

THE HIGH ORDER BLOCK RIP CONDITION FOR SIGNAL RECOVERY*

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Abstract

In this paper, we consider the recovery of block sparse signals, whose nonzero entries appear in blocks (or clusters) rather than spread arbitrarily throughout the signal, from incomplete linear measurements. A high order sufficient condition based on block RIP is obtained to guarantee the stable recovery of all block sparse signals in the presence of noise, and robust recovery when signals are not exactly block sparse via mixed l_2/l_1 minimization. Moreover, a concrete example is established to ensure the condition is sharp. The significance of the results presented in this paper lies in the fact that recovery may be possible under more general conditions by exploiting the block structure of the sparsity pattern instead of the conventional sparsity pattern.

Mathematics subject classification: 90C59, 94A12, 94A20.

Key words: Block sparsity, Block restricted isometry property, Compressed sensing, Mixed l_2/l_1 minimization.

1. Introduction

Compressed sensing (CS), a new type of sampling theory, is a fast growing field of research. It has attracted considerable interest in a number of fields including applied mathematics, statistics, seismology, signal processing and electrical engineering. Interesting applications include radar system [26,50], coding theory [1,13], DNA microarrays [39], color imaging [33], magnetic resonance imaging [31]. Up to now, there are already many works on CS [3,4,15–17,29,30,40–44]. The key problem in CS is to recover an unknown high-dimensional sparse signal $x \in \mathbb{R}^N$ using an efficient algorithm through a sensing matrix $A \in \mathbb{R}^{n \times N}$ and the following linear measurements

$$y = Ax + z \tag{1.1}$$

where the observed signal $y \in \mathbb{R}^n$, $n \ll N$ and the vector of measurement errors $z \in \mathbb{R}^n$. In general, the solutions to the underdetermined system of linear equations (1.1) are not unique. But now suppose that x is known to be sparse in the sense that it contains only a small number of nonzero entries, which can occur in anywhere in x . This premise fundamentally changes the problem such that there is a unique sparse solution under regularity conditions. It is well known the l_1 minimization approach, a widely used algorithm, is an effective way to recover sparse signals in many settings. One of the most widely used frameworks to depict recovery

* Received July 7, 2017 / Revised version received September 14, 2017 / Accepted October 10, 2017 /
Published online August 22, 2018 /

ability of l_1 minimization in CS is the restricted isometry property (RIP) introduced by Candès and Tao [13]. Let $A \in \mathbb{R}^{n \times N}$ be a matrix and $1 \leq k \leq N$ is an integer, the restricted isometry constant (RIC) δ_k of order k is defined as the smallest nonnegative constant that satisfies

$$(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2,$$

for all k -sparse vectors $x \in \mathbb{R}^N$. A vector $x \in \mathbb{R}^N$ is k -sparse if $|\text{supp}(x)| \leq k$, where $\text{supp}(x) = \{i : x_i \neq 0\}$ is the support of x . When k is not an integer, we define δ_k as $\delta_{\lceil k \rceil}$. It has been shown l_1 minimization can recover a sparse signal with a small or zero error under some appropriate RIC met by the measurement matrix A [5–12, 23, 24, 37]. As far as we know, a sharp sufficient condition based on RIP for exact and stable recovery of signals in both noiseless and noisy cases by l_1 minimization was established by Cai and Zhang [8].

However, in practical examples, there are signals which have a particular sparsity pattern, where the nonzero coefficients appear in some blocks (or clusters). Such signals are referred to as block sparse [19, 20, 46]. In practice, the block sparse structure is very common, such as reconstruction of multi-band signals [35], equalization of sparse communication channels [18] and multiple measurement vector (MMV) model [20, 21, 36]. Actually, the notion of block sparsity was already introduced in statistics literature and was named the group Lasso estimator [2, 14, 27, 34, 38, 48]. Recently, block sparsity pattern has attracted significant attention. Various efficient methods and explicit recovery guarantees [19, 20, 22, 25, 28, 32, 45–47] have been proposed.

In this paper, our goal is to recover the unknown signal x from linear measurements (1.1). But at the moment, nonzero elements of signal x are occurring in blocks (or clusters) instead of spreading arbitrarily throughout the signal vector. To this end, firstly, we need the concept of block sparsity. In order to emphasize the block structure, similar to [20, 46], we view x as a concatenation of blocks over $\mathcal{I} = \{d_1, d_2, \dots, d_M\}$. Then x can be expressed as

$$x = \underbrace{(x_1, \dots, x_{d_1})}_{x[1]}, \underbrace{(x_{d_1+1}, \dots, x_{d_1+d_2})}_{x[2]}, \dots, \underbrace{(x_{N-d_M+1}, \dots, x_N)}_{x[M]} \in \mathbb{R}^N,$$

where $x[i]$ denotes the i th block of x with the length d_i and $N = \sum_{i=1}^M d_i$. A vector $x \in \mathbb{R}^N$ is called block k -sparse over $\mathcal{I} = \{d_1, d_2, \dots, d_M\}$ if the number of nonzero vectors $x[i]$ is at most k for $i \in \{1, 2, \dots, M\}$. Define

$$\|x\|_{2,0} = \sum_{i=1}^M I(\|x[i]\|_2 > 0),$$

where $I(\cdot)$ is an indicator function that it equals to 1 if its argument is larger than zero and 0 elsewhere. Then the block k -sparse vector over $\mathcal{I} = \{d_1, d_2, \dots, d_M\}$ can be cast as $\|x\|_{2,0} \leq k$. If $d_i = 1$ for all i , block sparsity is just the conventional sparsity. Next, one of the efficient methods to recover block sparse signals is mixed l_2/l_1 minimization

$$\min_x \|x\|_{2,1}, \quad \|y - Ax\|_2 \leq \varepsilon, \quad (1.2)$$

where $\|x\|_{2,1} = \sum_{i=1}^M \|x[i]\|_2$. Moreover, mixed norm $\|x\|_{2,2} = (\sum_{i=1}^M \|x[i]\|_2^2)^{1/2}$ and $\|x\|_{2,\infty} = \max_i \|x[i]\|_2$. Note that $\|x\|_{2,2} = \|x\|_2$. It is easy to know the mixed norm minimization is a generalization of conventional norm minimization. To ensure uniqueness and stability of solution for the system (1.1) via mixed l_2/l_1 minimization, Eldar and Mishali [20] generalized the notion of standard restricted isometry property to block sparse vectors, and obtained the following concept of block restricted isometry property (block RIP).

Definition 1.1 (block RIP) Let $A \in \mathbb{R}^{n \times N}$ be a matrix, then A has the k order block restricted isometry property over $\mathcal{I} = \{d_1, d_2, \dots, d_M\}$ with nonnegative parameter $\delta_{k|\mathcal{I}}$ if

$$(1 - \delta_{k|\mathcal{I}})\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_{k|\mathcal{I}})\|x\|_2^2$$

holds for all block k -sparse vectors $x \in \mathbb{R}^N$ over \mathcal{I} . The smallest constant $\delta_{k|\mathcal{I}}$ is called block restricted isometry constant (block RIC). When k is not an integer, we define $\delta_{k|\mathcal{I}}$ as $\delta_{\lceil k \rceil|\mathcal{I}}$.

For simplicity, we use δ_k for the block RIP constant $\delta_{k|\mathcal{I}}$ in the remainder of this paper. The block RIP plays a role similar to standard RIP. The block RIP provides recovery guarantee for block sparse signals. For example, Eldar and Mishali [20] proved that if matrix A satisfies block restricted isometry constant (block RIC) $\delta_{2k} < \sqrt{2} - 1$, the mixed l_2/l_1 minimization can recover exactly the block k -sparse signals in noiseless case, and can approximate the best block k -sparse solution in the presence of noise and mismodeling errors. Furthermore, they illustrated the advantage of block RIP over standard RIP. That is, the probability to satisfy the standard RIP is less than that of satisfying the block RIP. Meanwhile, a specific example is given to account for the advantage. They also experimentally demonstrated the advantage of their algorithm (mixed l_2/l_1 minimization) over standard basis pursuit. This explained the performance advantage of block sparse recovery over standard sparse recovery. Later, Lin and Li [28] improved the bound of block RIC to $\delta_{2k} < 0.4931$, and also gave another one order sufficient condition of recovery based on block RIC $\delta_k < 0.307$. There are many other recovery guarantees and efficient recovery methods to ensure the recovery of signals with special structure. For example, block coherence [19], strong group sparsity [27], l_2/l_p ($0 < p < 1$) minimization [46], BOMP [25, 47].

In this paper, we investigate the high order block RIP conditions for the exact or stable recovery of signals with block structure from (1.1) via solving mixed l_2/l_1 minimization in noiseless and noisy cases. Using ideas similar to [8], we establish a sufficient condition on δ_{tk} to ensure the stable or exact recovery of signals with nonzero entries occurring in blocks (or clusters) rather than being arbitrarily spread throughout the signal vector. The key is to generalize the technique of sparse representation of a polytope [8] to the block setting. We show that block RIC $\delta_{tk} < \sqrt{\frac{t-1}{t}}$ for any $t > 1$ can ensure exact and stable recovery for all block sparse signals and robust recovery for nearly block sparse signals via mixed l_2/l_1 minimization. Moreover, it is sharp when $t \geq 4/3$. A concrete example is given to illustrate the optimality. Actually, our results are a generalization of that of Cai and Zhang [8] in the block setting. When $d_i = 1$ for $i \in \{1, \dots, M\}$, our results return to those of Cai and Zhang [8]. The significance of our results lies in the fact that taking advantage of explicit block sparsity has better reconstruction performance than viewing the signals as being standard sparsity, accordingly ignoring the additional structure in the problem.

The rest of the paper is organized as follows. In Section 2, we will introduce some notations and establish some basic lemmas that will be used. The main results and their proofs are given in Section 3. Finally, we summarize this paper in Section 4.

2. Preliminaries

Throughout this paper, we adopt the following notations unless otherwise stated. For any $x \in \mathbb{R}^N$, we model it over $\mathcal{I} = \{d_1, d_2, \dots, d_M\}$. $x[i]$ denotes the i th block of x . Let $\mathbf{0}$ be the zero vector whose dimension may be different. $\Gamma \subset \{1, 2, \dots, M\}$ indicates block indices, Γ^c is the

complement of Γ in $\{1, 2, \dots, M\}$. $x[\Gamma] \in \mathbb{R}^N$ denotes a vector which equals to x on block indices Γ and 0 otherwise. For example, if $\Gamma = \{1, 3, M\}$, then $x[\Gamma] = (x[1], \mathbf{0}, x[3], \mathbf{0}, \dots, \mathbf{0}, x[M])^T \in \mathbb{R}^N$, and $x[\Gamma][i]$ denotes i th block of $x[\Gamma]$. We denote by $\text{supp}[x] = \{i : \|x[i]\|_2 \neq 0\}$ the block support of x , and \mathcal{I}_0 the block indices of the k largest block in l_2 norm of x , i.e., $\|x[i]\|_2 \geq \|x[j]\|_2$ for any $i \in \mathcal{I}_0$ and $j \in \mathcal{I}_0^c$. We also denote $x[\max(k)]$ as x with all but the largest k blocks in l_2 norm set to zero. From now on, we always take that $h = \hat{x} - x$, where \hat{x} is the minimizer of l_2/l_1 minimization problem (1.2) and x is the original signal.

The following lemma provides a key technical tool for the proof of our main result. It is an extension of Lemma 1.1 introduced by Cai and Zhang [8]. We extend sparse representation of a polytope to the block setting.

Lemma 2.1. *For a positive number α and a positive integer k , define the block polytope $T(\alpha, k) \subset \mathbb{R}^N$ by*

$$T(\alpha, k) = \left\{ v \in \mathbb{R}^N : \|v\|_{2,\infty} \leq \alpha, \|v\|_{2,1} \leq k\alpha \right\}.$$

For any $v \in \mathbb{R}^N$, define the set of block sparse vectors $U(\alpha, k, v) \subset \mathbb{R}^N$ by

$$U(\alpha, k, v) = \left\{ u \in \mathbb{R}^N : \text{supp}(u) \subseteq \text{supp}(v), \|u\|_{2,0} \leq k, \|u\|_{2,1} = \|v\|_{2,1}, \|u\|_{2,\infty} \leq \alpha \right\}.$$

Then any $v \in T(\alpha, k)$ can be expressed as

$$v = \sum_{i=1}^J \lambda_i u_i,$$

where $u_i \in U(\alpha, k, v)$ and $0 \leq \lambda_i \leq 1, \sum_{i=1}^J \lambda_i = 1$.

Proof. First of all, what we have to prove is that $v \in T(\alpha, k)$ is in the convex hull of $U(\alpha, k, v)$. To show the statement, we proceed by induction.

Suppose $v \in T(\alpha, k)$. If v is block k -sparse, v itself is in $U(\alpha, k, v)$. Thus, assume that the assertion is true for all block $(s-1)$ -sparse vectors v ($s-1 \geq k$), then we show that the assertion is also true for any block s -sparse vectors v . For any block s -sparse vectors $v \in T(\alpha, k)$, (without loss of generality, suppose that v is not block $(s-1)$ -sparse, otherwise the result holds by assumption of block $(s-1)$ -sparse), we have $\|v\|_{2,\infty} \leq \alpha, \|v\|_{2,1} \leq k\alpha$. Furthermore, v can be expressed as $v = \sum_{i=1}^s c_i E_i$ with $c_1 \geq c_2 \geq \dots \geq c_s > 0$, where c_1 equals to the largest $\|v[i]\|_2$ for every $i \in \{1, 2, \dots, M\}$, c_2 equals to the next largest $\|v[i]\|_2$, and so on, where E_i is a unit vector in \mathbb{R}^N , which equals to v/c_i on the i th largest block of v and zero elsewhere. Let

$$\Omega = \left\{ 1 \leq l \leq s-1 : c_l + c_{l+1} + \dots + c_s \leq (s-l)\alpha \right\}.$$

Owing to $\sum_{i=1}^s c_i = \|v\|_{2,1} \leq k\alpha$, Ω is not empty for $1 \in \Omega$. We denote by l the largest element in Ω . It is easy to get

$$c_l + c_{l+1} + \dots + c_s \leq (s-l)\alpha, \quad (2.1a)$$

$$c_{l+1} + c_{l+2} + \dots + c_s > (s-l-1)\alpha. \quad (2.1b)$$

It is worthy of noting that (2.1) also holds when the largest element in Ω is $s-1$. Take

$$b_j = \frac{\sum_{i=l}^s c_i}{s-l} - c_j, \quad l \leq j \leq s. \quad (2.2)$$

By direct calculations, we have $(s-l) \sum_{i=l}^s b_i = \sum_{i=l}^s c_i$ and $b_j \geq b_l$ for all $l \leq j \leq s$. Moreover, for any $l \leq j \leq s$,

$$\begin{aligned} b_j \geq b_l &= \frac{\sum_{i=l+1}^s c_i - (s-l-1)c_l}{s-l} \\ &\geq \frac{\sum_{i=l+1}^s c_i - (s-l-1)\alpha}{s-l} > 0, \end{aligned}$$

where the second inequality follows from the fact that $\|v\|_{2,\infty} \leq \alpha$, the last inequality is a result of the second inequality in (2.1). Next, define

$$\lambda_j = \frac{b_j}{\sum_{i=l}^s b_i}, \quad v_j = \sum_{i=1}^{l-1} c_i E_i + \left(\sum_{i=l}^s b_i \right) \sum_{i=l, i \neq j}^s E_i \in \mathbb{R}^N,$$

for every $l \leq j \leq s$, we have

$$v = \sum_{j=l}^s \lambda_j v_j, \quad 0 < \lambda_j \leq 1, \quad \sum_{j=l}^s \lambda_j = 1, \quad \text{supp}(v_j) \subseteq \text{supp}(v).$$

In addition, using the fact that $(s-l) \sum_{i=l}^s b_i = \sum_{i=l}^s c_i$, $\|v\|_{2,\infty} \leq \alpha$ and the first inequality in (2.1), we have

$$\begin{aligned} \|v_j\|_{2,1} &= \sum_{i=1}^{l-1} c_i + (s-l) \sum_{i=l}^s b_i = \sum_{i=1}^{l-1} c_i + \sum_{i=l}^s c_i = \|v\|_{2,1}, \\ \|v_j\|_{2,\infty} &= \max \left\{ c_1, \dots, c_{l-1}, \sum_{i=l}^s b_i \right\} \leq \max \left\{ \alpha, \frac{\sum_{i=l}^s c_i}{s-l} \right\} \leq \alpha. \end{aligned}$$

Finally, since v_j is block $(s-1)$ -sparse, under the induction assumption, we have $v_j = \sum_{i=1}^J \mu_{j,i} u_{j,i}$, where $u_{j,i}$ is block k -sparse, $\|u_{j,i}\|_{2,1} = \|v_j\|_{2,1} = \|v\|_{2,1}$, $\|u_{j,i}\|_{2,\infty} \leq \alpha$ and $0 \leq \mu_{j,i} \leq 1$, $\sum_{i=1}^J \mu_{j,i} = 1$. Hence, $v = \sum_{i=1}^J \sum_{j=l}^s \lambda_j \mu_{j,i} u_{j,i}$, which implies that the statement is true for s .

On the other hand, if v is in the convex hull of $U(\alpha, k, v)$, then $v = \sum_{i=1}^J \lambda_i u_i$, $u_i \in U(\alpha, k, v)$ and $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^J \lambda_i = 1$. It follows immediately that

$$\begin{aligned} \|v\|_{2,1} &= \left\| \sum_{i=1}^J \lambda_i u_i \right\|_{2,1} \leq \sum_{i=1}^J \lambda_i \|u_i\|_{2,1} \leq \sum_{i=1}^J \lambda_i \|u_i\|_{2,0} \|u_i\|_{2,\infty} \leq k\alpha, \\ \|v\|_{2,\infty} &= \left\| \sum_{i=1}^J \lambda_i u_i \right\|_{2,\infty} \leq \sum_{i=1}^J \lambda_i \|u_i\|_{2,\infty} \leq \alpha, \end{aligned}$$

which completes the proof. \square

The following lemma is a useful elementary inequality, which will be used in proving our main results.

Lemma 2.2 ([9], **Lemma 5.3**) *Assume $m \geq k$, $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$, $\sum_{i=1}^k a_i \geq \sum_{i=k+1}^m a_i$. Then for all $\alpha \geq 1$,*

$$\sum_{j=k+1}^m a_j^\alpha \leq \sum_{i=1}^k a_i^\alpha.$$

More generally, assume $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$, $\lambda \geq 0$ and $\sum_{i=1}^k a_i + \lambda \geq \sum_{i=k+1}^m a_i$, then for all $\alpha \geq 1$,

$$\sum_{j=k+1}^m a_j^\alpha \leq k \left(\sqrt[\alpha]{\frac{\sum_{i=1}^k a_i^\alpha}{k}} + \frac{\lambda}{k} \right)^\alpha.$$

From the definition of h , \hat{x} and x , we have the following lemma.

Lemma 2.3. *For any $\Gamma \subset \{1, 2, \dots, M\}$, it holds that*

$$\|h[\Gamma^c]\|_{2,1} \leq \|h[\Gamma]\|_{2,1} + 2\|x[\Gamma^c]\|_{2,1}.$$

Proof. Recall that $h = \hat{x} - x$. From the minimality of \hat{x} , it follows that

$$\|\hat{x}\|_{2,1} \leq \|x\|_{2,1} = \|x[\Gamma]\|_{2,1} + \|x[\Gamma^c]\|_{2,1}.$$

By the reverse triangle inequalities of $\|\cdot\|_{2,1}$, we obtain

$$\begin{aligned} \|\hat{x}\|_{2,1} &= \|x + h\|_{2,1} = \|x[\Gamma] + h[\Gamma]\|_{2,1} + \|x[\Gamma^c] + h[\Gamma^c]\|_{2,1} \\ &\geq \|x[\Gamma]\|_{2,1} - \|h[\Gamma]\|_{2,1} + \|h[\Gamma^c]\|_{2,1} - \|x[\Gamma^c]\|_{2,1}. \end{aligned}$$

The lemma follows from above inequalities immediately. \square

3. Main Results

With the preparations given in Section 2, we establish the main results in this section—a sharp high order block RIP condition for the robust recovery of arbitrary signals with block pattern. When the signal is block sparse, the sharp condition ensures the exact recovery and stable recovery in the noiseless case and in the noisy case, respectively. First, the following theorem provides a sufficient condition of recovery when x is not block sparse and the observation is noisy.

Theorem 3.1. *Suppose that $x \in \mathbb{R}^N$ is an arbitrary vector consistent with (1.1) and $\|z\|_2 \leq \varepsilon$. If the measurement matrix A satisfies the block RIP with $\delta_{tk} < \sqrt{\frac{t-1}{t}}$ for $t > 1$, the solution \hat{x} to (1.2) obeys*

$$\begin{aligned} \|\hat{x} - x\|_2 &\leq \frac{2\sqrt{2t(t-1)(1+\delta_{tk})}}{t(\sqrt{(t-1)/t} - \delta_{tk})} \varepsilon \\ &\quad + \left(\frac{\sqrt{2}\delta_{tk} + \sqrt{t(\sqrt{(t-1)/t} - \delta_{tk})\delta_{tk}}}{t(\sqrt{(t-1)/t} - \delta_{tk})} + 1 \right) \frac{2\|x[\mathcal{I}_0^c]\|_{2,1}}{\sqrt{k}}. \end{aligned} \quad (3.1)$$

Proof. First of all, suppose that tk is an integer. Let $\hat{x} = x + h$, where \hat{x} is a solution to l_2/l_1 minimization problem and x is the original signal. From Lemma 2.3 and the definition of \mathcal{I}_0 , $\|h[\mathcal{I}_0^c]\|_{2,1} \leq \|h[\mathcal{I}_0]\|_{2,1} + 2\|x[\mathcal{I}_0^c]\|_{2,1}$. We assume that T_0 is the block index set over the k blocks with largest l_2 norm of h . Hence,

$$\|h[T_0^c]\|_{2,1} \leq \|h[T_0]\|_{2,1} + 2\|x[\mathcal{I}_0^c]\|_{2,1}. \quad (3.2)$$

Denote $r = (\|h[T_0]\|_{2,1} + 2\|x[\mathcal{I}_0^c]\|_{2,1})/k$. Next, we decompose $h[T_0^c]$ as $h[T_0^c] = h[T_1] + h[T_2]$, where $h[T_1]$ remains the blocks of $h[T_0^c]$ whose l_2 norms are greater than $\frac{r}{t-1}$ and 0 elsewhere, $h[T_2]$ retains the blocks of $h[T_0^c]$ whose l_2 norms are not more than $\frac{r}{t-1}$ and 0 otherwise. Combining above definitions and (3.2), we have

$$\|h[T_1]\|_{2,1} \leq \|h[T_0^c]\|_{2,1} \leq kr.$$

Denote $\|h[T_1]\|_{2,0} = l$. From the definition of $h[T_1]$, we get

$$kr \geq \|h[T_1]\|_{2,1} = \sum_{i \in \text{supp}[h[T_1]]} \|h[T_1][i]\|_2 > \frac{lr}{t-1}.$$

Namely, $l < k(t-1)$. Thus, it is clear that

$$\begin{aligned} \|h[T_2]\|_{2,1} &= \|h[T_0^c]\|_{2,1} - \|h[T_1]\|_{2,1} \\ &< kr - \frac{lr}{t-1} = (k(t-1) - l) \cdot \frac{r}{t-1}, \end{aligned} \quad (3.3a)$$

$$\|h[T_2]\|_{2,\infty} \leq \frac{r}{t-1}, \quad (3.3b)$$

and $\|h[T_0] + h[T_1]\|_{2,0} = k + l < tk$. From the definition of δ_k ,

$$\begin{aligned} \langle A(h[T_0] + h[T_1]), Ah \rangle &\leq \|A(h[T_0] + h[T_1])\|_2 \|Ah\|_2 \\ &\leq \sqrt{1 + \delta_{tk}} \|h[T_0] + h[T_1]\|_2 \|Ah\|_2. \end{aligned} \quad (3.4)$$

Due to

$$\|Ah\|_2 \leq \|A\hat{x} - Ax\|_2 \leq \|A\hat{x} - y\|_2 + \|y - Ax\|_2 \leq 2\varepsilon, \quad (3.5)$$

so (3.4) becomes

$$\langle A(h[T_0] + h[T_1]), Ah \rangle \leq \sqrt{1 + \delta_{tk}} \|h[T_0] + h[T_1]\|_2 \cdot (2\varepsilon). \quad (3.6)$$

Using (3.3a) and Lemma 2.1, we have $h[T_2] = \sum_{i=1}^J \lambda_i u_i$, where u_i is block $(k(t-1) - l)$ -sparse, $\sum_{i=1}^J \lambda_i = 1$ with $0 \leq \lambda_i \leq 1$, and $\text{supp}(u_i) \subseteq \text{supp}(h[T_2])$, $\|u_i\|_{2,1} = \|h[T_2]\|_{2,1}$, $\|u_i\|_{2,\infty} \leq \frac{r}{t-1}$. Hence,

$$\begin{aligned} \|u_i\|_2 &= \|u_i\|_{2,2} \leq \sqrt{\|u_i\|_{2,0}} \|u_i\|_{2,\infty} \\ &\leq \sqrt{k(t-1) - l} \cdot \frac{r}{t-1} \leq \sqrt{\frac{k}{t-1}} r, \end{aligned}$$

where the first inequality follows from the fact that for any block k -sparse vector v ,

$$\|v\|_{2,2}^2 = \sum_i \|v[i]\|_2^2 \leq k \|v\|_{2,\infty}^2.$$

Let $X = \|h[T_0] + h[T_1]\|_{2,2}$, $P = 2\|x[\mathcal{I}_0^c]\|_{2,1}/\sqrt{k}$. Clearly,

$$\begin{aligned} \|u_i\|_2 &\leq \sqrt{\frac{k}{t-1}}r = \sqrt{\frac{k}{t-1}} \cdot \frac{\|h[T_0]\|_{2,1} + 2\|x[\mathcal{I}_0^c]\|_{2,1}}{k} \\ &\leq \frac{\|h[T_0]\|_{2,2}}{\sqrt{t-1}} + \frac{2\|x[\mathcal{I}_0^c]\|_{2,1}}{\sqrt{k(t-1)}} \leq \frac{\|h[T_0] + h[T_1]\|_{2,2}}{\sqrt{t-1}} + \frac{2\|x[\mathcal{I}_0^c]\|_{2,1}}{\sqrt{k(t-1)}} \\ &= \frac{X + P}{\sqrt{t-1}}, \end{aligned} \quad (3.7)$$

where the second inequality follows from applying Cauchy-Schwarz to any block k -sparse vector v , $\|v\|_{2,1} = \sum_i \|v[i]\|_2 \cdot 1 \leq \sqrt{k}\|v\|_{2,2}$. Take $\beta_i = h[T_0] + h[T_1] + \mu u_i$, where $0 \leq \mu \leq 1$. We observe that

$$\begin{aligned} \sum_{j=1}^J \lambda_j \beta_j - \frac{1}{2}\beta_i &= h[T_0] + h[T_1] + \mu h[T_2] - \frac{1}{2}\beta_i \\ &= \left(\frac{1}{2} - \mu\right)(h[T_0] + h[T_1]) + \mu h - \frac{\mu}{2}u_i. \end{aligned} \quad (3.8)$$

Moreover, β_i and $\sum_{j=1}^J \lambda_j \beta_j - \frac{1}{2}\beta_i - \mu h$ are block tk -sparse, due to the facts that $h[T_0]$ is block k -sparse, $h[T_1]$ is block l -sparse, and u_i is block $k(t-1) - l$ -sparse. Note that the following identity (see (25) in [8]) holds

$$\sum_{i=1}^J \lambda_i \left\| A \left(\sum_{j=1}^J \lambda_j \beta_j - \frac{1}{2}\beta_i \right) \right\|_2^2 = \sum_{i=1}^J \frac{\lambda_i}{4} \|A\beta_i\|_2^2. \quad (3.9)$$

We first bound the left hand side of (3.9). Substituting (3.8) into the left hand side of (3.9) and combining (3.6) and the definition of block RIP, we have the upper bound

$$\begin{aligned} &\sum_{i=1}^J \lambda_i \left\| A \left(\sum_{j=1}^J \lambda_j \beta_j - \frac{1}{2}\beta_i \right) \right\|_2^2 = \sum_{i=1}^J \lambda_i \left\| A \left(\left(\frac{1}{2} - \mu\right)(h[T_0] + h[T_1]) + \mu h - \frac{\mu}{2}u_i \right) \right\|_2^2 \\ &= \sum_{i=1}^J \lambda_i \left\| A \left(\left(\frac{1}{2} - \mu\right)(h[T_0] + h[T_1]) - \frac{\mu}{2}u_i \right) \right\|_2^2 \\ &\quad + 2 \left\langle A \left(\left(\frac{1}{2} - \mu\right)(h[T_0] + h[T_1]) - \frac{\mu}{2}h[T_2] \right), \mu Ah \right\rangle + \mu^2 \|Ah\|_2^2 \\ &= \sum_{i=1}^J \lambda_i \left\| A \left(\left(\frac{1}{2} - \mu\right)(h[T_0] + h[T_1]) - \frac{\mu}{2}u_i \right) \right\|_2^2 + \mu(1 - \mu) \langle A(h[T_0] + h[T_1]), Ah \rangle \\ &\leq (1 + \delta_{tk}) \sum_{i=1}^J \lambda_i \left\| \left(\frac{1}{2} - \mu\right)(h[T_0] + h[T_1]) - \frac{\mu}{2}u_i \right\|_2^2 + \mu(1 - \mu) \sqrt{1 + \delta_{tk}} \|h[T_0] + h[T_1]\|_2 \cdot (2\varepsilon) \\ &= (1 + \delta_{tk}) \sum_{i=1}^J \lambda_i \left(\left(\frac{1}{2} - \mu\right)^2 \|h[T_0] + h[T_1]\|_2^2 + \frac{\mu^2}{4} \|u_i\|_2^2 \right) + \mu(1 - \mu) \sqrt{1 + \delta_{tk}} \|h[T_0] + h[T_1]\|_2 \cdot (2\varepsilon). \end{aligned}$$

On the other hand, using the expression of β_i , the block RIP results in the lower bound

$$\begin{aligned} \sum_{i=1}^J \frac{\lambda_i}{4} \|A\beta_i\|_2^2 &= \sum_{i=1}^J \frac{\lambda_i}{4} \|A(h[T_0] + h[T_1] + \mu u_i)\|_2^2 \\ &\geq \sum_{i=1}^J \frac{\lambda_i}{4} (1 - \delta_{tk}) \|h[T_0] + h[T_1] + \mu u_i\|_2^2 \\ &= (1 - \delta_{tk}) \sum_{i=1}^J \frac{\lambda_i}{4} (\|h[T_0] + h[T_1]\|_2^2 + \mu^2 \|u_i\|_2^2). \end{aligned}$$

Combining the above two inequalities, we have

$$\begin{aligned} &\left[(\mu^2 - \mu) + \left(\frac{1}{2} - \mu + \mu^2 \right) \delta_{tk} \right] \|h[T_0] + h[T_1]\|_2^2 \\ &\quad + \mu(1 - \mu) \sqrt{1 + \delta_{tk}} \|h[T_0] + h[T_1]\|_2 \cdot (2\varepsilon) + \sum_{i=1}^J \lambda_i \frac{\delta_{tk}}{2} \mu^2 \|u_i\|_2^2 \geq 0. \end{aligned}$$

From (3.7) and the expression of X with the fact $\|\cdot\|_{2,2} = \|\cdot\|_2$, we obtain

$$\begin{aligned} &\left[(\mu^2 - \mu) + \left(\frac{1}{2} - \mu + \left(1 + \frac{1}{2(t-1)}\right) \mu^2 \right) \delta_{tk} \right] X^2 \\ &\quad + \left[2\varepsilon\mu(1 - \mu) \sqrt{1 + \delta_{tk}} + \frac{\mu^2 \delta_{tk} P}{t-1} \right] X + \frac{\mu^2 P^2 \delta_{tk}}{2(t-1)} \geq 0. \end{aligned}$$

Taking $\mu = \sqrt{t(t-1)} - (t-1)$, we have

$$\frac{\mu^2}{t-1} \left[-t \left(\sqrt{\frac{t-1}{t}} - \delta_{tk} \right) X^2 + \left(2\varepsilon \sqrt{t(t-1)(1 + \delta_{tk})} + P\delta_{tk} \right) X + \frac{P^2 \delta_{tk}}{2} \right] \geq 0.$$

The condition $\delta_{tk} < \sqrt{(t-1)/t}$ ensures that the above inequality is a second-order inequality for X and the quadratic coefficient is negative. Thus, we obtain

$$\begin{aligned} X &\leq \left\{ \left(2\varepsilon \sqrt{t(t-1)(1 + \delta_{tk})} + P\delta_{tk} \right) + \left[\left(2\varepsilon \sqrt{t(t-1)(1 + \delta_{tk})} + P\delta_{tk} \right)^2 \right. \right. \\ &\quad \left. \left. + 2t \left(\sqrt{(t-1)/t} - \delta_{tk} \right) P^2 \delta_{tk} \right]^{1/2} \right\} \cdot \left(2t \left(\sqrt{(t-1)/t} - \delta_{tk} \right) \right)^{-1} \\ &\leq \frac{2\sqrt{t(t-1)(1 + \delta_{tk})}}{t \left(\sqrt{(t-1)/t} - \delta_{tk} \right)} \varepsilon + \frac{2\delta_{tk} + \sqrt{2t \left(\sqrt{(t-1)/t} - \delta_{tk} \right) \delta_{tk}}}{2t \left(\sqrt{(t-1)/t} - \delta_{tk} \right)} P, \end{aligned}$$

where the last inequality is a result of the fact that $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ for any nonnegative constants a and b . With (3.2) and the representation of P , it is clear that

$$\|h[T_0^c]\|_{2,1} \leq \|h[T_0]\|_{2,1} + P\sqrt{k}.$$

From Lemma 2.2, we conclude that

$$\|h[T_0^c]\|_{2,2} \leq \|h[T_0]\|_{2,2} + P.$$

Therefore, it is not hard to see that

$$\begin{aligned}
\|h\|_2 &= \sqrt{\|h[T_0]\|_2^2 + \|h[T_0^c]\|_2^2} \leq \sqrt{\|h[T_0]\|_2^2 + (\|h[T_0]\|_2 + P)^2} \\
&\leq \sqrt{2}\|h[T_0]\|_2 + P \leq \sqrt{2}(\|h[T_0] + h[T_1]\|_2) + P = \sqrt{2}X + P \\
&\leq \frac{2\sqrt{2t(t-1)(1+\delta_{tk})}}{t(\sqrt{(t-1)/t} - \delta_{tk})} \varepsilon \\
&\quad + \left(\frac{\sqrt{2}\delta_{tk} + \sqrt{t(\sqrt{(t-1)/t} - \delta_{tk})\delta_{tk}}}{t(\sqrt{(t-1)/t} - \delta_{tk})} + 1 \right) \frac{2\|x[I_0^c]\|_{2,1}}{\sqrt{k}}.
\end{aligned}$$

If tk is not an integer, we denote $t' = \lceil tk \rceil / k$, then $t'k$ is an integer and $t < t'$. So we have $\delta_{t'k} = \delta_{tk} < \sqrt{\frac{t-1}{4-t}} < \sqrt{\frac{t'-1}{t'}}$. Using the method similar to the proof above, we can prove the result by working on $\delta_{t'k}$. Hence, we complete the proof of the theorem. \square

Remark 3.1. Recently, Zhang and Li in [49] have shown that when $0 < t < \frac{4}{3}$, the condition $\delta_{tk} < \frac{t}{4-t}$ is sufficient to guarantee the exact recovery for all k -sparse signals in the noiseless case via the constrained l_1 -norm minimization. These bounds are sharp as in the following theorem. These results give a complete answer to the conjecture on restricted isometry property constants δ_{tk} ($0 < t < \frac{4}{3}$) which was proposed by Cai and Zhang in [8]. We will establish analogous block restricted isometry conditions for the exact and stable recovery of signals with block pattern by mixed l_2/l_1 minimization. Due to the big difference as in [8] and [49], we postpone the details to a separate paper.

Remark 3.2. Theorem 3.1 and Remark 3.1 indicate that as long as measurement matrix A meets with the block RIP with a suitable constant, the mixed l_2/l_1 minimization method can robustly recover any signals with block structure from noisy measurements $y = Ax + z$. Moreover, if x is block k -sparse, then Theorem 3.1 guarantees perfect and stable recovery of x from its samples y in the noise-free and noisy setting.

The following theorem shows the condition $\delta_{tk} < \sqrt{\frac{t-1}{t}}$ with $t \geq \frac{4}{3}$ is sharp for exact and stable recovery in noiseless and noisy case, respectively.

Theorem 3.2. Let $t \geq \frac{4}{3}$. For any $\varepsilon > 0$ and $k \geq \frac{5}{\varepsilon}$, then there exist a sensing matrix $A \in \mathbb{R}^{n \times N}$ with $\delta_{tk} < \sqrt{\frac{t-1}{t}} + \varepsilon$ and some block k -sparse signal x_0 such that

- (1) In the noiseless case, i.e., $y = Ax_0$, the mixed l_2/l_1 minimization (1.2) can not exactly recover the block k -sparse signal x_0 , i.e., $\hat{x} \neq x_0$, where \hat{x} is the solution to (1.2).
- (2) In the noise case, i.e., $y = Ax_0 + z$, the mixed l_2/l_1 minimization (1.2) can not stably recover the block k -sparse signal x_0 , i.e., $\hat{x} \not\rightarrow x_0$ as $z \rightarrow 0$, where \hat{x} is the solution to (1.2).

Proof. For all $\varepsilon > 0$ and $k \geq \frac{5}{\varepsilon}$, let $a' = ((t-1) + \sqrt{t(t-1)})k$ and $N \geq (k + a')d$. Since $t \geq \frac{4}{3}$, we have $a' \geq k$. Suppose that a is the largest integer strictly smaller than a' , then $a < a'$

and $a' - a \leq 1$. Denote

$$\gamma = \frac{1}{\sqrt{kd + \frac{ak^2}{a'^2}d}} \left(\underbrace{1, \dots, 1}_{k \text{ blocks}}, \underbrace{1, \dots, 1}_{d \text{ blocks}}, \underbrace{-\frac{k}{a'}, \dots, -\frac{k}{a'}}_{a \text{ blocks}}, \underbrace{-\frac{k}{a'}, \dots, -\frac{k}{a'}}_{d \text{ blocks}}, \mathbf{0}, \dots, \mathbf{0} \right) \in \mathbb{R}^N,$$

where $k + a \leq M$, $\mathcal{I} = \{d_1 = d, \dots, d_{k+a} = d, d_{k+a+1}, \dots, d_M\}$ and $\|\gamma\|_2 = 1$. Define the linear map $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$Ax = \sqrt{1 + \sqrt{\frac{t-1}{t}}} (x - \langle \gamma, x \rangle \gamma),$$

for all $x \in \mathbb{R}^N$. Then for any block $\lceil tk \rceil$ -sparse signal x , we have

$$\|Ax\|_2^2 = \left(1 + \sqrt{\frac{t-1}{t}}\right) \left(\|x\|_2^2 - |\langle \gamma, x \rangle|^2\right).$$

From the Cauchy-Schwarz inequality and $a' \geq k$, $a' - a \leq 1$ as well as the fact $\frac{a'^2 + k^2(t-1)}{a'^2 + a'k} = 2\sqrt{t-1}(\sqrt{t} - \sqrt{t-1})$ (see the proof of Theorem 2.2, [8]), therefore,

$$\begin{aligned} 0 \leq |\langle \gamma, x \rangle|^2 &= |\langle \gamma[\text{supp}[x]], x \rangle|^2 \leq \|x\|_2^2 \cdot \|\gamma[\max(\lceil tk \rceil)]\|_2^2 \\ &\leq \|x\|_2^2 \cdot \frac{a'^2 + k(\lceil tk \rceil - k)}{a'^2 + ak} \leq \frac{a'^2 + k^2(t-1) + k}{a'^2 + ak} \|x\|_2^2 \\ &= \frac{a'^2 + k^2(t-1) + k}{a'^2 + a'k} \cdot \frac{a'^2 + a'k}{a'^2 + ak} \|x\|_2^2 \\ &= \frac{a'^2 + k^2(t-1) + k}{a'^2 + a'k} \cdot \frac{1}{1 - \frac{k(a'-a)}{a'^2 + a'k}} \|x\|_2^2 \\ &= \frac{a'^2 + k^2(t-1)}{a'^2 + a'k} \cdot \frac{a'^2 + k^2(t-1) + k}{a'^2 + k^2(t-1)} \cdot \frac{1}{1 - \frac{k(a'-a)}{a'^2 + a'k}} \|x\|_2^2 \\ &\leq 2\sqrt{t-1}(\sqrt{t} - \sqrt{t-1}) \cdot \left(1 + \frac{1}{tk}\right) \cdot \frac{1}{1 - \frac{1}{2k}} \|x\|_2^2 \\ &\leq 2\left(\sqrt{t(t-1)} - (t-1)\right) \cdot \left(1 + \frac{5}{2k}\right) \|x\|_2^2 \\ &\leq \left(2\sqrt{t(t-1)} - 2(t-1) + \frac{5}{2k}\right) \|x\|_2^2, \end{aligned}$$

where $\gamma[\max(\lceil tk \rceil)]$ denotes a vector remaining the $\lceil tk \rceil$ blocks with largest l_2 norm of γ and zero elsewhere and $\|\gamma[\max(\lceil tk \rceil)]\|_2^2 \leq \frac{a'^2 + k(\lceil tk \rceil - k)}{a'^2 + ak}$. The last inequality follows that $2\sqrt{t(t-1)} -$

$2(t-1) \leq 1$. Consequently,

$$\begin{aligned}
& \left(1 + \sqrt{\frac{t-1}{t}} + \varepsilon\right) \|x\|_2^2 \geq \left(1 + \sqrt{\frac{t-1}{t}}\right) \|x\|_2^2 \geq \|Ax\|_2^2 \\
& \geq \left(1 + \sqrt{\frac{t-1}{t}}\right) \left(1 - 2\sqrt{t(t-1)} + 2(t-1) - \frac{5}{2k}\right) \|x\|_2^2 \\
& = \left[\left(1 + \sqrt{\frac{t-1}{t}}\right) \left(1 - 2\sqrt{t(t-1)} + 2(t-1)\right) - \left(1 + \sqrt{\frac{t-1}{t}}\right) \frac{5}{2k} \right] \|x\|_2^2 \\
& = \left[1 - \sqrt{\frac{t-1}{t}} - \left(1 + \sqrt{\frac{t-1}{t}}\right) \frac{5}{2k} \right] \|x\|_2^2 \geq \left(1 - \sqrt{\frac{t-1}{t}} - \varepsilon\right) \|x\|_2^2,
\end{aligned}$$

where the last inequality follows from that $1 + \sqrt{\frac{t-1}{t}} \leq 2$ and $k \geq \frac{5}{\varepsilon}$. It follows immediately that $\delta_{tk} \leq \sqrt{\frac{t-1}{t}} + \varepsilon$. Next, we define

$$\begin{aligned}
x_0 &= \underbrace{(1, \dots, 1, \dots, 1, \dots, 1)}_{k \text{ blocks}}, \underbrace{(0, \dots, 0, \dots, 0, \dots, 0)}_{a \text{ blocks}}, \mathbf{0}, \dots, \mathbf{0} \in \mathbb{R}^N, \\
\gamma_0 &= \underbrace{(0, \dots, 0, \dots, 0, \dots, 0)}_{k \text{ blocks}}, \underbrace{\left(\frac{k}{a'}, \dots, \frac{k}{a'}, \dots, \frac{k}{a'}, \dots, \frac{k}{a'}\right)}_{a \text{ blocks}}, \mathbf{0}, \dots, \mathbf{0} \in \mathbb{R}^N.
\end{aligned}$$

Note that $\|x_0\|_{2,1} = k\sqrt{d}$, $\|\gamma_0\|_{2,1} \leq \frac{a}{a'} \cdot k\sqrt{d} < k\sqrt{d}$ and x_0 is block k -sparse, $\gamma = \frac{1}{\sqrt{kd + \frac{ak^2}{a'^2}d}}$ ($x_0 - \gamma_0$). Since $A\gamma = 0$, we have $Ax_0 = A\gamma_0$.

Thus, in the noiseless case $y = Ax_0$, suppose that the mixed l_2/l_1 minimization method (1.2) can exactly recover x_0 , i.e., $\hat{x} = x_0$. From the definition of \hat{x} and $y = A\gamma_0$, it contradicts that $\|\gamma_0\|_{2,1} < \|x_0\|_{2,1}$.

In the noisy case $y = Ax_0 + z$, suppose that the mixed l_2/l_1 minimization method (1.2) can stably recover x_0 , i.e., $\lim_{z \rightarrow 0} \hat{x} = x_0$. We observe that $y - A(\hat{x} - x_0 + \gamma_0) = y - A\hat{x}$, thus $\|\hat{x}\|_{2,1} \leq \|\hat{x} - x_0 + \gamma_0\|_{2,1}$. So $\|x_0\|_{2,1} \leq \|\gamma_0\|_{2,1}$ as $z \rightarrow 0$. It contradicts that $\|\gamma_0\|_{2,1} < \|x_0\|_{2,1}$.

Therefore, the mixed l_2/l_1 minimization method (1.2) fails to exactly and stably recover x_0 based on y and A . \square

4. Conclusion

In this paper, we consider the problem of recovering for an unknown signal with additional structure-its entries are not dispersing over all the signal vector but arising in clusters (or blocks)-from a given set of noisy linear measurements. Based on block RIP, we mainly investigate the recovery guarantee for the mixed l_2/l_1 minimization method. By extending the technique of sparse representation of a polytope to the block setting, we establish a high order block RIP condition for robust recovery of signals with block pattern by mixed l_2/l_1 minimization in the presence of noise. We also proved under the same condition, the block sparse signals

can be exactly and stably recovered in the noiseless and noisy cases, respectively. Furthermore, in order to clarify its optimality, we also give a specific measurement matrix and block sparse signal such that the given concrete signal can not be exactly and stably recovered from its samples via mixed l_2/l_1 minimization, when the upper bound of δ_{tk} increases any ε . Also, if $d_i = 1$ for $i \in \{1, \dots, M\}$, our results return to those of Cai and Zhang [8].

Acknowledgments. The authors thank the referees for their valuable suggestion and comments. This work was supported by the NSF of China (Nos.11801509, 11871109).

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