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PARAREAL ALGORITHMS APPLIED TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONSERVED QUANTITIES*

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Abstract

In this paper, we couple the parareal algorithm with projection methods of the trajectory on a specific manifold, defined by the preservation of some conserved quantities of stochastic differential equations. First, projection methods are introduced as the coarse and fine propagators. Second, we apply the projection methods for systems with conserved quantities in the correction step of original parareal algorithm. Finally, three numerical experiments are performed by different kinds of algorithms to show the property of convergence in iteration, and preservation in conserved quantities of model systems.

Mathematics subject classification: 60H10, 60H35, 65Y05
Key words: Stochastic differential equation, Parareal algorithm, Conserved quantity, Structure-preserving method.

1. Introduction

Designing highly efficient algorithms is an important subject of numerical computation due to the computational time and memory issues in the solution of large scale problems. The technique of parallel algorithms attracted more and more attention in the past few years, containing domain decomposition method in spatial direction and the parallel in time direction generally. The parareal algorithm, our focus in the sequel, was first introduced by Lions et al. [1], further work modified by Bal and Maday in [2], and has attracted vast attention in the last decade. Compared with other parallel approaches, this algorithm belongs to time-parallel category. The general idea of parareal algorithm contains roughly three steps as follows. First, we obtain an approximate solution on a coarse time-step by a rough solver. Second, we use another more accurate solver to get the approximation on each coarse time interval (splitting the coarse time interval into more fine time domain) performed in parallel with initial values computed in the first step. Finally, combining the values of the above two steps in the coarse time grids, we obtain a new approximation value by a prediction and correction iteration. In

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general, this algorithm has better parallel performance and is easy to perform, which motivates the development of efficient parallel methods for time dependent problems. Since the parareal algorithm was proposed, many efforts have been made to analyze it theoretically [3] and numerically, which verify the effectiveness of the parareal algorithm for a large various of problems, including control theory [4], Navier-Stokes problem [5] and Hamiltonian differential equations [6, 7] for instance.

Stochastic differential equations (SDEs) have attracted considerable attention in order to obtain much more realistic mathematical models in many scientific disciplines, such as physics, molecular biology, population dynamics and finance [8,9]. However, it is difficult to find explicit solutions of SDEs analytically; therefore, there has been tremendous interest in developing effective and reliable numerical methods for SDEs (e.g. [10–12] and references therein). It is also a significant issue whether some geometric features of SDEs are preserved in performing reliable numerical methods, especially for long-time simulation, which is as important as the deterministic case [13, 14]. In practice, they are time consuming, so the parallel techniques can be considered to speed up the original integrator. For stochastic problem, the application of parallel algorithm are relatively few. For example, the parareal algorithm has been applied to stochastic ordinary differential equations with filter problems [15] and stochastic models in chemical kinetics [16]. However, to the best of our knowledge, no results on parareal algorithm focusing on stochastic differential equations with conserved quantities. In order to apply the parareal algorithm to SDEs with conserved quantities, as mentioned in [6, 7, 17], the original algorithm are not able to share this kind of conservative property, namely, the preservation of conserved quantities along the sample path of the exact solution, even though when the coarse and fine integrators all have adequate conservative properties. Therefore, the behavior of long time numerical simulation is not enjoyed as the original system itself has. In this paper, we mainly utilize the projection methods for SDEs with conserved quantities as the basic propagators and the parareal algorithm with a projection corrector, which preserve some conserved quantities of the exact flow as proposed in [6].

The rest of the paper is organized as follows. Section 2 briefly recalls the parareal algorithm for general time-dependent problem. Section 3 discusses the procedure projection methods for SDEs with conserved quantities, and gives the corresponding mean-square convergence. Next in Section 4, we consider the parareal algorithm focusing on the SDEs with certain conserved quantities, which combines the ideas of the previous two sections. Finally, three typical SDE examples are chosen to perform numerical tests in Section 5.

2. The Original Parareal Algorithm

In this section, we first review the original parareal algorithm for a general initial-value problem:

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a suitable function to ensure the well-posedness of (2.1). To perform the parareal algorithm, we first divide time interval $[0, T]$ into N uniform large time intervals $[T_n, T_{n+1}]$, with step-size $\Delta T = T_{n+1} - T_n$ $n = 0, 1, \ldots, N-1$, and $N = \frac{T}{\Delta T}$. Then, we further divide every large interval $[T_n, T_{n+1}]$ into J small time intervals $[t_{n+\frac{j}{J}}, t_{n+\frac{j+1}{J}}]$, $j = 0, 1, \ldots, J-1$. With that, two numerical propagators, the coarse propagator $\mathscr{G}$ and the fine propagator $\mathscr{F}$, are needed here. In fact, $\mathscr{G}$ is usually easy to solve with low convergence