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THE QUADRATIC SPECHT TRIANGLE*

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Abstract

We propose a class of 12 degrees of freedom triangular plate bending elements with quadratic rate of convergence. They may be viewed as the second order Specht triangle, while the Specht triangle is one of the best first order plate bending element. The convergence result is proved under minimal smoothness assumption on the solution. Numerical results for both the smooth solution and nonsmmoth solution confirm the theoretical prediction.

Mathematics subject classification: 65N38, 65N30.
Key words: Specht triangle, Plate bending element, Basis functions

1. Introduction

Numerical approximation of the plate bending problem as well as other fourth order elliptic boundary value problems usually demand certain special devices. C^1 finite elements are required for the conforming finite elements [1,2], which can be quite complicated, in particularly in three dimension. This stimulates the develop of the nonconforming finite elements. The Specht triangle [3] is a successful plate bending element, which passes all the patch tests and performs excellently, and is one of the best thin plate triangles with 9 degrees of freedom that currently available [2, citation in p. 345]. The Specht triangle employs quadratic polynomial approximation and hence is a first order plate bending element, which likes many practical nonconforming plate bending elements such as Zienkwiecz triangle [4], Morley triangle [5], just name a few of them. There are some second order nonconforming plate bending elements scattered in the literature, such as the one proposed by one of the author in [6] from the notion of the double set parameter method and the one in [7], both elements have 12 degrees of freedom. Other quadratic plate bending elements, such as those in [8] and [9], have 16 degrees of freedom. When we consider the rectangular mesh, the second order plate bending elements include the famous Adini element [10], the one proposed by one of the author in [11], and the one proposed in [12], in which a family of rectangular plate bending element is constructed. Compared to the first order plate bending elements, the choice for the second order plate bending elements is quite limited. It is worthwhile to mention that there are some higher order finite elements for

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the biharmonic problem in the framework of mixed finite elements and discontinuous Galerkin method; see; e.g., [13–16].

Motivated by the bubble function method, we propose a family of second order plate bending elements with 12 degrees of freedom, which could be regarded as the second order Specht triangle. The original motivation for the bubble function method is to design the stable finite element pair for the Stokes problem [17]. The basic idea of this method is to augment the finite element space by a bubble function space. The augmented bubble function space helps out in dealing with the extra constraints such as the divergence stability in Stokes problem and the high order consistency error. Besides being widely used in design stable finite elements in Stokes problem, the bubble function method has also been used to design efficient mass lumping method [18], to design robust finite elements for a singularly perturbed fourth order problem [8, 19], and it has been exploited by the authors to design robust finite elements for the strain gradient elasticity model [20]. In the context of the plate bending elements, certain classical elements such as the Zienkiewicz triangle and the Specht triangle can be derived by the bubble function method.

In the present work, the bubble function method is exploited to improve both the approximating error and the consistency error to the second order, which naturally yields a class of second order plate bending elements, which recovers the element in [7] as a special case. These elements are C^0 continuous. Therefore, they may be used to approximate the singular perturbation problem of fourth order as shown in [19], and to be exploited to construct robust strain gradient element as shown in [20, 21]. Based on the enriching operator technique in the discontinuous Galerkin method [22–24], we prove the convergence of the proposed elements under minimal smoothness assumption of the solution. Optimal rate of convergence is derived for solution in various Sobolev norms and broken norms. We also derived the optimal rate of convergence for the problem with Dirac-delta source term, which is particularly important for plate bending problem, because it corresponds to an idealization of a point load [25]. Numerical results for both the smooth solution and the nonsmooth solution support the theoretical prediction.

The structure of the paper is as follows. In the next section, we introduce the nonconforming finite element approximation of the plate bending problem. Detailed derivation of the new elements is presented in §3. The error estimates under minimal smoothness assumption are proved in §4. The numerical results for both the smooth solution as well as the nonsmooth solution are reported in the last section.

Throughout this paper, the constant C may differ at different occurrence, while it is independent of the mesh size h.

2. Finite Element Approximation of the Plate Bending Problem

To introduce the plate bending problem, we introduce some notations. The space $L^2(\Omega)$ of the square-integrable functions defined on a bounded polygon Ω is equipped with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_{L^2(\Omega)}$. Let $H^m(\Omega)$ be the standard Sobolev space [26] with the norm and seminnorm defined as

$$\|v\|^2_{H^m(\Omega)} = \sum_{k=0}^{m} |v|^2_{H^k(\Omega)} \quad \text{and} \quad |v|^2_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha|=k} |\nabla^\alpha v|^2 \mathrm{d}x,$$

where $\alpha=(\alpha_1,\alpha_2)$ is a multi-index whose components α_i are nonnegative integers, $|\alpha|=\alpha_1+\alpha_2$ and $\nabla^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}$. We may drop Ω in the Sobolev norm $\|\cdot\|_{H^m(\Omega)}$ when there is no confusion may occur. Denote by $H^{-m}(\Omega)$ the dual space of $H_0^m(\Omega)$ with the dual pair $\langle\cdot,\cdot\rangle$. Define

$$
\begin{aligned}
H_0^1(\Omega)&:=\{\,v\in H^1(\Omega)\ \mid\ v=0 \text{ on } \partial\Omega\,\},\\
H_0^2(\Omega)&:=\{\,v\in H^2(\Omega)\ \mid\ v=\partial_n v=0 \text{ on } \partial\Omega\,\},
\end{aligned}
$$

where $\partial_n v$ and $\partial_t v$ are the normal derivative of v and the tangential derivative of v, respectively.

We consider the plate bending problem with clamped boundary conditions

$$
\begin{cases}
\Delta^2 u=f & x\in\Omega,\\
u=\partial_n u=0 & x\in\partial\Omega,
\end{cases}
\tag{2.1}
$$

where $\Omega\subset\mathbb{R}^2$ is a polygon and f is the surface (or body) load. The corresponding variational problem is to find $u\in H_0^2(\Omega)$ such that

$$
a(u,v)=(f,v)\quad\text{for all}\quad v\in H_0^2(\Omega),
\tag{2.2}
$$

where for $w,v\in H_0^2(\Omega)$,

$$
a(w,v)=\int_\Omega\big(\nu\Delta w\Delta v+(1-\nu)\nabla^2 w:\nabla^2 v\big)\,\mathrm{d}x\quad\text{and}\quad (f,v)=\int_\Omega fv\mathrm{d}x,
$$

and $0<\nu<\frac{1}{2}$ is the Poisson's ratio and the inner product for Hessian matrix is defined as $\nabla^2 w:\nabla^2 v=\sum_{i,j=1}^2\partial^2_{x_ix_j}w\partial^2_{x_ix_j}v$.

Let $\mathcal{T}_h$ be a shape-regular family of triangulations of Ω with h the diameter of the element and V_h be a finite element space associated with $\mathcal{T}_h$. The finite element approximation of (2.2) is to find $u_h\in V_h$ such that

$$
a_h(u_h,v)=(f,v)\quad\text{for all}\quad v\in V_h,
\tag{2.3}
$$

where

$$
a_h(v,w)=\sum_{K\in\mathcal{T}_h}\int_K\big(\nu\Delta v\Delta w+(1-\nu)\nabla^2 v:\nabla^2 w\big)\,\mathrm{d}x\qquad\text{for all}\quad v,w\in V_h.
$$

By the Second Strang's lemma [27], we have the error estimate

$$
\|u-u_h\|_h\le\inf_{v\in V_h}\|u-v\|_h+\sup_{w\in V_h}\frac{|E_h(u,w)|}{\|w\|_h},
\tag{2.4}
$$

where the consistency error functional $E_h(u,w)=a_h(u,w)-(f,v)$ and $\|\cdot\|_h$ is a broken seminorm that is defined for any $v\in V_h$ as $\|v\|_h^2:=a_h(v,v)$. To improve the accuracy of the element, we need to use higher order polynomial and improve the order of the consistency error simultaneously. The following result gives a sufficient condition for high order consistency error estimate [6]. We refer to [28] for related discussion.

Lemma 2.1 ([6, Lemma 1]). *Let K be an element and $\widehat{K}$ be its reference. For every K, there exists an affine mapping from K to $\widehat{K}$. Assume that the following three conditions are fulfilled.*

1. *Function* $v_h \in V_h$ *is continuous at vertices of* K *and vanishing at vertices lying on* $\partial\Omega$.

2. *For every polynomial* $p(s) \in \mathbb{P}_{m-1}(F)$, *the integral* $\int_F p(s)v_h ds$ *is continuous over each interelement side* F *and vanishing when* $F \subset \partial\Omega$, *where* $\mathbb{P}_r(F)$ *is a polynomial of degree* r *on the side* F.

3. *For every* $p(s) \in \mathbb{P}_m(F)$, *the integral* $\int_F p(s)\frac{\partial v_h}{\partial n} ds$ *is continuous over each interelement side* F *and vanishing when* $F \subset \partial\Omega$.

Then if the solution $u \in H^{m+3}(\Omega) \cap H_0^2(\Omega)$, *we have*

$$\sup_{w\in V_h} \frac{|E_h(u,w)|}{\|w\|_h} \le Ch^{m+1}\|u\|_{H^{m+3}(\Omega)}, \quad m \ge 1.$$

If V_h has the following aproximating estimate

$$\inf_{v\in V_h} \|u-v\|_h \le Ch^{m+1}, \tag{2.5}$$

then

$$\|u-u_h\|_h \le Ch^{m+1}, \quad m \ge 1. \tag{2.6}$$

This means the element is a plate bending element of order $m+1$. The above lemma is the starting point for constructing the new elements.

3. A Family of New Plate Element of Second Order

Given a triangle K with vertices $\{A_i\}_{i=1}^3$ and coordinates (x_i, y_i), let λ_i be the area coordinate associated with the vertices A_i. Define $b_1 = y_2 - y_3, c_1 = x_3 - x_2$, and b_2, b_3, c_2 and c_3 may be defined by cyclic permutation of the indices. Let $\ell_i = (b_i^2 + c_i^2)^{1/2}$ be the length of the edge $e_i := A_jA_k$ opposite to A_i. Let $\triangle$ be the area of K. It is clear to see that for $i = 1,2,3$, $\nabla\lambda_i = (b_i/(2\triangle), c_i/(2\triangle))$. We may write the outer normal of the edge e_i as $n_i = -\nabla\lambda_i/|\nabla\lambda_i|$, where $|\nabla\lambda_i|$ is the length of the vector $\nabla\lambda_i$. A direct calculation gives

$$\frac{\partial\lambda_i}{\partial n_j} = -\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|}.$$

In particular,

$$\frac{\partial\lambda_i}{\partial n_i} = -|\nabla\lambda_i|.$$

We shall frequently used these two identities later on.

The new element is described by the finite element triple $(K, \widehat{P}_K, \Sigma_K)$ [29] with K a triangle and

$$\begin{cases} \widehat{P}_K = Z_K + b_K\mathbb{P}_2(K), \\ \Sigma_K = \left\{ p(A_i), \partial_x p(A_i), \partial_y p(A_i), \quad \fint_{e_i} \partial_n p, 1 \le i \le 3 \right\}, \end{cases}$$

where $b_K = \lambda_1\lambda_2\lambda_3$ is the cubic bubble function, and Z_K is the Zienkiewicz space [4] that is defined by

$$Z_K = \mathbb{P}_2(K) + \text{Span}\left\{\lambda_1^2\lambda_2 - \lambda_2^2\lambda_1, \lambda_2^2\lambda_3 - \lambda_3^2\lambda_2, \lambda_3^2\lambda_1 - \lambda_1^2\lambda_3\right\}.$$

Note that $\dim\widehat{P}_K = 15 > \dim\Sigma_K = 12$. Certain constraints have to be imposed on $\widehat{P}_K$ to enforce unisolvability, and we denote the reduced finite element space as P_K.

To ensure second order approximating error, we require that $\mathbb{P}_3(K) \subset P_K$. Since $Z_K \cap b_K\mathbb{P}_2(K) = \emptyset$ and $Z_K + b_K = \mathbb{P}_3(K)$, then we only need

$$b_K \in P_K \cap b_K\mathbb{P}_2(K). \tag{3.1}$$

It is clear to see that the element is C^0-continuous. Therefore, any $v \in P_K$ satisfies the hypothesises (1) and (2) of Lemma 2.1. For any $v \in \mathbb{P}_3(K)$, we apply *Simpson's formula* to $(\lambda_j - 1/2)\partial_n v$ and obtain

$$⨍_{e_i}\left(\lambda_j - \frac{1}{2}\right)\frac{\partial v}{\partial n} = \frac{1}{12}\left(\frac{\partial v}{\partial n}(A_j) - \frac{\partial v}{\partial n}(A_k)\right),$$

where A_j and A_k are two endpoints of the edge e_i in the counterclockwise manner. Here $1 \le i,j,k \le 3, j \ne i$ and $k \ne i,j$. Since $Z_K \subset \mathbb{P}_3(K)$ and by the condition (3) of Lemma 2.1, the following identity should be valid for any $v \in b_K\mathbb{P}_2(K)$,

$$⨍_{e_i}\lambda_j\frac{\partial v}{\partial n} = \frac{1}{12}\left(\frac{\partial v}{\partial n}(A_j) - \frac{\partial v}{\partial n}(A_k)\right) + \frac{1}{2}⨍_{e_i}\frac{\partial v}{\partial n} \tag{3.2}$$

to ensure the second order consistency error.

Remark 3.1. The original derivation of the Specht triangle is carried out by removing the quadratic part of the normal derivative of the shape function along each edge [3]. It may be recovered by imposing the constraint

$$⨍_{e_i}\frac{\partial v}{\partial n} = \frac{1}{2}\left(\frac{\partial v}{\partial n}(A_j) + \frac{\partial v}{\partial n}(A_k)\right), \quad i = 1,2,3.$$

We refer to [30] for a detailed derivation and the proof for the equivalence between the Specht triangle and certain elements scattered in the engineering literatures. It is worthwhile to point out that the new Zienkiewicz type triangle proposed in [31] is just the Specht triangle, and the authors exploited the above constraints in an essential way.

Denote q_i the basis function associated with the degree of freedom $⨍_{e_i}\partial_n v$, which is assumed to be

$$q_i = b_K(a\lambda_i^2 + b\lambda_j^2 + c\lambda_k^2 + d\lambda_i\lambda_j + e\lambda_j\lambda_k + f\lambda_k\lambda_i).$$

It is clear to verify that q_i as well as ∇q_i vanishes at all the vertices of K. It remains to check

$$⨍_{e_j}\frac{\partial q_i}{\partial n} = \delta_{ij}, \quad j = 1,2,3.$$

This implies

$$\begin{cases} \dfrac{b+c}{20} + \dfrac{e}{30} = -\dfrac{1}{|\nabla\lambda_i|}, \\ \dfrac{c+a}{20} + \dfrac{f}{30} = 0, \\ \dfrac{a+b}{20} + \dfrac{d}{30} = 0. \end{cases}$$

Solving the above linear system, we obtain

$$\begin{cases} a = \dfrac{1}{3}(e-d-f) + \dfrac{10}{|\nabla\lambda_i|}, \\ b = \dfrac{1}{3}(f-d-e) - \dfrac{10}{|\nabla\lambda_i|}, \\ c = \dfrac{1}{3}(d-e-f) - \dfrac{10}{|\nabla\lambda_i|}. \end{cases}$$

Therefore, we rewrite q_i as

$$q_i = b_K\Big\{\Big(\frac{e}{3}+\frac{10}{|\nabla\lambda_i|}\Big)(\lambda_i^2-\lambda_j^2-\lambda_k^2)+e\lambda_j\lambda_k+\frac{f}{3}(\lambda_j^2-\lambda_k^2-\lambda_i^2)$$
$$+f\lambda_k\lambda_i+\frac{d}{3}(\lambda_k^2-\lambda_i^2-\lambda_j^2)+d\lambda_i\lambda_j\Big\}.$$

By (3.2), we obtain

$$⨍_{e_i}\lambda_j\frac{\partial q_i}{\partial n}=\frac{1}{2},\quad ⨍_{e_j}\lambda_k\frac{\partial q_i}{\partial n}=0,\quad ⨍_{e_k}\lambda_i\frac{\partial q_i}{\partial n}=0.$$

This immediately implies

$$d=f,\quad d-e=\frac{30}{|\nabla\lambda_i|},\quad e-f=-\frac{30}{|\nabla\lambda_i|}.$$

We let

$$d=f=e+\frac{30}{|\nabla\lambda_i|}.$$

Now we reshape q_i as

$$q_i=b_K\Big(-\Big(\frac{10}{|\nabla\lambda_i|}+\frac{e}{3}\Big)(\lambda_1^2+\lambda_2^2+\lambda_3^2)+e(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+\frac{30}{|\nabla\lambda_i|}\lambda_i(\lambda_j+\lambda_k)\Big),$$

which may be rewritten as

$$q_i=\frac{b_K}{|\nabla\lambda_i|}\Big(-\Big(10+\frac{\alpha_i}{3}\Big)(\lambda_1^2+\lambda_2^2+\lambda_3^2)+\alpha_i(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+30\lambda_i(\lambda_j+\lambda_k)\Big),$$

where $\{\alpha_i\}_{i=1}^3$ are three parameters, and $1\le i,j,k\le 3$ with $i\ne j, j\ne i, k\ne i$. The above form of q_i can be further simplified to

$$q_i=\frac{b_K}{|\nabla\lambda_i|}\Big(\Big(10+\frac{\alpha_i}{3}\Big)(5(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)-1)-30\lambda_j\lambda_k\Big). \tag{3.3}$$

It is straightforward to find

$$\sum_{i=1}^3|\nabla\lambda_i|q_i=b_K\Big(-30-\frac{1}{3}\sum_{i=1}^3\alpha_i+\Big(\frac{5}{3}\sum_{i=1}^3\alpha_i+120\Big)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\Big).$$

If we let

$$\sum_{i=1}^3\alpha_i=-72, \tag{3.4}$$

then

$$-\frac{1}{6}\sum_{i=1}^3|\nabla\lambda_i|q_i=b_K,$$

which immediately implies that the constraint (3.1) is valid.

Next, we derive the basis functions associated with the degrees of freedom $v(A_i)$, $\partial_x v(A_i)$ and $\partial_y v(A_i)$, respectively. We start from the basis functions of Zienkiewicz's element [29, p. 350], which reads as

$$\widetilde{\zeta}_i=\lambda_i^2(3-2\lambda_i),\quad \widetilde{\theta}_i=\lambda_i^2(c_k\lambda_j-c_j\lambda_k),\quad \widetilde{\vartheta}_i=\lambda_i^2(b_j\lambda_k-b_k\lambda_j).$$

They satisfy

$$\begin{cases} \widetilde{\zeta}_i(A_j) = \delta_{ij}, \quad \nabla\widetilde{\zeta}_i(A_j) = 0, \\ \widetilde{\theta}_i(A_j) = 0, \quad \partial_x\widetilde{\theta}_i(A_j) = \delta_{ij}, \quad \partial_y\widetilde{\theta}_i(A_j) = 0, \\ \widetilde{\vartheta}_i(A_j) = 0, \quad \partial_x\widetilde{\vartheta}_i(A_j) = 0, \quad \partial_y\widetilde{\vartheta}_i(A_j) = \delta_{ij}. \end{cases}$$

A direct calculation yields

$$\fint_{e_j} \frac{\partial\widetilde{\zeta}_i}{\partial n_j} = 6\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|}\fint_{e_j}(\lambda_i^2-\lambda_i) = -\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|}(1-\delta_{ij}).$$

By definition, we have

$$\frac{\partial\widetilde{\theta}_i}{\partial n_j} = 2\lambda_i\frac{\partial\lambda_i}{\partial n_j}(c_k\lambda_j - c_j\lambda_k) + \lambda_i^2\left(c_k\frac{\partial\lambda_j}{\partial n_j} - c_j\frac{\partial\lambda_k}{\partial n_j}\right).$$

Using the fact that $c_i + c_j + c_k = 0$, we obtain

$$\begin{aligned} \fint_{e_j}\frac{\partial\widetilde{\theta}_i}{\partial n_j} &= \frac{1}{3}\left(-c_j\frac{\partial\lambda_i}{\partial n_j} + c_k\frac{\partial\lambda_j}{\partial n_j} - c_j\frac{\partial\lambda_k}{\partial n_j}\right) \\ &= \frac{1}{3}\left(-c_j\frac{\partial(\lambda_i+\lambda_k)}{\partial n_j} + c_k\frac{\partial\lambda_j}{\partial n_j}\right) \\ &= \frac{c_k+c_j}{3}\frac{\partial\lambda_j}{\partial n_j} = \frac{c_i}{3}|\nabla\lambda_j|. \end{aligned}$$

Proceeding along the same line that leads to the above equation, we obtain

$$\fint_{e_i}\frac{\partial\widetilde{\theta}_i}{\partial n_i} = 0 \quad \text{and} \quad \fint_{e_k}\frac{\partial\widetilde{\theta}_i}{\partial n_k} = -\frac{c_i|\nabla\lambda_k|}{3}.$$

Similarly, we have

$$\fint_{e_i}\frac{\partial\widetilde{\vartheta}_i}{\partial n_i} = 0, \quad \fint_{e_j}\frac{\partial\widetilde{\vartheta}_i}{\partial n_j} = -\frac{b_i|\nabla\lambda_j|}{3}, \quad \fint_{e_k}\frac{\partial\widetilde{\vartheta}_i}{\partial n_k} = \frac{b_i|\nabla\lambda_k|}{3}.$$

We are ready to prove the main theorem of this paper.

Theorem 3.1. *Let ζ_i, θ_i, ϑ_i and q_i be the basis functions associated with the degrees of freedom $v(A_i)$, $\partial_x v(A_i)$, $\partial_y v(A_i)$ and $\fint_{e_i}\partial_n v$, respectively. Then*

$$\begin{cases} \zeta_i = \lambda_i^2(3-2\lambda_i) + \sum_{j\neq i}\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|^2}\widetilde{q}_j, \\ \theta_i = \lambda_i^2(c_k\lambda_j - c_j\lambda_k) - \frac{c_i}{3}(\widetilde{q}_j - \widetilde{q}_k), \\ \vartheta_i = \lambda_i^2(b_j\lambda_k - b_k\lambda_j) + \frac{b_i}{3}(\widetilde{q}_j - \widetilde{q}_k), \\ q_i = \frac{\widetilde{q}_i}{|\nabla\lambda_i|}, \end{cases} \tag{3.5}$$

where

$$\widetilde{q}_i = b_K\left((\alpha_i/3+10)\left(5(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)-1\right) - 30\lambda_j\lambda_k\right)$$

with α_i satisfying (3.4)*, i.e., $\sum_{i=1}^3 \alpha_i = -72$.*

Proof. Since b_K and ∇b_K vanish at all the vertices of the element, then

$$\zeta_i(A_j) = \widetilde{\zeta}_i(A_j) = \delta_{ij} \quad \text{and} \quad \nabla\zeta_i(A_j) = \nabla\widetilde{\zeta}_i(A_j) = 0.$$

A direct calculation gives

$$\begin{aligned}
\fint_{e_j} \frac{\partial \zeta_i}{\partial n_j} &= \fint_{e_j} \frac{\partial \widetilde{\zeta}_i}{\partial n_j} + \sum_{l\neq i} \frac{\nabla\lambda_i \cdot \nabla\lambda_l}{|\nabla\lambda_l|} \fint_{e_j} \frac{\partial q_l}{\partial n_j} \\
&= -\frac{\nabla\lambda_i \cdot \nabla\lambda_j}{|\nabla\lambda_j|} + \sum_{l\neq i} \frac{\nabla\lambda_i \cdot \nabla\lambda_l}{|\nabla\lambda_l|}\delta_{jl} = 0.
\end{aligned}$$

This implies that ζ_i is the basis function associate with the degree of freedom $v(A_i)$.

Proceeding along the same line that leads to $(3.5)_1$, we may verify the remaining basis functions. This completes the proof. $\square$

There are infinitely choices of α_i that satisfy (3.4). For any $\alpha, \beta \in \mathbb{R}$, if we let

$$\alpha_i = -24 + 3\alpha, \quad \alpha_j = -24 + 3\beta, \quad \alpha_k = -24 - 3\alpha - 3\beta,$$

then

$$\begin{cases}
q_i = \dfrac{b_K}{|\nabla\lambda_i|}\big((2+\alpha)\,(5\lambda_i(1-\lambda_i)-1) - 5(4-\alpha)\lambda_j\lambda_k\big), \\
q_j = \dfrac{b_K}{|\nabla\lambda_j|}\big((2+\beta)\,(5\lambda_j(1-\lambda_j)-1) - 5(4-\beta)\lambda_k\lambda_i\big), \\
q_k = \dfrac{b_K}{|\nabla\lambda_k|}\big((2-\alpha-\beta)\,(5\lambda_k(1-\lambda_k)-1) - 5(4+\alpha+\beta)\lambda_i\lambda_j\big).
\end{cases} \tag{3.6}$$

Furthermore, if we let $\alpha = \beta = 0$ in the above equations, then

$$q_i = \frac{2b_K}{|\nabla\lambda_i|}\left(5\lambda_i(1-\lambda_i) - 1 - 10\lambda_j\lambda_k\right), \tag{3.7}$$

and q_j, q_k may be obtained by cyclic permutation of the indices. This is exactly the same with [7, equation $(3.4)_1$]. We hereby recover the element in [7].

Next if we let $\alpha = \beta = -2$ in (3.6), then

$$q_i = -\frac{30b_K}{|\nabla\lambda_i|}\lambda_j\lambda_k, \quad q_j = -\frac{30b_K}{|\nabla\lambda_j|}\lambda_k\lambda_i, \quad q_k = \frac{6b_K}{|\nabla\lambda_k|}\left(5\lambda_k(1-\lambda_k) - 1\right). \tag{3.8}$$

It seems that such unsymmetrical choice of the parameters α_i leads to a new element with slightly simpler basis functions compared to the most symmetrical choice of α_i; cf., (3.7).

Another unsymmetrical choice of α_i would be $\alpha = \beta = 1$. In this case,

$$\begin{aligned}
&q_i = \frac{3b_K}{|\nabla\lambda_i|}\left(5\left(\lambda_i(1-\lambda_i) - \lambda_j\lambda_k\right) - 1\right), \\
&q_j = \frac{3b_K}{|\nabla\lambda_j|}\left(5\left(\lambda_j(1-\lambda_j) - \lambda_k\lambda_i\right) - 1\right), \quad q_k = -\frac{30b_K}{|\nabla\lambda_k|}\lambda_i\lambda_j.
\end{aligned} \tag{3.9}$$

The Hermite element space is defined by $H_K = Z_K + b_K$, with the values of v and ∇v at the vertices, and the value of v at the barycenter as the degrees of freedom. Define the finite element space V_h and W_h as

$$\begin{aligned}
V_h\colon = \Big\{ & v \in L^2(\Omega) \mid v|_K \in P_K, v, \nabla v \text{ are continuous at each node, vanish at the boundary} \\
& \text{nodes}, \int_e \partial_n v \text{ is continuous across each edge, vanishes at the boundary edge} \Big\},
\end{aligned}$$

and

$W_h \colon = \{v \in L^2(\Omega) v|_K \in H_K, v, \nabla v$ are continuous at each node, vanish at the boundary nodes$\}$.

For any $v \in V_h$ and $w \in W_h$, the broken seminorms $\|v\|_h$ and $\|w\|_h$ are indeed broken norms by [32].

The interpolate operator Π and π are defined locally as $\Pi|_K = \Pi_K$ and $\pi|_K = \pi_K$ respectively for any $v \in H^s(K)$ with $s > 2$ as

$$\begin{cases} \Pi_K v(a) = v(a), & \text{for all vertices} \quad a, \\ \nabla \Pi_K v(a) = \nabla v(a), & \text{for all vertices} \quad a, \\ \displaystyle\int_e \frac{\partial \Pi_K v}{\partial n} = \int_e \frac{\partial v}{\partial n}, & \text{for all edges} \quad e, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \pi_K v(a) = v(a), & \text{for all vertices and barycentre} \quad a, \\ \nabla \pi_K v(a) = \nabla v(a), & \text{for all vertices} \quad a. \end{cases}$$

By the general interpolation theory [33] (see also [34]), we obtain, for any $\phi \in H^{2+s}(\Omega)$ with $s \in (0, 2]$, there exists C such that

$$\|\phi - \Pi\phi\|_{L^2} + h^2 \|\phi - \Pi\phi\|_h \leq C h^{2+s} \|\phi\|_{H^{2+s}}, \tag{3.11}$$

$$\|\phi - \pi\phi\|_{L^2} + h^2 \|\phi - \pi\phi\|_h \leq C h^{2+s} \|\phi\|_{H^{2+s}}. \tag{3.12}$$

4. Error Estimate for Nonsmooth Data

It follows from the standard error estimate (2.6) that the new element converges quadratically in the broken H^2 norm provided that $u \in H^4(\Omega)$, which is usually invalid for point load f or when the domain Ω is a polygon [35]. In this part we revisit the error estimate under minimal regularity assumption on u.

The oscillation of f is defined as

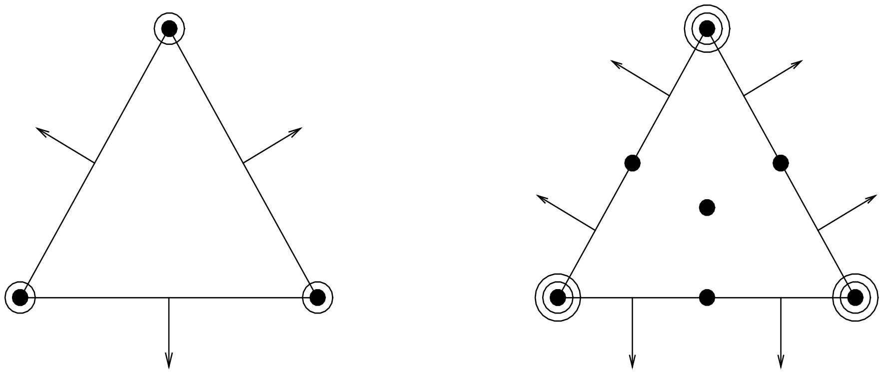
$$\mathrm{Osc}(f) \colon = \left(\sum_{K \in \mathcal{T}_h} h_K^4 \inf_{\overline{f} \in \mathbb{P}_0(K)} \|f - \overline{f}\|^2_{L^2(K)} \right)^{1/2}.$$

4.1. Properties of enriching operator and error estimate for nonsmooth solution

To state the result, we need the following enriching operator, which measures how far the nonconforming finite element space V_h departs from $H^2(\Omega)$. Enriching operator was firstly introduced by Brenner [36,37] to analyze nonconforming elements in the context of fast solvers, which was exploited recently to study the convergence properties of discontinuous Galerkin method in [23, 24]. An adaption of this operator has been employed to derive sharp error estimate of Morley's triangle [38]. The enriching operator used here is a combination of the averaging type enriching operator in [22] and the one in [38].

The enriching operator $E_h : V_h \to H_0^2(\Omega)$ is constructed with the aid of the quintic ($\mathbb{P}_6$) Argyris triangle [1]; See Figure 4.1. Let p be a vertex, we define

$$(\nabla^\alpha E_h v)(p) = \frac{1}{|\mathcal{T}_p|} \sum_{K \in \mathcal{T}_a} (\nabla^\alpha v|_K)(p), \qquad |\alpha| = 0, 1, 2, \tag{4.1}$$



(a) (b)

Fig. 4.1. (a) The degrees of freedom of quadratic Specht triangle are point evaluations of the function value and the first derivatives at the vertex, and the moments of the normal derivative along each edge; and (b) The degrees of freedom of quintic ($\mathbb{P}_6$) Argyris triangle are the point evaluations of the function value, the first derivatives and the second order derivatives at the vertex, and the means of the function along each edge, and the moments of the normal derivatives against $\mathbb{P}_1$ along each edge, and the mean of the function over the element.

where $\mathcal{T}_p$ is the union of triangles that share a common vertex p, and $|\mathcal{T}_p|$ is the cardinality of $\mathcal{T}_p$. Define

$$\begin{cases} \displaystyle\int_e E_h v w = \int_e v w, & \text{for any} \quad w \in \mathbb{P}_0(e), \\ \displaystyle\int_e \frac{\partial E_h v}{\partial n} w = \int_e \frac{\partial v}{\partial n} w & \text{for any} \quad w \in \mathbb{P}_1(e), \\ \displaystyle\int_K E_h v w = \int_K v w, & \text{for any} \quad w \in \mathbb{P}_0(K). \end{cases} \tag{4.2}$$

If e is on the boundary $\partial\Omega$, or p is both a boundary node and a vertex of Ω, then we set the corresponding degree of freedom as zero. If p is a vertex of $\mathcal{T}_h$ interior to an edge of Ω, then we define

$$(\partial_t^2 E_h v)(p) = (\partial_{tn}^2 E_h v)(p) = 0,$$

and $(\partial_n^2 E_h v)(p)$ is defined as (4.1). We refer to [22] for more details.

The properties of the enriching operator is summarized in the following lemma.

Lemma 4.1. *The enriching operator E_h defined above has the following properties:*

1. *Galerkin orthogonality: For any $v \in V_h$,*

$$a_h(v - E_h v, w) = 0 \qquad \textit{for all} \quad w \in W_h. \tag{4.3}$$

2. *E_h is stable in the sense that*

$$\|E_h v\|_h \le \alpha \|v\|_h \qquad \textit{for all} \quad v \in V_h. \tag{4.4}$$

3. *For any $v \in V_h$, we have*

$$\|v - E_h v\|_{L^2} \le \beta h^2 \|v\|_h, \tag{4.5}$$

and for any $s\in(\frac{1}{2},2]$, *there holds*

$$
\begin{aligned}
&\|v-E_hv\|_{H^{2-s}}\le\beta h^s\|v\|_h,\\
&\|v-E_hv\|_{H^{2-s}}\le\beta h^s\big(\|u-v\|_h+\mathrm{Osc}(f)\big).
\end{aligned}
\tag{4.6}
$$

4. *For any interpolate operator* Π, $E_h\circ\Pi$ *is a quasi-interpolant in the sense that*

$$
\|\phi-E_h\Pi\phi\|_{L^2}+h^2\|\phi-E_h\Pi\phi\|_h\le Ch^{2+s}\|\phi\|_{H^{2+s}}
\tag{4.7}
$$

for any $\phi\in H^{2+s}(\Omega)$ *with* $s=1,2$.

The stability estimate (4.4) and the interpolate estimates (4.5) and (4.7) may be readily proved by adopting the argument in [22, Lemma 3 and Lemma 4]. We omit the details. The Galerkin orthoganality (4.3) differs from the one for the Morley's triangle because the test function lies in W_h instead of V_h. This seems the trouble brings in by the higher order degree of the element. It remains to prove (4.6), which seems new. A similar inequality for the Morley's triangle has appeared in [39]. The proof combines the techniques in [22] and the efficiency estimate in the a posteriori error estimate.

Proof. Due to the construction of the enriching operator, the Galerkin orthogonality (4.3) may be obtained by an integration by parts.

$$
\begin{aligned}
&a_h(v-E_hv,w)\\
&=\sum_{K\in\mathcal{T}_h}\int_K(v-E_hv)\Delta^2w-\sum_{K\in\mathcal{T}_h}\int_{\partial K}(v-E_hv)\left(\nu\partial_n\Delta w+(1-\nu)\partial_tM_{tn}(w)\right)\\
&\quad+\sum_{K\in\mathcal{T}_h}\int_{\partial K}\partial_n(v-E_hv)\left(\nu\Delta w+(1-\nu)M_{nn}(w)\right),
\end{aligned}
$$

where $M_{nn}=n^{\mathrm{T}}\cdot\nabla^2w\cdot n$ and $M_{tn}=t^{\mathrm{T}}\cdot\nabla^2w\cdot n$. Since

$$
\Delta^2w=0\quad\text{and}\quad\partial_n\Delta w,\partial_tM_{tn}(w)\in\mathbb{P}_0,
$$

by $(4.2)_1$, we obtain

$$
a_h(v-E_hv,w)=\sum_{e\in\mathcal{E}_h}\int_e[\![\partial_n(v-E_hv)]\!]\{\{M_{nn}(w)\}\}+\{\{\partial_n(v-E_hv)\}\}[\![M_{nn}(w)]\!].
$$

Since $M_{nn}(w)\in\mathbb{P}_1$, and employing the fact

$$
[\![\partial_n(v-E_hv)]\!]=[\![\partial_nv]\!]\quad\text{and}\quad\{\{\partial_n(v-E_hv)\}\}=\{\{\partial_nv\}\}-\partial_nE_hv,
$$

and by the constraint (3.2) and $(4.2)_2$, we obtain (4.3).

For any element $K\in\mathcal{T}_h$, we let $\mathcal{N}(K),\mathcal{E}(K)$ and $\mathcal{V}(K)$ be the set of the nodal variables, edge variables, and the set of the volume variables of the $\mathbb{P}_6$ Argyris triangle, respectively. For any $v\in V_h$, $v-E_hv\in\mathbb{P}_6$, and it follows from the scaling argument that

$$
\begin{aligned}
\|v-E_hv\|^2_{L^2(K)}\le{}&C\sum_{N\in\mathcal{N}(K)}h_K^{2(1+\mathrm{order}(N))}\left(N(v-E_hv)\right)^2\\
&+C\sum_{E\in\mathcal{E}(K)}h_K^{2(1+\mathrm{order}(E))}\left(E(v-E_hv)\right)^2\\
&+C\sum_{V\in\mathcal{V}(K)}h_K^{2(1+\mathrm{order}(V))}\left(V(v-E_hv)\right)^2.
\end{aligned}
$$

Using the facts that $N(v) = N(E_h(v))$ if $\text{order}(N) = 0$ and 1, $E(v) = E(E_h(v))$ if $\text{order}(E) = 0$ and 1, and $V(v) = V(E_h(v))$, which is a direct consequence of the construction of E_h, we obtain

$$\|v - E_h v\|^2_{L^2(K)} \le C h_K^6 \sum_{N\in\mathcal{N}(K),\text{order}(N)=2} \left(N(v - E_h v)\right)^2.$$

Let N be a second order node variable at a vertex p of K. It follows from an inverse estimate that

$$\begin{aligned}\left(N(v - E_h v)\right)^2 &\le |\nabla^2 (v - E_h v)(p)|^2 \\ &\le C \sum_{K'\in\mathcal{T}_p} |\nabla^2 v|_{K'}(p)|^2 \le C \sum_{K'\in\mathcal{T}_p} h_{K'}^{-2} |v|^2_{H^2(K')}.\end{aligned}$$

Then we obtain

$$\|v - E_h v\|^2_{L^2(K)} \le C \sum_{K'\in\mathcal{T}_p} h_{K'}^4 |v|^2_{H^2(K')}, \tag{4.8}$$

which immediately implies (4.5).

When $s \in (\frac{1}{2}, 2)$, we note an inverse estimate (cf. [40]):

$$\|v - E_h v\|^2_{H^{2-s}(\Omega)} \le C \sum_{K\in\mathcal{T}_h} h_K^{2s-4} \|v - E_h v\|^2_{L^2(K)}, \tag{4.9}$$

which together with (4.5) implies $(4.6)_1$.

By the inverse inequality and the continuity of v and $E_h v$, we have, for N a second order node variable at a vertex p of K, there holds

$$\left(N(v - E_h v)\right)^2 \le C \sum_{e\in\mathcal{E}_{\mathcal{V}(p)}} |e|^{-1} \left(\|[\![\partial_n^2 v]\!]\|^2_{L^2(e)} + \|[\![\partial_{tn}^2 v]\!]\|^2_{L^2(e)} \right),$$

where $\mathcal{E}_{\mathcal{V}(p)}$ is the set of edges of $\mathcal{T}_h$ emanating from the vertex p. Then we obtain

$$\|v - E_h v\|^2_{L^2(K)} \le C \sum_{e\in\mathcal{E}_{\mathcal{V}(K)}} h_K^5 \left(\|[\![\partial_n^2 v]\!]\|^2_{L^2(e)} + \|[\![\partial_{tn}^2 v]\!]\|^2_{L^2(e)} \right), \tag{4.10}$$

where $\mathcal{E}_{\mathcal{V}(K)}$ is the set of edges of $\mathcal{T}_h$ emanating from the vertices of K. By [23, Eq. (5.27)], we obtain

$$\sum_{e\in\mathcal{E}_h^i} h_K^5 \|[\![\partial_n^2 v]\!]\|^2_{L^2(e)} \le C h^4 \left(\|u - v\|_h^2 + [\text{Osc}(f)]^2 \right).$$

By [39, Proposition 2.3], we obtain

$$\sum_{e\in\mathcal{E}_h^i} h_K^5 \|[\![\partial_{tn}^2 v]\!]\|^2_{L^2(e)} \le C h^4 \|u - v\|_h^2.$$

Substituting the above two estimates into (4.10) and (4.9), we obtain (4.6). This completes the proof. $\square$

Theorem 4.1. *Let u and u_h be the solutions of Problem* (2.2) *and Problem* (2.3)*, respectively. Then*

$$\|u - u_h\|_h \le (1 + \alpha) \left(2 \inf_{v\in V_h} \|u - v\|_h + \inf_{w\in W_h} \|u - w\|_h \right) + \beta \,\text{Osc}(f). \tag{4.11}$$

Note that if $u \in H^4(\Omega)$ and $f \in L^2(\Omega)$, then

$$\|u - u_h\|_h \le Ch^2 \left(\|u\|_{H^4} + \|f\|_{L^2}\right).$$

This indicates that the proposed element converges quadratically for smooth solution, and it is natural to dub the element as the quadratic Specht triangle.

Remark 4.1. Error estimate similar to (4.11) has been proved in [41, Theorem 3.3] for the element in [7] (cf. [41, Example 5.2.3]) except an extra term $\|\nabla^2 u - \Pi_0 \nabla^2 u\|_{L^2}$ appears in the right-hand side of (4.11), where Π_0 is the L^2 projection operator to piecewise constant space. We do not know whether this term can be dropped by the techniques therein.

Proof. For any $v \in V_h$, we denote $w = v - u_h$. By the Galerkin orthogonality of the enriching operator (4.3), we obtain, for any $z \in W_h$,

$$\begin{aligned}\|w\|_h^2 &= a_h(v, w) - a_h(u_h, w) = a_h(v, w - E_h w) + a_h(v, E_h w) - (f, w)\\ &= a_h(v - z, w - E_h w) + a_h(v - u, E_h w) + (f, E_h w - w)\\ &= a_h(v - z, w - E_h w) + a_h(v - u, E_h w) + (f - \overline{f}, E_h w - w),\end{aligned} \tag{4.12}$$

where we have used $(4.2)_3$ in the last step. The energy estimate (4.11) follows from (4.4), (4.5) and the triangle inequality and the estimate

$$|a_h(v - z, w - E_h w)| \le (1 + \alpha)\big(\|u - v\|_h + \|u - z\|_h\big)\|w\|_h. \qquad \square$$

The only information of the solution we used in the proof of the energy estimate (4.11) is that $u \in H_0^2(\Omega)$ satisfies the variational problem, by contrast to the classical analysis of the nonconforming finite elements [34], which requires that $u \in H^s(\Omega) \cap H_0^2(\Omega)$ with $s > \frac{5}{2}$ for justifying the integration by parts, which is key to estimate the consistency error functional $E_h(u, w)$.

The next theorem gives the error bounds in the H^1 norm and L^2 norm under the regularity assumption $u \in H^{2+s}(\Omega)$ with $s \in (0, 2]$. This contrasts the classical results established in [34], which are valid for smooth solution $u \in H^3(\Omega)$, such smoothness assumption is not fullfilled in general [35]. The new property (4.6) for the enriching operator allows for the proof of the theorem below.

Theorem 4.2. *If we assume the regularity estimate*

$$\|u\|_{H^{2+s}} \le C_{reg}\|f\|_{H^{s-2}} \quad s \in (0, 2], \tag{4.13}$$

then

$$\|\nabla(u - u_h)\|_{L^2} \le Ch^{s\wedge 1}\left(\inf_{v \in V_h}\|u - v\|_h + \inf_{w \in W_h}\|u - w\|_h + \mathrm{Osc}(f)\right), \tag{4.14a}$$

$$\|u - u_h\|_{L^2} \le Ch^{s\wedge 2}\left(\inf_{v \in V_h}\|u - v\|_h + \inf_{w \in W_h}\|u - w\|_h + \mathrm{Osc}(f)\right). \tag{4.14b}$$

Proof. We only prove this theorem for $0 < s < 1$. The case $1 \le s \le 2$ may be proceeded by the standard dual estimate as in [34]. An integration by parts gives

$$\begin{aligned}\|\nabla(u - u_h)\|_{L^2}^2 &= \langle -\Delta(u - u_h), u - u_h\rangle\\ &= \langle -\Delta(u - u_h), u - E_h u_h\rangle + \langle -\Delta(u - u_h), E_h u_h - u_h\rangle,\end{aligned}$$

where $\Delta(u-u_h) \in H^{-1}(\Omega)$ and $u-u_h \in H_0^1(\Omega)$, and we write $\langle \cdot,\cdot\rangle$ to denote the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Let $g\colon = -\Delta(u-u_h)$, and we find $\phi \in H_0^2(\Omega)$ such that

$$a(\phi, v) = \langle g, v\rangle \quad \text{for all} \quad v \in H_0^2(\Omega). \tag{4.15}$$

Using the Galerkin orthogonality of E_h, we obtain

$$\begin{aligned}\langle g, u-E_hu_h\rangle &= a(\phi, u-E_hu_h) = a_h(\phi-\Pi\phi, u-E_hu_h) + a_h(\Pi\phi, u-E_hu_h)\\ &= a_h(\phi-\Pi\phi, u-E_hu_h) + a_h(\Pi\phi, u-u_h) + a_h(\Pi\phi-\pi\phi, u_h-E_hu_h).\end{aligned}$$

Proceeding along the same line that leads to (4.12), we obtain that for any $w \in W_h$,

$$a_h(\Pi\phi, u-u_h) = a_h(u-w, \Pi\phi - E_h\Pi\phi) + (f-\overline{f}, E_h\Pi\phi - \Pi\phi).$$

Combining the above two equations, we obtain

$$\begin{aligned}\langle g, u-E_hu_h\rangle = {}& a_h(u-E_hu_h, \phi-\Pi\phi) + a_h(u_h-E_hu_h, \phi-\pi\phi)\\ &+ a_h(u-w, \Pi\phi - E_h\Pi\phi) + (f-\overline{f}, E_h\Pi\phi-\Pi\phi).\end{aligned}$$

Using $(4.6)_2$ with $s=1$ and the inverse inequality, we obtain

$$\|u_h - E_hu_h\|_h \le Ch^{-1}\|u_h-E_hu_h\|_{H^1} \le C\left(\|u-u_h\|_h + \mathrm{Osc}(f)\right),$$

which together with (4.7) and (4.11) yields

$$\begin{aligned}&|\langle g, u-E_hu_h\rangle|\\ \le{}& Ch^s\|\phi\|_{H^{2+s}}\left(\inf_{v\in V_h}\|u-v\|_h + \inf_{w\in W_h}\|u-w\|_h + \|u-u_h\|_h + \mathrm{Osc}(f)\right).\end{aligned}$$

Using (4.13), we obtain

$$\|\phi\|_{H^{2+s}} \le C_{\mathrm{reg}}\|g\|_{H^{s-2}} \le C\|u-E_hu_h\|_{H^s} \le C\|\nabla(u-E_hu_h)\|_{L^2},$$

where we have used the Poincaré inequality in the last step.

Using (4.6), we bound

$$\begin{aligned}&|\langle \Delta(u-u_h), E_hu_h-u_h\rangle|\\ \le{}&\|u-u_h\|_h\|u_h-E_hu_h\|_{L^2} \le Ch^2\|u-u_h\|_h\left(\|u-u_h\|_h + \mathrm{Osc}(f)\right).\end{aligned}$$

A combination of the above two inequalities and the energy estimate (4.11) yield (4.14a).

Proceeding along the same line that leads to (4.14a), we obtain (4.14b). □

4.2. Error estimate for nonsmooth source term

In this part, we consider the error estimate for the nonsmooth source term f. The first one is f is a Dirac-delta function at certain point $x_0 \in \Omega$, i.e., $f = \delta_{x_0}$ corresponds to an idealization of a point load at x_0, which is common for the plate bending problem, while it is not so well understood, by contrast to the vast work devoted to the Poisson problem with Dirac-delta source term. We refer to [42] and the references therein for a review.

We assume that the regularity estimate (4.13) is valid for $s = 2$ in this part.

For delta source term f, the variational problem (2.2) is still well-defined because of $H^2(\Omega) \hookrightarrow C^0(\Omega)$. The discrete approximation problem is also well-defined because $X_h \subset C^0(\Omega)$. The existence and uniqueness of the solutions u and u_h are a direct consequence of the Lax-Milgram theorem and the weak continuity of X_h. By the standard regularity theorem for elliptic problem, we obtain that $u \in H^{3-\epsilon}(\Omega)$ for $\epsilon > 0$. By standard error estimate in [34], we obtain the following sub-optimal error estimate

$$\|u - u_h\|_{L^2} + h\|\nabla(u - u_h)\|_{L^2} + h^2\|u - u_h\|_h \le Ch^{3-\epsilon}$$

for $\epsilon > 0$. The following theorem shows that the ϵ in the above estimate may be removed. Our proof is a combination of the regularized Green's function due to Scott [43] and a clever dual argument belongs to Casas [44].

Theorem 4.3. *Let u and u_h be the solutions of Problem (2.2) and Problem (2.3) with $f = \delta_{x_0}$, respectively. If the regularity assumption (4.13) is valid for $s = 2$, and if the mesh is quasi-uniform in the sense that there exists σ such that $h/h_K \le \sigma$ for all $K \in \mathcal{T}_h$, then there exists an C such that*

$$\|u - u_h\|_{L^2} + h\|\nabla(u - u_h)\|_{L^2} + h^2\|u - u_h\|_h \le Ch^3. \tag{4.16}$$

Proof. We define a regularized Green's function $\delta_h \in V_h$ that approximate δ_{x_0} and satisfies

1. $(\delta, v) = (\delta_h, v)$ for all $v \in V_h$;
2. $\|\delta_h\|_{L^2} = \mathcal{O}(h^{-1})$;
3. $\|\delta - \delta_h\|_{H^{-2}} = \mathcal{O}(h)$.

The construction of δ_h may be proceeded along the same line in [43]. We omit the details. Let $\widehat{u} \in H_0^2(\Omega)$ be the solution of

$$a(\widehat{u}, v) = (\delta_h, v) \quad \text{for all} \quad v \in H_0^2(\Omega).$$

Let $\widehat{u}_h \in V_h$ be the finite element approximation of the above variational problem. By the uniqueness of u_h and the first property of δ_h, we conclude $\widehat{u}_h = u_h$. Using (4.11) and the second property of δ_h, we obtain

$$\|\widehat{u} - u_h\|_h = \|\widehat{u} - \widehat{u}_h\|_h \le Ch^2\left(\|\widehat{u}\|_{H^4} + \|\delta_h\|_{L^2}\right) \le Ch^2\|\delta_h\|_{L^2} \le Ch. \tag{4.17}$$

Note that

$$a(u - \widehat{u}, v) = (\delta - \delta_h, v) \quad \text{for all} \quad v \in H_0^2(\Omega).$$

By the standard a-priori estimate for (2.2), we obtain

$$\|u - \widehat{u}\|_{H^2} \le C\|\delta - \delta_h\|_{H^{-2}} \le Ch,$$

where we have used the third property of δ_h.

A combination of the above two inequalities gives

$$\|u - u_h\|_h \le \|u - \widehat{u}\|_h + \|\widehat{u} - u_h\|_h \le \|u - \widehat{u}\|_{H^2} + \|\widehat{u} - u_h\|_h \le Ch.$$

To obtain the error estimate in the lower order norm, we resort to the auxiliary problem (4.15). Let $\widetilde{\phi} \in V_h$ be the Galerkin projection of ϕ with ϕ the solution of (4.15), i.e.,

$$a_h(\widetilde{\phi}, \psi) = a(\phi, \psi) \quad \text{for all} \quad \psi \in V_h.$$

Standard error estimate [34] implies that for $s = 1, 2$, there holds

$$\|\phi - \widetilde{\phi}\|_{L^2} + h\|\nabla(\phi - \widetilde{\phi})\|_{L^2} + h^2\|\phi - \widetilde{\phi}\|_h \le Ch^{2+s}\|\phi\|_{H^{2+s}}.$$

An application of the inverse estimate and the above error estimate implies that

$$\begin{aligned}
\|\phi - \widetilde{\phi}\|_{L^\infty} &\le \|\phi - \Pi\phi\|_{L^\infty} + \|\Pi\phi - \widetilde{\phi}\|_{L^\infty} \\
&\le Ch^{s+1}\|\phi\|_{H^{2+s}} + Ch^{-1}\|\Pi\phi - \widetilde{\phi}\|_{L^2} \\
&\le Ch^{s+1}\|\phi\|_{H^{2+s}} + Ch^{-1}\left(\|\phi - \Pi\phi\|_{L^2} + \|\phi - \widetilde{\phi}\|_{L^2}\right) \\
&\le Ch^{s+1}\|\phi\|_{H^{2+s}}.
\end{aligned} \tag{4.18}$$

It is clear that

$$\langle g, u - u_h\rangle = a_h(\phi, u - u_h) = a(\phi, u) - a_h(\phi, u_h) = a(\phi, u) - a_h(\widetilde{\phi}, u_h) = (\delta, \phi - \widetilde{\phi}).$$

Combining the above two equations, we obtain, for $s = 1, 2$,

$$|\langle g, u - u_h\rangle| \le \|\phi - \widetilde{\phi}\|_{L^\infty} \le Ch^{s+1}\|\phi\|_{H^{2+s}} \le Ch^{s+1}\|g\|_{H^{s-2}}.$$

This gives

$$\|u - u_h\|_{L^2} + h\|\nabla(u - u_h)\|_{L^2} \le Ch^3.$$

This completes the proof. □

If the solution u is less smooth, e.g., $u \in H^{2+s}(\Omega)$ for $s > 0$, we conclude from Theorem 4.1 that

$$\|u - u_h\|_h \le C\left(h^s\|u\|_{H^{2+s}} + h^2\|f\|_{L^2}\right).$$

By the regularity result (4.13), we may further rewrite the above estimate as

$$\|u - u_h\|_h \le C\left(h^s\|f\|_{H^{s-2}} + h^2\|f\|_{L^2}\right) \le Ch^s\|f\|_{L^2}.$$

This estimate is not optimal with respect to the smoothness assumption on the load. In fact, the above estimate may be improved for $s \in (\frac{1}{2}, 2]$ as follows.

Corollary 4.1. *Let u and u_h be the solutions of Problem* (2.2) *and Problem* (2.3)*, respectively. If we assume that*

$$\|u\|_{H^{s+2}} \le C\|f\|_{H^{s-2}} \quad s \in (\tfrac{1}{2}, 2],$$

then

$$\|u - u_h\|_h \le Ch^s\|f\|_{H^{s-2}}. \tag{4.19}$$

The regularity assumption is true for the hard clamped problem posed on polygon, while it may be false for variational problem with other boundary conditions [35]. The proof follows essentially the same line that leads to (4.11) with minor modification.

Proof. For any $v\in V_h$, we denote $w=v-u_h$. By (4.12), we obtain, for any $z\in W_h$, there holds

$$\|w\|_h^2=a_h(v-z,w-E_hw)+a_h(v-u,E_hw)+(f,E_hw-w).$$

For any $s\in(1/2,2]$, using $(4.6)_1$, we obtain

$$\begin{aligned}\|w\|_h^2&\le\|v-z\|_h\|w-E_hw\|_h+\|u-v\|_h\|E_hw\|_h+\|f\|_{H^{s-2}}\|w-E_hw\|_{H^{2-s}}\\&\le(1+\alpha)\|v-z\|_h\|w\|_h+\alpha\|u-v\|_h\|w\|_h+\beta\,h^s\|f\|_{H^{s-2}}\|w\|_h.\end{aligned}$$

This gives

$$\|w\|_h\le(1+\alpha)\|u-z\|_h+(1+2\alpha)\|u-v\|_h+\beta h^s\|f\|_{H^{s-2}},$$

which together with the triangle inequality implies

$$\|u-u_h\|_h\le 2(1+\alpha)\inf_{v\in V_h}\|u-v\|_h+(1+\alpha)\inf_{z\in W_h}\|u-z\|_h+\beta h^s\|f\|_{H^{s-2}}.\tag{4.20}$$

Using the interpolate estimate and the regularity estimate, we obtain (4.19). □

The discrete variational problem (2.3) is not well-defined for $f\in H^{s-2}(\Omega)$ with $s\in[0,1/2]$ because $V_h\subset H^{3/2-\epsilon}(\Omega)$ for $\epsilon>0$. The Spehct triangle can be equally applied to problem with rough load if we modify the source term in (2.3) by exploiting the enriching operator as follows. Find $\widetilde{u}_h\in V_h$ such that

$$a_h(\widetilde{u}_h,v)=(f,E_hv)\quad\text{for all}\quad v\in V_h.\tag{4.21}$$

The following result can be obtained in exactly the same way as that for Theorem 4.1.

Corollary 4.2. *Let u and $\widetilde{u}_h$ be the solutions of Problem (2.2) and Problem (4.21), respectively. Then*

$$\|u-\widetilde{u}_h\|_h\le(1+\alpha)\left(2\inf_{v\in V_h}\|u-v\|_h+\inf_{w\in W_h}\|u-w\|_h\right).\tag{4.22}$$

If the regularity estimate (4.13) is valid, then

$$\|\nabla(u-\widetilde{u}_h)\|_{L^2}\le Ch^{s\wedge 1}\left(\inf_{v\in V_h}\|u-v\|_h+\inf_{w\in W_h}\|u-w\|_h\right),\tag{4.23a}$$

$$\|u-\widetilde{u}_h\|_{L^2}\le Ch^{s\wedge 2}\left(\inf_{v\in V_h}\|u-v\|_h+\inf_{w\in W_h}\|u-w\|_h\right).\tag{4.23b}$$

The above error estimates in H^1 and L^2 norms can be obtained in a similar way; See [22]. We omit the details and leave it to the interested readers. Such modification may be traced back to [45] for Morley's triangle.

5. Numerical Examples

In this part, we test the proposed elements for problems with smooth solution and non-smooth solution. We assume that $\Omega=(0,1)^2$. The initial mesh is generated by the function "initmesh" of the partial differential equation toolbox of MATLAB. The initial mesh consists of 872 triangles and 469 vertices, and the maximum mesh size is $h=1/16$; see Fig. 5.1.

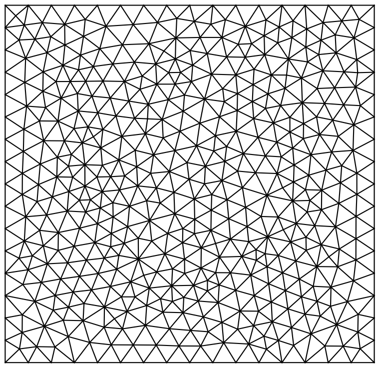
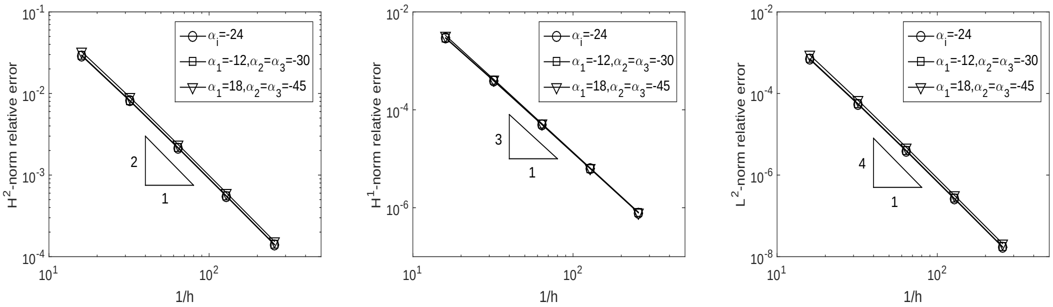


Fig. 5.1. Plots of the unstructured mesh.

5.1. Example with clamped boundary condition and full regularity

We consider the clamped boundary condition and assume the Poisson ratio $\nu = 1/2$ and the solution is given by $u = 4\sin^2(2\pi x)\sin^2(2\pi y)$. The surface load f is computed from the equation $(2.1)_1$. It is clear that u is smooth and satisfies the clamped boundary condition. We test the performance of the elements with different choice of parameters $\alpha_i, i = 1, 2, 3$ that satisfies the constraint (3.4). Besides the elements (3.7) and (3.8), we also test another element by setting $\alpha_1 = 18, \alpha_2 = \alpha_3 = -45$. In this case,

$$\begin{cases} \widetilde{q}_1 = b_K\,(80\lambda_1(1-\lambda_1) - 16 + 50\lambda_2\lambda_3)\,, \\ \widetilde{q}_2 = b_K\,(5 - 25\lambda_2(1-\lambda_2) + 55\lambda_3\lambda_1)\,, \\ \widetilde{q}_3 = b_K\,(5 - 25\lambda_3(1-\lambda_3) + 55\lambda_1\lambda_2)\,. \end{cases}$$



(a) (b) (c)

Fig. 5.2. Plots of the convergence rate for smooth solution. The subfigures (a), (b) and (c) show the rates of convergence in broken H^2 norm, H^1-norm and L^2-norm, respectively.

In Fig. 5.2, we report the convergence rate of the above elements measured in the relative Sobolev broken norm and Sobolev norms:

$$\frac{\|u-u_h\|_h}{\|u\|_{H^2}}, \quad \frac{\|u-u_h\|_{H^1}}{\|u\|_{H^1}}, \quad \frac{\|u-u_h\|_{L^2}}{\|u\|_{L^2}}.$$

The numerical results indicate that the rate of convergence tends to be quadratic, cubic and quartic with respect to H^2 broken norm, H^1 norm and L^2 norm, respectively, which is consistent with the theoretical prediction (4.14). The performance of the elements for different choices of the parameters α_i are almost the same.

5.2. Simply supported boundary value problem without full regularity

In the second example, we consider a plate which is simply supported and subjected to a point load at the center, i.e., $u = \triangle u = 0$ on the boundary. The problem is posed on the same domain, and with the same materials parameter as the previous example. The exact solution can be written as an infinite sum

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \sin(m\pi x) \sin(n\pi y)$$

with

$$u_{mn} = \frac{4 \sin(m\pi/2) \sin(n\pi/2)}{\pi^4 (m^2 + n^2)^2}.$$

This example is taken from [25]. Due to the point load, we cannot expect the full H^4 regularity for the solution; See [35]. In fact, we have $u \in H^{3-\epsilon}(\Omega)$ for any $\epsilon > 0$. In Figure 5.3, we report the convergence rate of the elements (3.7) and (3.8) in the relative broken H^2 norm, H^1-norm and L^2-norm. The exact solution is taken from the above series solution by truncating the infinite sum by $m = n = 1000$. In view of Fig. 5.3, the second order Specht triangle converges linearly in the broken H^2 norm, quadratically in H^1 norm, and cubically in L^2 norm, which is consistent with the error estimate in Theorem 4.3. For the sake of comparison, we also report the numerical results for the Specht triangle [3].

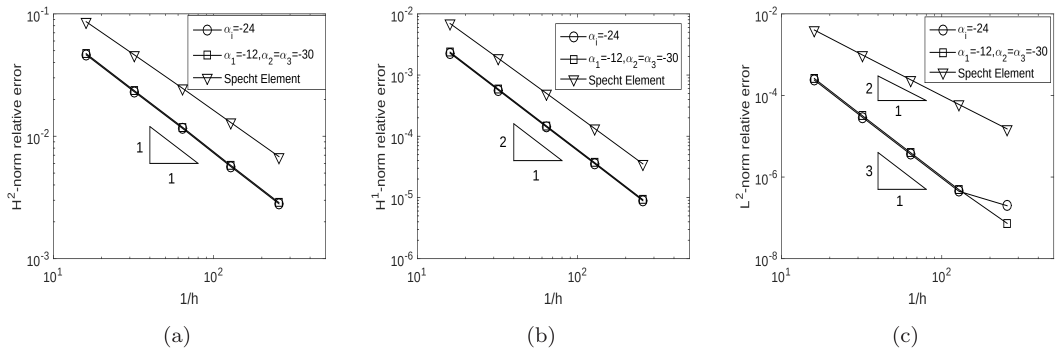


Fig. 5.3. Plots of the rate of convergence without full regularity. The subfigure (a), (b) and (c) show the rates of convergence in H^2-norm, H^1-norm and L^2-norm, respectively.

6. Conclusion

We derive a class of quadratic Specht triangle, which is not unique because there are infinitely possibilities for the choices of the parameters α_i that satisfy the constraint (3.4), and each choice gives one quadratic Specht triangle. It seems that the numerical performance of the element corresponding to certain typical choices of α_i are almost the same. It would be interesting to extend this element to higher order to see whether they come in a natural hierarchy, which is desirable because most existing nonconforimg plate bending elements are invented in an ad hoc manner and lack a natural hierarchy, as opposed to the discontinuous Galerkin method [23]. The method may also be extended to three-dimensional problem, which is our ongoing work [46].

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