t) = f(x,t) + \int_0^t k(t-\eta)u(g(x,t,\eta),\eta)\,d\eta, \qquad t > 0. \tag{2.1}$$

In this case, for the flow solution $g^t(X_0)$ of an ordinary differential equation, a monotonic property holds as follows. Let g_1^t and g_2^t be two solutions of $dX/dt = A(X)$ in the interval $[t_0,t]$ to the initial value problem $X(t_0) = X_0$. Supposing that $g_1^{t_0} < g_2^{t_0}$ and that $A(X)$ is (weakly) increasing or decreasing in X, then (i) $g_1^t < g_2^t$ in $[t_0,t]$ and (ii) the difference $g_2^t - g_1^t$ is increasing or decreasing, respectively. The generalization of these properties to the multi-dimensional case has been proved in the Müller-Kamke theorem [33, 34]. As mentioned in Section 1, we define the function $g(x,t,\eta)$ as representative of the inverse flux solution of the single ODE of Eq. (1.3), i.e.

$$g(x,t,\eta)|_{x=X} := g^{-1,\eta}(X).$$

We start to recall the following theorem (see [3]), which states the conditions for the existence, uniqueness and regularity properties of the solution of (2.1) for any interval $I \subset \mathbb{R}^+$.

Theorem 2.1. *Suppose that the functions $g(x,t,\eta) \in C^{\bar{r},\bar{q},\bar{q}}(\Omega \times I \times I)$, $k(t) \in C^{\bar{q}}(I)$, $f(x,t) \in C^{\bar{r},\bar{q}}(\Omega \times I)$, then the integral equation* (2.1) *has an unique solution $u(x,t) \in C^{\bar{r},\underline{q}}(\Omega \times I)$, with $\underline{q} = \min(\bar{r}, \bar{q})$.*

From now on, we assume that the following hypotheses on the functions involved in (2.1), for the existence and boundedness of the solution, are satisfied (see [3, 4] and the bibliography therein):

- **h1)** $f(x,t) \in C(\Omega \times \mathbb{R}^+) \bigcap L^1(\Omega \times \mathbb{R}^+)$;
- **h2)** $\int_0^{+\infty} |f(x,t)| dt < \alpha < +\infty$, $\forall x \in \Omega$;
- **h3)** $k(t) \in C(\mathbb{R}^+) \bigcap L^1(\mathbb{R}^+)$, $k(t) \geq 0$, $\int_0^{+\infty} k(t) dt = 1$;
- **h4)** $g(x,t,\eta) \in C(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+)$ and $g(x,t,\eta) \in \Omega$, for $x \in \Omega, t \geq \eta \geq 0$;
- **h5)** $\int_0^{+\infty} t\, k(t) dt = \mu > 0$;
- **h6)** $f(x,t) \geq 0$ for $x \in \Omega,\ t \geq 0$;
- **h7)** $f(x+y,t) - f(x,t) \geq 0$ for $x \in \Omega, y > 0, x+y \in \Omega$;
- **h8)** $g(x+y,t,\eta) - g(x,t,\eta) \geq 0$, for $x \in \Omega, y > 0, x+y \in \Omega$.

These hypotheses are consistent with the model described in Section 1.

Theorem 2.2 (Non-negativity) *Assume that $h1) - h6)$ hold then the solution $u(x,t)$ of Eq.* (2.1) *is non-negative for all $x \in \Omega$ and $t \geq 0$.*

Proof. As proved in [3], the solution $u(x,t)$ is continuous, so it is bounded. Furthermore, $\lim_{t\to\infty} \max_{x\in\Omega} |u(x,t)| < \infty$, as proved in [4]. Set $\bar{u}(t) = \min_{x\in\Omega} u(x,t)$, and $\bar{f}(t) = \min_{x\in\Omega} f(x,t)$: it is well defined because f is continuous in the bounded set Ω. Then from Eq. (2.1), since k is non-negative, we get:

$$u(x,t) \geq \bar{f}(t) + \int_0^t k(t-\eta)\bar{u}(\eta) d\eta \tag{2.2}$$

for all $x \in \Omega$, $t \geq 0$. Therefore,

$$\bar{u}(t) \geq \bar{f}(t) + \int_0^t k(t-\eta)\bar{u}(\eta) d\eta. \tag{2.3}$$

So, since $\bar{f} \geq 0$ for $h6)$, we can write

$$\bar{u}(t) \geq \bar{f}(t) + \int_0^t r(t-\eta)\bar{f}(\eta) d\eta \tag{2.4}$$

([8, pg. 80, Theorem 2.1.16]), where $r(t)$ is the resolvent for $k(t)$. Finally, by taking into account that $k \geq 0$ implies r non-negative, we obtain $\bar{u}(t) \geq 0$, for all $t \geq 0$ and hence

$$u(x,t) \geq \bar{u}(t) \geq 0, \quad \forall x \in \Omega,\ t \geq 0.$$

This completes the proof of the lemma. □

Theorem 2.3 (Monotonicity) *Assume that $h1) - h5)$ and $h7) - h8)$ hold, then the solution $u(x,t)$ of Eq.* (2.1) *is non-decreasing in space, i.e. $u(x+y,t) - u(x,t) \geq 0$, for $y > 0$.*

Proof. Let consider the following functions: $E(x,y,t) := f(x+y,t) - f(x,t)$, $G(x,y,t,\eta) := g(x+y,t,\eta) - g(x,t,\eta)$, $U(x,y,t) := u(x+y,t) - u(x,t)$, all of them defined on the domain $(x,y) \in \Omega \times \{y \geq 0\}$, $x+y \in \Omega$, $t \geq 0, \eta \geq 0$, $t - \eta \geq 0$. According to the hypothesis the following properties hold:

2b) $E(x,y,t) \geq 0$ for $(x,y) \in \Omega \times \{y \geq 0\}$

2c) $E(x,0,t) = 0$, for $x \in \Omega$

3b) $G(x,y,t,\eta) + g(x,t,\eta) \in \Omega$ for $(x,y) \in \Omega \times \{y \geq 0\}$, $x+y \in \Omega$

4b) $G(x,y,t,\eta) \geq 0$ for $(x,y) \in \Omega \times \{y \geq 0\}$, $t,\eta \geq 0$

4c) $G(x,0,t,\eta) = 0$ for $x \in \Omega$.

We replace x with $x+y$ in Eq. (2.1) and subtract it side by side, so

$$U(x,y,t) = E(x,y,t) + \int_0^t k(t-\eta)[u(g(x+y,t,\eta),\eta) - u(g(x,t,\eta),\eta)]d\eta, \tag{2.5}$$

and by using 3b), we get

$$U(x,y,t) = E(x,y,t) + \int_0^t k(t-\eta)U(g(x,t,\eta), G(x,y,t,\eta),\eta)d\eta. \tag{2.6}$$

For any fixed positive value of y, we define the set $\Omega_y := \{x : x+y \in \Omega\} \subset \Omega$, and the functions:

$$\underline{E}(y,t) = \min_{x \in \Omega_y} E(x,y,t) \geq 0, \quad \underline{U}(y,t) = \min_{x \in \Omega_y} U(x,y,t) \geq 0$$

that exists for $y \geq 0$, because it is the difference of regular functions $u(x,t)$. Further, $g(x,t,\eta) + G(x,y,t,\eta) \in \Omega$, for $x+y \in \Omega$, so that

$$U(x,y,t) \geq \underline{E}(y,t) + \int_0^t k(t-\eta)\underline{U}(G(x,y,t,\eta),\eta)d\eta. \tag{2.7}$$

Where, since $G \geq 0$ for 4b), it belongs to the definition domain of $\underline{U}(y,t)$. Let

$$\underline{\underline{E}}(t) = \min_{y \geq 0} \underline{E}(y,t) \geq 0, \quad \underline{\underline{U}}(t) = \min_{y \geq 0} \underline{U}(y,t) \geq 0.$$

Then

$$U(x,y,t) \geq \underline{\underline{E}}(t) + \int_0^t k(t-\eta)\underline{\underline{U}}(\eta)d\eta. \tag{2.8}$$

We have already mentioned in Theorem 2.2 that the resolvent $r(t)$ for the kernel $k(t)$ is non negative, furthermore $\underline{\underline{E}} \geq 0$ for $h7)$, so we conclude that

$$U(x,y,t) \geq \underline{\underline{U}}(t) \geq \underline{\underline{E}}(t) + \int_0^t r(t-\eta)\underline{\underline{E}}(\eta)d\eta \geq 0.$$

This completes the proof of the theorem. □

3. Bernstein Discretization Analysis

In this section we propose a discretization of Eq. (2.1) based on Bernstein approximation in space, with the aim to obtain a numerical solution which preserves the structural characteristics of the continuous one.

Consider a spatial mesh for Ω, i.e. $\Omega_h := (x_0, x_1, \dots, x_M)$ of size h and a time discretization $t_0, t_1, \dots$ with constant stepsize τ. Let

$$B^l_{mi}(x) = h^{-m}\binom{m}{i}(x - x_l)^i(x_{l+1} - x)^{m-i} \tag{3.1}$$

denote the $i-$th Bernstein polynomial of degree m in the interval $[x_l, x_{l+1}]$. Further, for each interval $[x_l, x_{l+1}]$, we introduce a finer mesh at the points $x_{l,i} = x_l + ih/m,\ \ i = 0, \dots, m$, note that $x_{l,m} = x_{l+1,0}$. In the following we shall use the short notation $x_l = x_{l,0}$. The m-th degree Bernstein polynomial for a function $f(x)$, defined on $[x_l, x_{l+1}]$, is given by

$$\sum_{i=0}^{m} B^l_{mi}(x) f(x_{li}). \tag{3.2}$$

For the theory about Bernstein polynomials we refer, for example, to [20, 25]. Here we mainly use non-negativity, monotonicity and the partition of unity properties. So, for all $m,\ l \geq 0$ and $x \in [x_l, x_{l+1}]$,

$$B^l_{mi}(x) \geq 0, \quad \sum_{i=0}^{m} B^l_{m,i}(x) f_1(x_{l,i}) \geq \sum_{i=0}^{m} B^l_{m,i}(x) f_2(x_{l,i})$$

when $f_1(x) \geq f_2(x)$ and $\sum_{i=0}^{m} B^l_{mi}(x) = 1$.

To discretize (2.1), we write down the equation on the mesh points (x_{sk}, t_n), $s = 0, \dots, M$, $k = 1, \dots, m$ and $n \geq 0$, then we integrate along time by the classical n_0-step DQ (Direct Quadrature) method with convolution weights ω_{n-j} (see, e.g., [9]) and approximate $u(g(x_{sk}, t_n, t_j,))$, $0 \leq j \leq n$, by Bernstein polynomials as in (3.2). This yields,

$$u_{sk,n} = f_{sk,n} + \tau \sum_{j=0}^{n} \tilde{\omega}_{nj} k_{n-j} \sum_{i=0}^{m} B^l_{mi}(g(x_{sk}, t_n, t_j))\, u_{li,j}, \qquad n = n_0, n_0 + 1, \dots, \tag{3.3}$$

where τ is the constant time stepsize. Here, $k_{n-j} = k(t_n - t_j)$ and, for each $n \geq n_0$,

$$\tilde{\omega}_{nj} = \begin{cases} w_{nj}, & \text{for } 0 \leq j < n_0, \\ \omega_{n-j}, & \text{for } n_0 \leq j \leq n, \end{cases}$$

where w_{nj} are the starting weights. For $s = 0, \dots, M$, $k = 1, \dots, m$ and $n \geq n_0$, $f_{sk,n} = f(x_{sk}, t_n)$, and the starting values $u_{sk,0} = f(x_{sk}, 0)$, $u_{sk,1}, \dots, u_{sk,n_0-1}$ are given. Furthermore, l is chosen such that $g(x_{sk}, t_n, t_j) \in [x_l, x_{l+1}]$, and $u_{sk,n}$ represents an approximation to the exact solution u of (2.1) at point (x_{sk}, t_n), i.e. $u_{sk,n} \approx u(x_{sk}, t_n)$. From now on we make the following assumptions on the starting values, $\forall j = 0, \dots, n_0 - 1$,:

- $u_{sk,j} \geq 0, \quad \forall\, s = 0, \dots, M,\ k = 1, \dots, m$, and
- $u_{sk+1,j} - u_{sk,j} \geq 0, \quad \forall\, s = 0, \dots, M,\ k = 0, \dots, m - 1,$

and on the weights, $\forall n \geq n_0,\ 0 \leq j \leq n$:

- $\tilde{\omega}_{nj} > 0$,
- $\sup_{n,j} \tilde{\omega}_{nj} \leq Z < \infty$.

Now we define the Bernstein piecewise interpolation operator as follows:

$$\mathcal{B}(f(\cdot);x) := \sum_{l=0}^{M} \sum_{i=0}^{m} B_{mi}^{l}(x) I_l(x) f(x_{l,i}) \qquad x \in \Omega, \tag{3.4}$$

where $I_l(x)$ is the indicator function on the interval $[x_l, x_{l+1})$. Given a function $f(x)$, then $\mathcal{B}(f;x) \approx f(x)$ and $f(x_l) = \mathcal{B}(f;x_l)$, so x_l are continuity points for $l = 0, \dots, M$. Furthermore, if $f(x)$ is non-decreasing then $\mathcal{B}(f;x)$ is also non-decreasing and the partition of unit property reads as $\mathcal{B}(1;x) = 1$.

By using (3.4), the discrete Volterra equation recasts to

$$u_{sk,n} = f_{sk,n} + \tau \sum_{j=0}^{n} \tilde{\omega}_{nj} k_{n-j}\, \mathcal{B}(u_j(\cdot);g)_{sk,n,j}. \tag{3.5}$$

Here $u_j(x)$ is the step function defined as $u_j(x) = \sum_{l,i} u_{li,j} \chi_{li}(x)$, where χ_{li} is the indicator function on the interval $(x_{li-1}, x_{li}]$. In (3.5), $\mathcal{B}(u_j(\cdot);g)_{sk,n,j}$=$\mathcal{B}(u_j(\cdot);g(x_{sk}, t_n, t_j))$ is the short notation for the evaluated Bernstein operator on the mesh, that includes the transformation of the coordinates induced by the flux g.

The boundedness of the solution of Eq. (3.3) for all $n = 0, 1, \dots,$ is provided by Theorem 12 in [4] under hypotheses $h1) - h5)$ and

h9) $k(t)$ and $\max_{x\in\Omega} |f(x,t)|$ asymptotically decreasing in time.

The following theorem represents the discrete equivalent of Theorem 2.2 in Section 2 about the positivity of the solution.

Theorem 3.1 (Discrete non-negativity) *Assume that* $h1) - h6)$ *and* $h9)$ *hold, then the solution* $u_{l,n}$ *of Eq.* (3.3) *(i.e.* (3.5)*) is non-negative for all* $l = 0, \dots, M$ *and* $n \geq 0$.

Proof. The solution $u_{l,n}$ of Eq. (3.5) is bounded and $\lim_{n\to\infty} \max_{l=0,\dots,M} |u_{ln}| < \infty$, as proved in [4]. Set $\bar{u}_j = \min_{r,k} u_{rk,j}$, and $\bar{f}_n = \min_{r,k} f_{rk,n}$. Then from Eq. (3.5), since $\bar{f}$, k and B_{mi}^{l} are non-negative, we get:

$$u_{sk,n} \geq \bar{f}_n + \tau \sum_{j=0}^{n} \tilde{\omega}_{nj} k_{n-j} \bar{u}_j \mathcal{B}(1;g)_{sk,n,j},$$

for all $s = 0, \dots, M,\ k = 1, \dots, m$. Therefore, for the partition of unity property,

$$\bar{u}_n \geq \bar{f}_n + \tau \sum_{j=0}^{n} \tilde{\omega}_{nj} k_{n-j} \bar{u}_j.$$

So we can write

$$\bar{u}_n \geq \bar{f}_n + \sum_{j=0}^{n} r_{n-j} \bar{f}_j$$

([18, Cor. 1.6.1, pg.15]), where r_j is the discrete resolvent for $k(t)$. Finally, by taking into account that $k \geq 0$ implies r non-negative, we obtain $\bar{u}_n \geq 0$, for all $n \geq n_0$ and hence

$$u_{sk,n} \geq \bar{u}_n \geq 0, \quad \forall s = 0, \dots, M, \quad k = 1, \dots, m,\ n \geq n_0.$$

This completes the proof of this theorem. □

Theorem 3.2 (Discrete monotonicity) *Assume that* $h1)-h5)$ *and* $h7)-h9)$ *hold. Then the solution* $u_{sk,n}$ *of Eq.* (3.3) *(i.e. Eq.* (3.5)*) is non-decreasing in space, i.e.* $u_{sk+1,n} - u_{sk,n} \geq 0$, *for all* $s = 0, \dots, M,\ k = 0, \dots, m$.

Proof. We distinguish two cases.

When $k_0 = 0$ the discrete equation (3.5) is explicit, hence by using the monotonic property of the operator $\mathcal{B}$ and the monotony of the forcing term $f_{sk,n}$ we get that the values $u_{ks,n}$ are monotonic with respect to k and s for all $n \geq 0$.

When $k_0 > 0$, the equation is implicit to the unknown $u_{ks,n}$. If we suppose that all $u_{ks,j}, j < n$ are monotone, then $\sum_{j=0}^{n-1} \tilde{\omega}_{n-j} k_{n-1} \mathcal{B}(u_j, g)_{ks,n,j}$ is monotone because sum of monotone operators with positive weights. Hence the system of equation for $u_{sk,n}$ has monotone known vector, and since also $\tau\omega_0 k_0 \mathcal{B}(u_n, x_{sk})$ is monotone, then the solution is monotone. □

Theorem 3.3 (Convergence order of the numerical method) *Assume that* $h1)-h5)$ *hold. Furthermore, let* k, f *and* g *be continuous functions that are* $\bar{r} \geq 2$ *times continuously differentiable with respect to* x *on* Ω *and* $\bar{q}$ *times continuously differentiable with respect to* t *on* $I \subseteq \mathbb{R}^+$. *Let* $q \leq \min(\bar{q}, \bar{r})$ *be the order of the DQ method and* m *the degree of the Bernstein polynomials used in* (3.3). *Then, the global error* $e_n = \max_{s,k} |u(x_{sk}, t_n) - u_{sk,n}|$ *of Eq.* (3.3) *satisfies*

$$\max_{n=0,\dots,N} e_n = C_1 \tau^q + C_2 h^2 + \tau C_3 \delta(h, \tau), \tag{3.6}$$

for $h \to 0$, $\tau \to 0$ *and* $N\tau = T$, $Mh = b - a$. C_1, C_2 *and* C_3 *are positive constants, with* $C_2 \propto m^{-1}$, *and*

$$\delta(h, \tau) = \max_{s,k} \sum_{j=0}^{n_0 - 1} |u(x_{sk}, t_j) - u_{sk,j}|$$

contains the starting errors.

Proof. From (2.1) and (3.3) we have

$$u(x_{sk}, t_n) - u_{sk,n} = T_n(\xi, \tau) + \tau \sum_{j=0}^{n} \tilde{\omega}_{n-j} k_{n-j} \theta_m(\xi),$$

for any $n = 0, \dots, N$ and $s = 0, \dots, M$, $k = 1, \dots, m$. Here, $\xi = g(x_{sk}, t_n, t_j) \in [x_l, x_{l+1}]$, and

$$T_n(\xi, \tau) = \int_0^{t_n} k(t_n - \eta) u(g(x_{sk}, t_n, \eta), \eta) d\eta - \tau \sum_{j=0}^{n} \tilde{\omega}_{n-j} k_{n-j} u(\xi, t_j), \tag{3.7}$$

$$\theta_m(\xi) = u(\xi, t_j) - \sum_{i=0}^{m} B^l_{mi}(\xi) u_{li,j}. \tag{3.8}$$

From Theorem 2.1 u is $\underline{q} = \min(\bar{r}, \bar{q})$ times continuously differentiable with respect to t and $\bar{r}$ times with respect to x, since $q \leq \min(\bar{q}, \bar{r})$, the consistency of DQ methods (see for example [19]) gives

$$|T_n(\xi, \tau)| \leq c\tau^q, \qquad (\xi, \tau) \in \Omega \times I, \tag{3.9}$$

with c independent on t_n, t_j, x_{sk}. Furthermore,

$$\theta_m(\xi) = u(\xi, t_j) - \sum_{i=0}^{m} B^l_{mi}(\xi) u(x_{li}, t_j) + \sum_{i=0}^{m} B^l_{mi}(\xi)(u(x_{li}, t_j) - u_{li,j}). \tag{3.10}$$

Consider [20, p.21, Th.1.6.2], taking into account that ξ belongs to the interval $[x_l, x_{l+1}]$ of length h, let $\xi = x_l + \frac{s}{m}h$, $s \in [0, m]$, $x_{li} = x_l + \frac{i}{m}h$, $i = 0, 1, \ldots, m$ and define

$$\beta_{mi}(z) = \begin{pmatrix} m \\ i \end{pmatrix} z^i (1-z)^{m-i}, \tag{3.11}$$

for $z \in [0, 1]$, as the Bernstein polynomials related to the interval $[0, 1]$. One can easily check that

$$B^l_{mi}(\xi) = \beta_{mi}\left(\frac{\sigma}{m}\right), \qquad \sigma \in [0, m]. \tag{3.12}$$

For the properties of Bernstein polynomials (see [25]), (3.12) holds and $\sum_{i=0}^m \beta_{mi}(z) = 1$, $\forall\, z \in [0, 1]$, then

$$\begin{aligned}&\left|u(\xi, t_j) - \sum_{i=0}^m B^l_{mi}(\xi) u(x_{li}, t_j)\right| \\ =&\left|\sum_{i=0}^m \left(u\left(x_l + \frac{\sigma}{m}h, t_j\right) - u\left(x_l + \frac{i}{m}h, t_j\right)\right)\beta_{mi}\left(\frac{\sigma}{m}\right)\right| \\ =&\left|\sum_{i=0}^m \left(\frac{\sigma}{m} - \frac{i}{m}\right) h \frac{\partial u}{\partial x}\left(x_l + \frac{s_i}{m}h, t_j\right)\beta_{mi}\left(\frac{\sigma}{m}\right)\right|,\end{aligned}$$

where s_i belongs to the interval with endpoints σ and i. The last equality in the previous expression has been obtained by using the mean value theorem for $u(x_l + \frac{\sigma}{m}h, t_j) - u(x_l + \frac{i}{m}h, t_j)$, since $\bar{r} \geq 2$ and thus $\frac{\partial u}{\partial x}(x, t)$ is differentiable in Ω with respect to x. From there, by adding and subtracting the same quantity we get

$$\begin{aligned}&\left|\sum_{i=0}^m \left(\frac{\sigma}{m} - \frac{i}{m}\right) h \frac{\partial u}{\partial x}\left(x_l + \frac{s_i}{m}h, t_j\right)\beta_{mi}\left(\frac{\sigma}{m}\right)\right| \\ \leq&\left|\sum_{i=0}^m \left(\frac{\sigma}{m} - \frac{i}{m}\right) h \frac{\partial u}{\partial x}\left(x_l + \frac{\sigma}{m}h, t_j\right)\beta_{mi}\left(\frac{\sigma}{m}\right)\right| \\ &+\left|\sum_{i=0}^m \left(\frac{\sigma}{m} - \frac{i}{m}\right) h \left(\frac{\partial u}{\partial x}\left(x_l + \frac{s_i}{m}h, t_j\right) - \frac{\partial u}{\partial x}\left(x_l + \frac{\sigma}{m}h, t_j\right)\right)\beta_{mi}\left(\frac{\sigma}{m}\right)\right|.\end{aligned} \tag{3.13}$$

The first term in the sum is 0. As a matter of fact,

$$\sum_{i=0}^m \frac{\sigma}{m}\beta_{mi}\left(\frac{\sigma}{m}\right) = \frac{\sigma}{m}, \quad \sum_{i=0}^m \frac{i}{m}\beta_{mi}\left(\frac{\sigma}{m}\right) = \frac{\sigma}{m}\sum_{i=1}^m \beta_{m-1 i-1}\left(\frac{\sigma}{m}\right) = \frac{\sigma}{m}.$$

For the second term in (3.13), we use once again the mean value theorem, this time for $\frac{\partial u}{\partial x}\left(x_l + \frac{s_i}{m}h, t_j\right) - \frac{\partial u}{\partial x}\left(x_l + \frac{\sigma}{m}h, t_j\right)$, to obtain finally

$$\left|u(\xi, t_j) - \sum_{i=0}^m B^l_{mi}(\xi) u(x_{li}, t_j)\right| \leq \sum_{i=0}^m \left(\frac{\sigma}{m} - \frac{i}{m}\right)^2 h^2 \bar{U} \beta_{mi}\left(\frac{\sigma}{m}\right) \leq h^2 \frac{\bar{U}}{4m},$$

where $\bar{U}$ is a bound for $\left|\frac{\partial^2 u}{\partial x^2}(x, t)\right|$ in $\Omega \times I$ and $\frac{1}{4m}$ is the maximum for $\sum_{i=0}^m \left(\frac{\sigma}{m} - \frac{i}{m}\right)^2 \beta_{mi}\left(\frac{\sigma}{m}\right)$, as a function of σ/m. Since $\tilde{\omega}_{nj}$ and k_j are positive and $h3)$ holds, then

$$\tau \sum_{j=0}^n \tilde{\omega}_{nj} k_j \leq 1 - \varphi(\tau),$$

where $\varphi(\tau)$ is the quadrature error satisfying $\varphi(\tau) \to 0$ as $\tau \to 0$. So, we obtain

$$e_n \leq c\tau^q + \frac{h^2\bar{U}}{4m}(1-\varphi(\tau)) + \tau \sum_{j=0}^{n} \tilde{\omega}_{n-j} k_{n-j} e_j,$$

where, once again, we have used the fact that $\sum_{i=0}^{m} B_{mi}^{l}(x) = 1$. Then, the discrete Gronwall lemma ([8], p. 81)) leads to

$$e_n \leq \frac{c\tau^q + \frac{\bar{U}}{4m}(1-\varphi(\tau))h^2 + \tau Z K \delta(h,\tau)}{1-\tau\omega_0 k_0} e^{\frac{ZKT}{1-\tau\omega_0 k_0}}, \tag{3.14}$$

with $K = \max_{0\leq t\leq T} k(t)$. So, (3.6) comes true with

$$C_1 = \frac{c}{1-\tau\omega_0 k_0} e^{\frac{ZKT}{1-\tau\omega_0 k_0}}, \quad C_2 = \frac{\frac{\bar{U}}{4m}(1-\varphi(\tau))}{1-\tau\omega_0 k_0} e^{\frac{ZKT}{1-\tau\omega_0 k_0}}, \quad C_3 = \frac{ZK}{1-\tau\omega_0 k_0} e^{\frac{ZKT}{1-\tau\omega_0 k_0}}.$$

This completes the proof of the theorem. □

Remark 3.1. From the error estimate (3.14) it is clear that, if the starting values are computed with sufficiently high accuracy, the maximum order attainable for the method (3.3) is 2. We note that, the first degree interpolating polynomial used in [1] coincides with the case $m = 1$ here, therefore the convergence order of the numerical method is stated in Theorem 2.2 in [1]. Here, we have extended the investigation to polynomials of degree $m > 1$ and we have shown that, while (3.14) gives optimal convergence rates in the time steps, the contribution of the spatial part in the error converges if the polynomial degree is increased, with fixed spatial stepsize. In practice, the degree m of Bernstein polynomials in (3.14) has a not negligible influence on the error in terms of magnitude of the error constant C_2. In conclusion, high order Bernstein polynomials produce smaller errors and, at the same time, preserve the positivity and monotonicity properties in the numerical approximation. This effect will be shown later in the experiments.

Finally, we conclude the numerical analysis by proving that the Bernstein polynomial discretization is able to preserve the monotonicity and the positivity of Eq. (1.4) for the transformation of the solution of the Volterra equation back to the probability distribution of the original problem

$$F(x_{sk}, t_n, t_j) = u(\xi, t_n - t_j) e^{-\int_0^{t_j} \lambda(s) ds},$$

where $\xi = g(x_{sk}, t_n, t_n - t_j) \in [x_l, x_{l+1}]$. Set $b(t) = \int_0^t \lambda(s) ds$ ($\lambda(t) \geq 0$, so $b(t)$ is positive and increasing),

$$F_{sknj} = e^{-b(t_j)} \mathcal{B}(u_{n-j}(\cdot); \xi), \tag{3.15}$$

is the approximation of $F(x_{sk}, t_n, t_j)$ by Bernstein polynomials. The error of this approximation is given by

$$F(x_{sk}, t_n, t_j) - F_{sknj} = e^{-b(t_j)} \theta_m(\xi),$$

where $\theta_m(\xi)$ is defined in (3.10) and, in the hypotheses of Theorem 3.3,

$$|\theta_m(\xi)| \leq C_1 \tau^q + D_2 h^2 + \tau C_3 \delta(h,\tau),$$

for all $0 \leq j \leq n \leq N$, with $T = N\tau$, $\tau \to 0$, $h \to 0$, with $D_2 = C_2 + \frac{\bar{U}}{4m}$. Thus also

$$|F(x_{sk}, t_n, t_j) - F_{sknj}| \leq C_1 \tau^q + D_2 h^2 + \tau C_3 \delta(h,\tau).$$

Furthermore, since $\mathcal{B}$ is a monotone operator with respect to the components of u_n, then F is monotone.

Let $\mathcal{F}(x_{sk}, t_n) = \int_0^{t_n} F(x_{sk}, \eta, t_n) d\eta$, we approximate the integral by the quadrature formula used in (3.3) and the F samples $F(x_{sk}, t_j, t_n)$, by F_{sknj} defined in (3.15)

$$\mathcal{F}_{sk,n} = \tau \sum_{j=0}^{n} \omega_{n-j} F_{sknj}.$$

Then

$$\begin{aligned}
&\mathcal{F}(x_{sk}, t_n) - \mathcal{F}_{sk,n} \\
&= \int_0^{t_n} F(x_{sk}, \eta, t_n) d\eta - \tau \sum_{j=0}^{n} \omega_{n-j} F(x_{sk}, t_j, t_n) + \tau \sum_{j=0}^{n} \omega_{n-j} (F(x_{sk}, t_j, t_n) - F_{sknj}), \\
&|\mathcal{F}(x_{sk}, t_n) - \mathcal{F}_{sk,n}| \leq C\tau^q + (C_1 \tau^q + D_2 h^2 + \tau C_3 \delta(h, \tau)) \tau \sum_{j=0}^{n} \omega_j.
\end{aligned}$$

For $n \to \infty$, $\tau,\ h \to 0$, $N\tau = T$, we have

$$|\mathcal{F}(x_{sk}, t_n) - \mathcal{F}_{sk,n}| \leq \bar{D}_1 \tau^q + \bar{D}_2 h^2 + \tau \bar{D}_3 \delta(h, \tau),$$

with $\bar{D}_1$, $\bar{D}_2$ and $\bar{D}_3$ positive constants. Thus, the convergence order of the discretization is preserved.

4. Numerical Tests

4.1. Convergence test

In this paragraph we perform a test to show the convergence order of the numerical method with trapezoidal quadrature formula and second degree Bernstein polynomials.

For this reason we choose the following setting for the *single* Eq. (2.1): $k(t) = e^{-t}$, $g(x, t, \eta) = x + x^2(e^{\eta - t} - 1)$, $f(x, t) = e^{-2t}[(x-1)(2x^3 + e^t(-1 - x - tx^2 + (t-2)x^3)) + x^4 \sinh(t)]$, in the domain $\Omega = [0, 1]$, for $\eta < t \in [0, T]$, then the solution is

$$u(x, t) = 1 - x^2 e^{-t}.$$

We solve this problem up to time $T = 1$, by using the Bernstein polynomials of second degree as described in Eq. (3.3). We report the error of the calculated solution with respect to the true solution versus computation with variable time and space mesh sizes, that confirms the theoretical finding of the second order convergence.

N	M	$\max_{sk} \lvert u(x_{sk}, T) - u_{sk,N} \rvert$
31	20	$2.00 \cdot 10^{-4}$
61	40	$4.84 \cdot 10^{-5}$
121	80	$1.19 \cdot 10^{-5}$
241	160	$2.95 \cdot 10^{-6}$
481	320	$7.36 \cdot 10^{-7}$

4.2. The statistics of a filtered telegraph signal.

Although the analysis described in the previous sections is confined to the scalar equation (2.1), a generalization to systems is possible. Here we perform a test for a complete model of real application with $S = 2$. We study the statistics of a filtered random telegraph signal with the McFadden setting [23]. The two dynamical states are described by the ODEs

$$\dot{X}(t) = -\gamma_i X + W_i \qquad X(t_0) = X_0, \quad i = 1, 2. \tag{4.1}$$

The forward flux solutions, denoted with Φ_i, are

$$\Phi_i(t; x_0, t_0) = (x_0 - W_i/\gamma_i)\, e^{-\gamma_i(t-t_0)} + W_i/\gamma_i$$

and the inverse are

$$\Phi_i^{-1}(t_0; x, t) = (x - W_i/\gamma_i)\, e^{\gamma_i(t-t_0)} + W_i/\gamma_i.$$

We set $W_1 = W$, $W_2 = -W$ and $\gamma_{1,2} = \gamma$, from that it is easy to recognize the invariant domain of the process is $\Omega := [-W/\gamma, +W/\gamma]$. In fact, from the forward map we see that $X(t) \in \Omega$ if the initial data is into the same domain $X_0 \in \Omega$. Since the codomain must be equal to the domain, from the inverse maps we define

$$g_i(x, t, \eta) = \min(W/\gamma, \max(-W/\gamma, \Phi_i^{-1}(\eta; x, t))).$$

The driving semi-Markov process is set with the stochastic matrix $q_{11} = q_{22} = 0$, $q_{12} = q_{21} = 1$, and with the switching time distributions $k_1(t) = k_2(t) = 3e^{-t}(1 - e^{-t})^2$. We set smooth initial distributions of (1.2) as follows

$$\tilde{F}_1(\xi) = \frac{(\xi + W/\gamma)^2}{2(2W/\gamma)^2}, \quad \tilde{F}_2(\xi) = \frac{1}{2} - \frac{(W/\gamma - \xi)^2}{2(2W/\gamma)^2}.$$

Thus, Eq. (1.1) reads as

$$u_1(x, t) = \tilde{F}_2(g_2(x, t, 0))\, k_2(t) + \int_0^t u_2(g_2(x, t, \eta), \eta)\; k_2(t - \eta)\, d\eta, \tag{4.2a}$$

$$u_2(x, t) = \tilde{F}_1(g_1(x, t, 0))\, k_1(t) + \int_0^t u_1(g_1(x, t, \eta), \eta)\; k_1(t - \eta)\, d\eta. \tag{4.2b}$$

In this function settings, all hypotheses $h1) - h5)$ and $h8)$ are accomplished for each equation in (4.2), hence we expect a positive and monotone behavior in the solution components.

When the solution of this system of integral equations is found, the marginal probability distribution functions are determined by (1.4). Then by using (1.5) and taking the derivative $\mathcal{P}_i(x, t) = \partial_x \mathcal{F}_i(x, t)$, the meaningful probability density distribution are determined.

We solve numerically Eq. (4.2), with $W = 1$ and $\gamma = 1$, i.e. $\Omega = [-1, 1]$, according to the numerical scheme proposed in this paper. The total probability density distribution function $\mathcal{P}(x, T) = \mathcal{P}_1(x, T) + \mathcal{P}_2(x, T)$, at the time $T = 1$, is plotted in Fig. 4.1.

Since we have no analytical solution available, we perform the convergence error test by numerically calculating a reference solution with a finer mesh, $M = 129$, on the space, then the error is evaluated with the numerical solution with coarses meshes. We note from Fig. 4.1 that there are two points where the required continuity C^2 of the solution $\mathcal{F}(x, T)$ is not satisfied. However, this fact does not affect the convergence results, because when evaluating the error on the coarses meshes these points are almost surely excluded.

Table 4.1: Errors $\max_{sk} |\mathcal{F}(x_{sk}, T) - \mathcal{F}_{sk,N}|$ of the numerical solution for the convergence test.

$M \diagdown m$	1	2	3	10
17	$1.63 \cdot 10^{-4}$	$8.78 \cdot 10^{-5}$	$6.06 \cdot 10^{-5}$	$1.83 \cdot 10^{-5}$
33	$4.15 \cdot 10^{-5}$	$2.06 \cdot 10^{-5}$	$1.37 \cdot 10^{-5}$	$4.18 \cdot 10^{-6}$
65	$8.16 \cdot 10^{-6}$	$4.11 \cdot 10^{-6}$	$2.76 \cdot 10^{-6}$	$8.32 \cdot 10^{-7}$

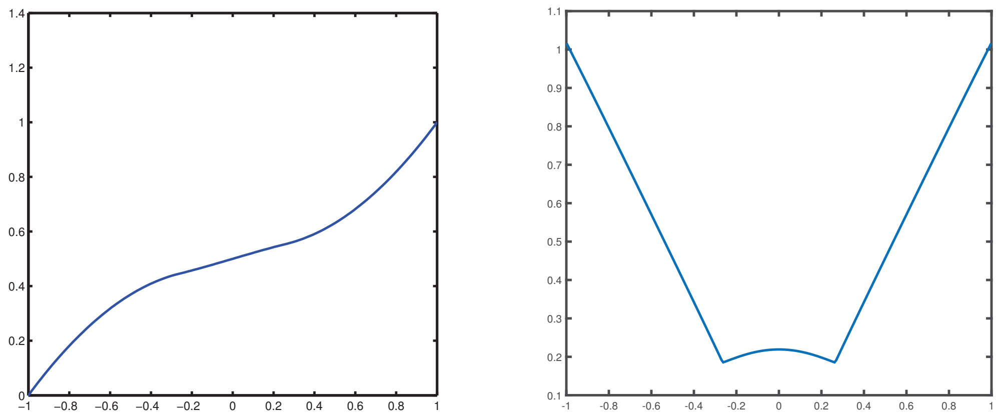

Fig. 4.1. Numerical solution of the probability distribution function $\mathcal{F}(x, T)$ (left) and its density $\mathcal{P}(x, T)$ (right) at the final time $T = 1$, for the integral formulation of the Chapman-Kolmogorov equation related to the filtered random telegraph signal with the McFadden switching time interval distribution. Degree of the Bernstein's polynomials $m = 2$, $M = 129$, $N = 400$.

In the following table we report such errors in the maximum norm, for the degrees of Bernstein's polynomials $m = 1, 2, 3, 10$.

From the values we can see that according to (3.6) and Remark 3.1, the second order convergence error is confirmed as well as the improvement of constant C_2 as the degree m of the Bernstein polynomial grows. Furthermore, Figure 4.1 show that, as expected from the theory developed in Section 3, the properties of positivity and monotonicity of the continuous solution are preserved.

5. Conclusions

This paper is devoted to the analysis of a Volterra-Renewal equation with space fluxes which represents an equation for the distribution function of a class of piecewise deterministic stochastic processes.

We have proved the positivity and monotonicity of its solution and we have carried out an analogous study on the numerical approximation obtained by a direct quadrature method along time and approximation via Bernstein polynomials in space. The quadratic convergence of the numerical method and the influence of the polynomial degree on the global error are proved both theoretically and by numerical experiments.

All the analysis carried out in this paper has been performed on a scalar equation, however a generalization to systems is straightforward. This motivates our experiments on more significative test examples such as the one related to the filtered random telegraph signal.

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