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EXTRAPOLATION METHODS FOR COMPUTING HADAMARD FINITE-PART INTEGRAL ON FINITE INTERVALS*

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Abstract

In this paper, we present the composite rectangle rule for the computation of Hadamard finite-part integrals in boundary element methods with the hypersingular kernel $1/(x-s)^2$ and we obtain the asymptotic expansion of error function of the middle rectangle rule. Based on the asymptotic expansion, two extrapolation algorithms are presented and their convergence rates are proved, which are the same as the Euler-Maclaurin expansions of classical middle rectangle rule approximations. At last, some numerical results are also illustrated to confirm the theoretical results and show the efficiency of the algorithms.

Mathematics subject classification: 65N30.
Key words: Hadamard finite-part integral, Extrapolation method, Composite rectangle rule, Superconvergence, Error functional.

1. Introduction

In recent years, much attention has been paid to the hypersigular integral of the form

$$\begin{aligned} I(f,s): \quad &= \,=\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-s)^2}dt \\ &= \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon}\frac{f(t)}{(t-s)^2}dt+\int_{s+\varepsilon}^b\frac{f(t)}{(t-s)^2}dt-\frac{2f(s)}{\varepsilon}\right\}, s\in(a,b), \end{aligned} \tag{1.1}$$

where $=\!\!\!\!\!\!\int_a^b$ denotes a Hadamard finite-part integral, $f(x)$ is the density function and s is the singular point.

Hypersinguler integral which must be considered in Hadamard finite-part sense usually appears in boundary element methods [21] and many physical problems [24], such as the calculation of stresses in elasticity problems; the crack problems in fracture mechanics, elasticity problems, acoustics and electromagnetic scattering problems and so on. Numerous work has been devoted in developing efficient quadrature formulas in recent years, such as the Gaussian method [7, 8], the Newton-Cotes rule [16, 22, 25–27, 31, 32], the transformation method [5, 6] and some other methods [3, 4, 20, 23, 30]. Amongst them the Newton-Cotes rule is a commonly used one in many areas due to its easiness of implementation and flexibility of mesh. As we know, the accuracy of the (composite) Newton-Cotes rules for Riemann integrals is $O(h^{k+1})$ for

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odd k and $O(h^{k+2})$ for even k. The (composite) Newton-Cotes rules for Hadamard finite-part integrals [9–15, 28] on interval with the superconvergence result is $O(h^{k+1})$ when the singular point coincides with some priori known point, and the existence of the superconvergence points occurring at the zeros of a special function is proved in [25].

The classic extrapolation method based on polynomial and rational function has been widely studied. The extrapolation methods as an accelerating convergence technique has been applied to many fields in computational mathematics [29]. The most famous one is Richardson extrapolation based on the error function as

$$T(h) - a_0 = a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots,$$

where $T(0) = a_0$ and a_j are constant independent of h.

In [2, 5], the Euler-Maclaurin formulae with sigmoidal transformation is used to deal of the second-order singularity, but the quadrature contains derivatives of the sigmoidal transformation. In [17, 18], Lyness studied the Euler-Maclaurin expansion technique for the evaluation of Cauchy principal integrals. In that paper [19], the integral was split into two parts. One can be calculated analytically and the other was evaluated by the trapezoidal rule with classical Euler-Maclaurin expansion. Sidi [34] presented numerical quadrature methods for integrals of periodic functions with algebraic, logarithmic, and Cauchy singularities at the interior points of the interval,then in [35, 36], Sidi have derived compact numerical quadrature formulas for finite-range integrals. The extrapolation method for the computation of Hadamard finite-part integrals on the interval and in a circle are studied in [9] and [14] which focus on the asymptotic expansion of error function. Based on the asymptotic expansion of the error functional, algorithm with theoretical analysis of the generalized extrapolation are given. In reference [19], quadrature formulae for hypersingular integrals and their asymptotic error expansions and the extrapolation methods for hypersingular integrals with either periodic integrand or non-periodic integrand are presented.

In this paper, we focus on the asymptotic error expansion of the middle rectangle rule for the computation of Hadamard finite-part integrals. The asymptotic error expansion takes the form of

$$E_n(f,s) = \sum_{i=0}^{\infty} \frac{h^{2i}}{2^{2i+1}} f^{(2i+1)}(s) a_{2i+1}(\tau), \tag{1.2}$$

where $a_{2i+1}(\tau)$ are certain special functions independent of h and τ is the local coordinate of the singular point.

Based on this asymptotic expansion (1.2), in order to avoid the computation of $a_i(\tau)$, we suggest an extrapolation algorithm for a given τ. Then, a series of s_j is selected to approximate the singular point s accompanied by the refinement of the meshes. Moreover, by means of the extrapolation technique, we not only obtain an approximation with higher order accuracy but also get a posteriori estimate of the error. In order to use the error expansion of (1.2), we also present a new algorithm to get the convergence rate the same as the Riemann integrals with extrapolation methods without to construct serie to approximate the singular point.

The rest of this paper is organized as follows. In Sect. 2, after introducing some basic formulas of the rectangle rule, we present the asymptotic error expansion and perform the proof of the error expansion. In Sect. 3, extrapolation algorithm and a posteriori asymptotic error estimation to compute Hadamard finite-part integral are obtained. Finally, several numerical examples are provided to validate our theoretical analysis.