

SUPERCONVERGENCE ANALYSIS OF THE POLYNOMIAL PRESERVING RECOVERY FOR ELLIPTIC PROBLEMS WITH ROBIN BOUNDARY CONDITIONS*

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Abstract

We analyze the superconvergence property of the linear finite element method based on the polynomial preserving recovery (PPR) for Robin boundary elliptic problems on triangulations. First, we improve the convergence rate between the finite element solution and the linear interpolation under the H^1 -norm by introducing a class of meshes satisfying the *Condition* (α, σ, μ) . Then we prove the superconvergence of the recovered gradients post-processed by PPR and define an asymptotically exact a posteriori error estimator. Finally, numerical tests are provided to verify the theoretical findings.

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Key words: Superconvergence, Polynomial preserving recovery, Finite element methods, Robin boundary condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygon with boundary $\Gamma := \partial\Omega$. Let \mathbf{n} be the unit normal vector to the boundary exterior to Ω . We consider the superconvergence analysis for the model problem: Find $u \in H^1(\Omega)$ such that

$$a(u, v) := \int_{\Omega} (\nabla u \cdot \nabla v + cuv) + \int_{\partial\Omega} quv = f(v) + g(v), \quad \forall v \in H^1(\Omega), \quad (1.1)$$

where $c \in L^\infty$, $q \in L^\infty(\Gamma)$, $f \in H^{-1}(\Omega)$ and $g \in H^{-\frac{1}{2}}(\partial\Omega)$. We note that most results hold for a general class of elliptic equations and (1.1) is for presenting the main idea and techniques in their simplest form.

For given a shape regular triangulation \mathcal{M}_h of $\bar{\Omega}$ with mesh size h , we denote

$$V_h := \left\{ v_h \in H^1(\Omega) : v_h|_{\tau} \in P_1(\tau) \quad \forall \tau \in \mathcal{M}_h \right\}$$

the space of all continuous, piecewise linear finite element functions corresponding to \mathcal{M}_h . Here P_1 denotes the set of polynomials with degree at most one. The finite element solution $u_h \in V_h$ satisfies

$$a(u_h, v_h) = f(v_h) + g(v_h), \quad \forall v_h \in V_h. \quad (1.2)$$

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It is well known that there are many superclose and superconvergent results for Dirichlet boundary problems [13, 16, 18, 19, 22]. The convergence analysis is for uniform grids or patch symmetric grids at first. However, since it is difficult to construct uniform grids on unstructured domains and the grids produced by grid generation algorithms are a small perturbation of uniform grids in the most region of the domain, one considered the so-called mildly structured grids where an $O(h^{1+\alpha})$ approximate parallelogram property is satisfied for pairs of adjacent triangles in most parts of Ω except for a region of size $O(h^{2\sigma})$ [6, 7, 16, 18]. Two finite element functions vanishing on $\partial\Omega$, the continuous linear finite element solution u_h^0 and the continuous linear nodal interpolation u_I^0 of u^0 , are superclose in the sense that

$$\|\nabla u_h^0 - \nabla u_I^0\|_{H^1(\Omega)} = O(h^{1+\min(\alpha, 1-\sigma)}).$$

Here we assume that u^0 is the exact solution to the Dirichlet boundary problem. Based on the supercloseness, various post-processing techniques, such as the global L^2 projection [6, 8, 11], the Zienkiewicz-Zhu (ZZ) method [24, 25], and the Polynomial Preserving Recovery [13, 14, 23], have been proposed to produce a new approximation $R_h(u_h^0)$ of ∇u^0 , which is superconvergent in the sense that

$$\|R_h(u_h^0) - \nabla u^0\|_{H^1(\Omega)} = O(h^{1+\min(\alpha, 1-\sigma)}).$$

Based on the superconvergence results, an asymptotically exact error estimator can be constructed [7, 16]. In the last decade the convergence proof for Dirichlet boundary problem has been well established. By contrast, there are only a few superconvergent works on the Robin boundary problem. [9] considered the Robin boundary condition and proved the superconvergent rate of $O(h^{3/2})$. [3] considered the case of Neumann boundary and $\alpha = 1$ (i.e. each of the ‘‘good’’ pairs of triangles forms an $O(h^2)$ approximate parallelograms) and proved the superconvergent rate of $O(h^{2-\sigma} |\log h|^{\frac{1}{2}})$.

In this work, we investigate the superconvergence property of the method (1.2) when being post-processed by the polynomial preserving recovery (PPR) for the Robin boundary problem. PPR was proposed by Zhang and Naga [23] in 2004 and has been successfully applied to finite element methods. COMSOL Multiphysics adopted PPR as a post-processing tool since 2008, see [1]. One important feature of PPR is its superconvergence property for the recovered gradient. To learn more about PPR, readers are referred to [13, 16, 20, 21]. Some theoretical results about recovery techniques and recovery-type error estimators can be found in [4, 12, 18, 19, 22].

We first extend the definition of mildly structured grids to the boundary by assuming that the two triangles associated to a ‘‘good’’ boundary node are $O(h^{1+\alpha})$ approximate congruent triangles and the number of ‘‘bad’’ boundary nodes is of order $O(h^{-2\mu})$ for some $0 \leq \mu < \frac{1}{2}$. Secondly, we prove the following supercloseness result:

$$\|u_h - u_I\|_{H^1(\Omega)} = O\left(h^{1+\min(\alpha, 1-\sigma)} + \min\left(h^{2-2\mu} |\log h|^{\frac{1}{2}}, h^{\frac{3}{2}}\right)\right),$$

which improves the estimates of [3, 9]. Denote $G_h : V_h \rightarrow V_h \times V_h$ as the gradient recovery operator from PPR. Thirdly, we obtain the following estimate:

$$\|\nabla u - G_h u_h\|_{L^2(\Omega)} \lesssim h^{1+\min\{\alpha, 1-\sigma\}} + \min\left(h^{2-2\mu} |\log h|^{\frac{1}{2}}, h^{3/2}\right). \quad (1.3)$$

Based on the superconvergent result, we define an asymptotically exact a posteriori error estimator $\|G_h u_h - \nabla u_h\|_{L^2(\Omega)}$. Readers are referred to [2, 5, 10, 15] for further theoretical results about recovery techniques and recovery-type error estimators.

The remainder of this paper is organized as follows: some notations and the mesh constraints are introduced in Section 2. In Section 3, we prove the supercloseness between the interpolant and the finite element solution to the Robin boundary problem (1.1). In Section 4, we prove the superconvergence property of G_h in the Sobolev space $H^3(\Omega)$ and define a posteriori error estimator. Finally, in Section 5 we verify the sharpness of our estimates by simulating some model problems on several specially designed meshes.

Throughout the paper, C is used to denote a generic positive constant which is independent of h, f and g . We also use the shorthand notation $A \lesssim B$ and $A \gtrsim B$ for the inequality $A \leq CB$ and $A \geq CB$, respectively. $A \approx B$ is a shorthand notation for the statement $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

The symbols $(\cdot, \cdot)_Q$ and $\langle \cdot, \cdot \rangle_\Sigma$ for $\Sigma = \partial Q$ denote the L^2 -inner products on $L^2(Q)$ and $L^2(\Sigma)$ spaces, respectively. For simplicity, denote by $(\cdot, \cdot) := (\cdot, \cdot)_\Omega$, $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\partial\Omega}$, $\|\cdot\|_j := \|\cdot\|_{H^j(\Omega)}$, and $|\cdot|_j := |\cdot|_{H^j(\Omega)}$. Throughout this paper, the norm $\|\cdot\|_{H^s(\Gamma)}$ (seminorm $|\cdot|_{H^s(\Gamma)}$) on the boundary of the polygon Ω are interpreted as the square root of the sum of squares of the H^s -norms (seminorms) on each sides of the polygon.

Let \mathcal{E}_h be the set of all edges of \mathcal{M}_h and \mathcal{N}_h be the set of all nodal points. For any $\tau \in \mathcal{M}_h$, we denote by h_τ its diameter and by $|\tau|$ its area. Similarly, for each edge $e \in \mathcal{E}_h$, define $h_e := \text{diam}(e)$. Let $h = \max_{\tau \in \mathcal{M}_h} h_\tau$. Assume that $h_\tau \approx h$. We denote all the boundary edges by $\mathcal{E}_h^B := \{e \in \mathcal{E}_h : e \subset \Gamma\}$ and the interior edges by $\mathcal{E}_h^I := \mathcal{E}_h \setminus \mathcal{E}_h^B$. Denote by \mathcal{N}_h^B the nodes on the physical boundary Γ .

Following the discussion in [6], We introduce some definitions regarding meshes at first. For an interior edge $e \in \mathcal{E}_h^I$, let τ_e and τ'_e be two elements sharing e and denote by $\Omega_e = \tau_e \cup \tau'_e$, see Figures 2.1–2.2. For an element $\tau \subset \Omega_e$, denote by θ_e the angle opposite of the edge e in τ , denote by \mathbf{t}_e the unit tangent vector of e with counterclockwise orientation and \mathbf{n}_e , the unit outward normal vector of e . Denote by h_e, h_{e+1}, h_{e-1} denote the lengths of the three edges of τ_e , respectively, where the subscript $e + 1$ or $e - 1$ is for orientation. We emphasize that all triangles in \mathcal{M}_h are orientated counterclockwise. An index $'$ is added for the corresponding quantities in τ'_e . Notice that $\mathbf{n}_e = -\mathbf{n}'_e$ and $\mathbf{t}_e = -\mathbf{t}'_e$ due to the orientation.

Let $e \in \mathcal{E}_h^I$ be an interior edge. Recall that Ω_e , the patch of e , consists of two adjacent triangles sharing e . We say that Ω_e is an ϵ approximate parallelogram if the lengths of any two opposite edges differ by at most ϵ , that is

$$|h_{e-1} - h'_{e-1}| + |h_{e+1} - h'_{e+1}| \leq \epsilon. \tag{2.1}$$

In general, the patch Ω_e is set to be an $O(h^{1+\alpha})$ approximate parallelogram in the superconvergence analysis for Dirichlet boundary problems. However, for Robin boundary condition, an additional restriction on elements pairs with common nodes in \mathcal{N}_h^B is necessary. For a node $z \in \mathcal{N}_h^B$ (cf. Fig. 2.2), let $e, e' \in \mathcal{E}_h^B$ be the edges sharing z . We say that τ_e and $\tau_{e'}$ are ϵ approximate congruent triangles if

$$|h_{e-1} - h_{e'-1}| + |h_{e+1} - h_{e'+1}| + |h_e - h_{e'}| \leq \epsilon. \tag{2.2}$$

Clearly, for an interior edge e , if Ω_e is an ϵ approximate parallelogram, the two triangles in Ω_e are ϵ approximate congruent triangles.

In order to deal with the Robin boundary condition, we introduce the following mesh condition (α, σ, μ) which is a modification of the usual condition (α, σ) , see [6, 18].

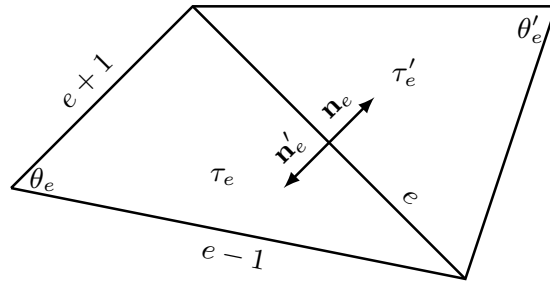


Fig. 2.1. Notation in the patch Ω_e .

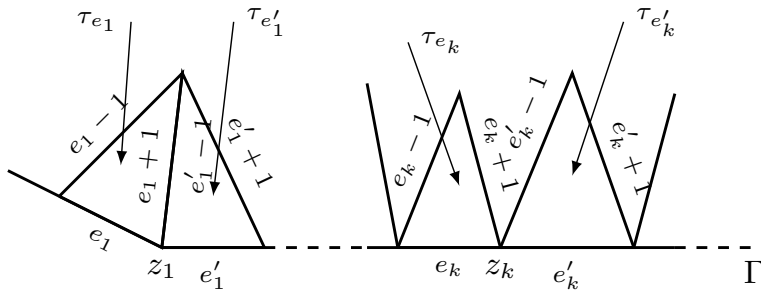


Fig. 2.2. Notation in the boundary elements.

Definition 2.1. The triangulation \mathcal{M}_h is said to satisfy condition (α, σ, μ) if \mathcal{E}_h^I can be divided into two parts $\mathcal{E}_{1,h}^I \oplus \mathcal{E}_{2,h}^I$, \mathcal{N}_h^B can be divided into two parts $\mathcal{N}_{1,h}^B \oplus \mathcal{N}_{2,h}^B$ and there exists constants $\alpha \geq 0$, $0 \leq \sigma < 1$ and $0 \leq \mu < \frac{1}{2}$ such that

- (i) for $e \in \mathcal{E}_{1,h}^I$, the patch Ω_e is an $O(h^{1+\alpha})$ approximate parallelogram;
- (ii) for $z \in \mathcal{N}_{1,h}^B$ shared by $e, e' \in \mathcal{E}_h^B$, the triangles τ_e and $\tau_{e'}$ are $O(h^{1+\alpha})$ approximate congruent triangles;
- (iii) the number of edges in $\mathcal{E}_{2,h}^I$ satisfies $\#\mathcal{E}_{2,h}^I = O(h^{-2\sigma})$;
- (iv) the number of nodes in $\mathcal{N}_{2,h}^B$ satisfies $\#\mathcal{N}_{2,h}^B = O(h^{-2\mu})$.

Remark 2.1. (a) Bank and Xu proposed some mesh conditions [3, Definition 2.4] for a superconvergence analysis for the Neumann boundary condition. Our conditions (i) and (iii) are similar to the corresponding conditions in [3] (or more precisely, the same as the mildly structured grids defined in the reference [18]), while the conditions (ii) and (iv) are new, which are weaker than the corresponding condition 2 in [3, Definition 2.4]. In fact, [3] assumed that the triangles associated with each node $z \in \mathcal{N}_{1,h}^B$ form an $O(h^2)$ approximate parallelogram and that $\#\mathcal{N}_{2,h}^B = O(1)$. Therefore, our conditions actually define “mildly structured grids” up to the boundary and are more practical.

(b) Although one can still get results of superconvergence for some meshes not satisfying the restriction (i), such as Chevron pattern uniform mesh, we construct in Section 5 a class of special grids to show that the restriction “ $h^{1+\alpha}$ approximate parallelogram” is necessary.

(c) However, since unlike the case of homogenous Dirichlet boundary conditions, $u_h - u_I \neq 0$ on Γ , we need the assumption (ii) for elements with one edge on Γ . Here u_I is the linear

interpolant of u . Note that whether the additional restriction (ii) in 2.1 is just technique for theoretical purpose or not is still an open problem. But if $\mathcal{E}_{2,h}^I = \emptyset$, we have $\mathcal{N}_{2,h}^B$ and the restriction (ii) can be removed, because, for any node $z \in \mathcal{N}_h^B$, all the triangles sharing the node z are approximate congruent triangles. In others words, the condition $(\alpha, 0, 0)$ is equivalent to classical condition $(\alpha, 0)$ and therefore, the results holds under the condition $(\alpha, 0)$ as well.

(d) The restrictions (iii) and (iv) in 2.1 means that $(\mathcal{E}_h^I, \mathcal{N}_h^B)$ can be grouped into “good” $(\mathcal{E}_{1,h}^I, \mathcal{N}_{1,h}^B)$ and “bad” $(\mathcal{E}_{2,h}^I, \mathcal{N}_{2,h}^B)$, respectively. The numbers of bad edges and bad boundary nodes are much smaller than those of good ones, respectively, where the ratios are

$$\frac{\#\mathcal{E}_{2,h}^I}{\#\mathcal{E}_{1,h}^I} \lesssim \frac{h^{-2\sigma}}{h^{-2}} = h^{2-2\sigma}, \quad \frac{\#\mathcal{N}_{2,h}^B}{\#\mathcal{N}_{1,h}^B} \lesssim \frac{h^{-2\mu}}{h^{-1}} = h^{1-2\mu}.$$

3. Supercloseness Between the FE Solution and the Interpolant

We investigate the supercloseness between the FE solution u_h and the linear interpolant u_I of u . Some special arguments are needed to establish the desired superclose result, because u_h is not equal to u_I on the physical boundary, that is $(u_h - u_I)|_\Gamma \neq 0$.

We introduce the quadratic interpolant $\phi_Q = \Pi_Q \phi$ of ϕ based on nodal values and moment conditions on edges,

$$(\Pi_Q \phi)(z) = \phi(z), \quad \int_e \Pi_Q \phi = \int_e \phi, \quad \forall z \in \mathcal{N}_h, e \in \mathcal{E}_h. \quad (3.1)$$

The following fundamental identity for $v_h \in P_1(\tau)$ has been proved in [6]:

$$\int_\tau \nabla(\phi - \phi_I) \cdot \nabla v_h = \sum_{e \in \partial\tau} \left(\beta_e \int_e \frac{\partial^2 \phi_Q}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 \phi_Q}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right), \quad (3.2)$$

where

$$\beta_e = \frac{1}{12} \cot \theta_e (h_{e+1}^2 - h_{e-1}^2), \quad \gamma_e = \frac{1}{3} \cot \theta_e |\tau|, \quad (3.3)$$

and $\phi_I \in P_1(\tau)$ is the linear interpolant of ϕ on τ .

Lemma 3.1. *We denote \mathbf{m}_e by \mathbf{t}_e or \mathbf{n}_e . Assume that \mathcal{M}_h satisfies the condition (α, σ, μ) . Then we have the following estimates:*

$$|\beta_e| + |\beta'_e| \lesssim h^2, \quad |\gamma_e| + |\gamma'_e| \lesssim h^2, \quad e \in \mathcal{E}_h^I, \quad (3.4)$$

$$|\beta_e - \beta'_e| \lesssim h^{2+\alpha}, \quad |\gamma_e - \gamma'_e| \lesssim h^{2+\alpha}, \quad e \in \mathcal{E}_{1,h}^I, \quad (3.5)$$

$$|\beta_e| \lesssim h^2, \quad |\gamma_e| \lesssim h^2, \quad e \in \mathcal{E}_h^B, \quad (3.6)$$

$$|\beta_e - \beta'_e| \lesssim h^{2+\alpha}, \quad |\gamma_e - \gamma'_e| \lesssim h^{2+\alpha}, \quad e \cap e' = z \in \mathcal{N}_{1,h}^B, \quad e, e' \in \mathcal{E}_h^B. \quad (3.7)$$

Moreover, for any $e \in \mathcal{E}_h$,

$$\int_e \frac{\partial^2 \phi}{\partial \mathbf{t}_e \partial \mathbf{m}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \lesssim \left(|\phi|_{H^3(\tau_e)} + h^{-1} |\phi|_{H^2(\tau_e)} \right) \|\nabla v_h\|_{L^2(\tau_e)}, \quad (3.8)$$

$$\int_e \frac{\partial^2 \phi}{\partial \mathbf{t}_e \partial \mathbf{m}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \lesssim |\phi|_{W^{2,\infty}(\Omega)} \|\nabla v_h\|_{L^2(\tau_e)}, \quad (3.9)$$

$$\int_e \frac{\partial^2(\phi - \phi_Q)}{\partial \mathbf{t}_e \partial \mathbf{m}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \lesssim |\phi|_{H^3(\tau_e)} \|\nabla v_h\|_{L^2(\tau_e)}. \quad (3.10)$$

See [6, 7] for the proof.

Throughout the paper, we assume that $c \geq c_0 > 0$, $q \in W^{2,\infty}(\Gamma)(q \geq 0)$, $f \in H^1(\Omega)$, $g \in H^2(\Gamma)$, and the exact solution $u \in H^3(\Omega) \cap W^{2,\infty}(\Omega)$.

Lemma 3.2. *Assume that \mathcal{M}_h satisfies the condition (α, σ, μ) . Then for any $v_h \in V_h$, we have*

$$\left| \int_{\Omega} \nabla(u - u_I) \cdot \nabla v_h \right| \lesssim \left(h^{1+\min(\alpha, 1-\sigma)} + \min(h^{2-2\mu} |\log h|^{\frac{1}{2}}, h^{\frac{3}{2}}) \right) \|v_h\|_1 C_{u,g,q}, \quad (3.11)$$

where $C_{u,g,q} = \|u\|_3 + |u|_{W^{2,\infty}(\Omega)} + \|g\|_{H^2(\Gamma)} + \|q\|_{W^{2,\infty}(\Gamma)} \|u\|_{H^2(\Gamma)} + \|u\|_{H^3(\Gamma)}$.

Proof. From (3.2), we have

$$\int_{\Omega} \nabla(u - u_I) \cdot \nabla v_h = \sum_{\tau \in \mathcal{M}_h} \sum_{e \subset \partial \tau} \left(\beta_e \int_e \frac{\partial^2 u_Q}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 u_Q}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right) =: I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \sum_{e \in \mathcal{E}_h^I} \left((\beta_e - \beta'_e) \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + (\gamma_e - \gamma'_e) \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right. \\ &\quad + \beta_e \int_e \frac{\partial^2 (u_Q - u)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 (u_Q - u)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \\ &\quad \left. + \beta'_e \int_e \frac{\partial^2 (u - u_Q)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma'_e \int_e \frac{\partial^2 (u - u_Q)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right), \\ I_2 &= \sum_{e \in \mathcal{E}_h^B} \left(\beta_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right. \\ &\quad \left. + \beta_e \int_e \frac{\partial^2 (u_Q - u)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 (u_Q - u)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right). \end{aligned}$$

We first estimate I_1 which is divided into $I_{1,1}$ and $I_{1,2}$, where

$$\begin{aligned} I_{1,j} &= \sum_{e \in \mathcal{E}_{j,h}^I} \left((\beta_e - \beta'_e) \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + (\gamma_e - \gamma'_e) \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right. \\ &\quad + \beta_e \int_e \frac{\partial^2 (u_Q - u)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 (u_Q - u)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \\ &\quad \left. + \beta'_e \int_e \frac{\partial^2 (u - u_Q)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma'_e \int_e \frac{\partial^2 (u - u_Q)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right). \end{aligned}$$

By Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned} |I_{1,1}| &\lesssim \sum_{e \in \mathcal{E}_{1,h}^I} \left((h^{2+\alpha} + h^2) |u|_{H^3(\tau_e)} + h^{1+\alpha} |u|_{H^2(\tau_e)} \right) \|\nabla v_h\|_{L^2(\tau_e)} \\ &\lesssim (h^2 |u|_3 + h^{1+\alpha} |u|_2) \|\nabla v_h\|_0. \end{aligned} \quad (3.12)$$

From Lemma 3.1 and noting that $\#\mathcal{E}_{2,h}^I = h^{-2\sigma}$,

$$\begin{aligned} |I_{1,2}| &\lesssim \sum_{e \in \mathcal{E}_{2,h}^I} h^2 |u|_{W^{2,\infty}(\tau_e)} \|\nabla v_h\|_{L^2(\tau_e)} \lesssim h^2 (\#\mathcal{E}_{2,h}^I)^{\frac{1}{2}} |u|_{W^{2,\infty}(\Omega)} \|\nabla v_h\|_0 \\ &\lesssim h^{2-\sigma} |u|_{W^{2,\infty}(\Omega)} \|\nabla v_h\|_0. \end{aligned} \quad (3.13)$$

Combining (3.12) and (3.13) yields

$$|I_1| \lesssim (h^{1+\alpha} |u|_2 + h^{2-\sigma} |u|_{W^{2,\infty}(\Omega)} + h^2 |u|_{H^3(\Omega)}) \|\nabla v_h\|_0. \quad (3.14)$$

We divide I_2 into two parts $I_{2,j}$ ($j = 1, 2$) to estimate, where

$$I_{2,1} := \sum_{e \in \mathcal{E}_h^B} \left(\beta_e \int_e \frac{\partial^2(u_Q - u)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2(u_Q - u)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right), \quad (3.15)$$

$$I_{2,2} := \sum_{e \in \mathcal{E}_h^B} \left(\beta_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right). \quad (3.16)$$

From Lemma 3.1, we can derive in analogy to (3.12),

$$\begin{aligned} |I_{2,1}| &\lesssim \sum_{e \in \mathcal{E}_{1,h}^B} (h^{1+\alpha} |u|_{H^2(\tau_e)} + (h^{2+\alpha} + h^2) |u|_{H^3(\tau_e)}) \|\nabla v_h\|_{L^2(\tau_e)} \\ &\lesssim (h^{1+\alpha} |u|_2 + h^2 |u|_3) \|\nabla v_h\|_0. \end{aligned} \quad (3.17)$$

Next we turn to the estimate $I_{2,2}$. First we have

$$\begin{aligned} I_{2,2} &\lesssim h^2 \sum_{e \in \mathcal{E}_h^B} |u|_{W^{2,\infty}(\Omega)} \|\nabla v_h\|_{L^2(\tau_e)} \lesssim h^2 (\#\mathcal{E}_h^B)^{\frac{1}{2}} |u|_{W^{2,\infty}(\Omega)} \|\nabla v_h\|_0 \\ &\lesssim h^{\frac{3}{2}} |u|_{W^{2,\infty}(\Omega)} \|\nabla v_h\|_0. \end{aligned} \quad (3.18)$$

In the following we give another estimate of $I_{2,2}$ by using the assumption on the elements with sides on the boundary Γ . For $z \in \mathcal{N}_h^B$, let e and e' be two edges in \mathcal{E}_h^B sharing z with counterclockwise orientation (cf. Figure 2.2). Denote by $[\beta_e]_z = \beta_{e'} - \beta_e$ and $[\gamma_e]_z = \gamma_{e'} - \gamma_e$. Let \mathcal{N}_h^V be the set of vertices of the domain Ω . By integration by parts, we have

$$\begin{aligned} I_{2,2} &= - \sum_{e \in \mathcal{E}_h^B} \left(\beta_e \int_e \frac{\partial^3 u}{\partial \mathbf{t}_e^3} v_h + \gamma_e \int_e \frac{\partial^3 u}{\partial \mathbf{t}_e^2 \partial \mathbf{n}_e} v_h \right) \\ &\quad + \sum_{z \in \mathcal{N}_{1,h}^B \setminus \mathcal{N}_h^V} \left([\beta_e]_z \frac{\partial^2 u}{\partial \mathbf{t}_e^2}(z) + [\gamma_e]_z \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(z) \right) v_h(z) \\ &\quad + \sum_{z \in \mathcal{N}_{2,h}^B \setminus \mathcal{N}_h^V} \left([\beta_e]_z \frac{\partial^2 u}{\partial \mathbf{t}_e^2}(z) + [\gamma_e]_z \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(z) \right) v_h(z) \\ &\quad + \sum_{z \in \mathcal{N}_h^V} \left(\beta_{e'} \frac{\partial^2 u}{\partial \mathbf{t}_{e'}^2}(z) - \beta_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2}(z) + \gamma_{e'} \frac{\partial^2 u}{\partial \mathbf{t}_{e'} \partial \mathbf{n}_{e'}}(z) - \gamma_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(z) \right) v_h(z) \\ &=: I_{2,2,1} + I_{2,2,2} + I_{2,2,3} + I_{2,2,4}. \end{aligned} \quad (3.19)$$

Since $\frac{\partial u}{\partial \mathbf{n}} + qu = g$ on Γ , we obtain

$$\begin{aligned} I_{2,2,1} &\lesssim h^2 \left(|u|_{H^3(\Gamma)} + \left| \frac{\partial u}{\partial \mathbf{n}} \right|_{H^2(\Gamma)} \right) \|v_h\|_{L^2(\Gamma)} \\ &\lesssim h^2 (|u|_{H^3(\Gamma)} + |g|_{H^2(\Gamma)} + |qu|_{H^2(\Gamma)}) \|v_h\|_0^{\frac{1}{2}} \|v_h\|_1^{\frac{1}{2}} \\ &\lesssim h^2 \|v_h\|_1 (|g|_{H^2(\Gamma)} + \|q\|_{W^{2,\infty}(\Gamma)} |u|_{H^2(\Gamma)} + |u|_{H^3(\Gamma)}). \end{aligned} \quad (3.20)$$

For any $w \in H^1([a, b])$, we have

$$w^2(b) = \int_a^b \left(\frac{x-a}{b-a} w^2(x) \right)' dx \lesssim \frac{1}{b-a} \|w\|_{L^2([a,b])}^2 + (b-a) \|w\|_{L^2([a,b])}^2,$$

which implies

$$\begin{aligned} I_{2,2,2} &\leq \sum_{z \in \mathcal{N}_{1,h}^B \setminus \mathcal{N}_h^V} \left(|[\beta_e]_z| (h_e^{-\frac{1}{2}} |u|_{H^2(e)} + h_e^{\frac{1}{2}} |u|_{H^3(e)}) \right. \\ &\quad \left. + |[\gamma_e]_z| \left(h_e^{-\frac{1}{2}} \left| \frac{\partial u}{\partial \mathbf{n}_e} \right|_{H^1(e)} + h_e^{\frac{1}{2}} \left| \frac{\partial u}{\partial \mathbf{n}_e} \right|_{H^2(e)} \right) \right) \\ &\quad \cdot \left(h_e^{-\frac{1}{2}} \|v_h\|_{L^2(e)} + h_e^{\frac{1}{2}} |v_h|_{H^1(e)} \right) \\ &\lesssim \max_{z \in \mathcal{N}_{1,h}^B \setminus \mathcal{N}_h^V} (|[\beta_e]_z| + |[\gamma_e]_z|) \left(h^{-\frac{1}{2}} |u|_{H^2(\Gamma)} + h^{\frac{1}{2}} |u|_{H^3(\Gamma)} \right. \\ &\quad \left. + h^{-\frac{1}{2}} \left| \frac{\partial u}{\partial \mathbf{n}} \right|_{H^1(\Gamma)} + h^{\frac{1}{2}} \left| \frac{\partial u}{\partial \mathbf{n}} \right|_{H^2(\Gamma)} \right) \cdot h^{-\frac{1}{2}} \|v_h\|_{L^2(\Gamma)} \\ &\lesssim h^{1+\alpha} \left(|u|_{H^2(\Gamma)} + h |u|_{H^3(\Gamma)} + |g|_{H^1(\Gamma)} + h |g|_{H^2(\Gamma)} \right. \\ &\quad \left. + \|q\|_{W^{1,\infty}(\Gamma)} |u|_{H^1(\Gamma)} + h \|q\|_{W^{2,\infty}(\Gamma)} |u|_{H^2(\Gamma)} \right) \|v_h\|_1, \end{aligned} \quad (3.21)$$

where we have used the second inequality in (3.7).

The following inequality [17] is going to be used to estimate $I_{2,2,3} + I_{2,2,4}$:

$$\|v_h\|_{L^\infty(\Omega)} \lesssim |\log h|^{\frac{1}{2}} \|v_h\|_1 \quad \forall v_h \in V_h.$$

Noting that $\#\mathcal{N}_{2,h}^B = h^{-2\mu}$, we have

$$\begin{aligned} &I_{2,2,3} + I_{2,2,4} \\ &\lesssim h^2 \sum_{z \in \mathcal{N}_{2,h}^B \cup \mathcal{N}_h^V} \left(\left| \frac{\partial^2 u}{\partial \mathbf{t}_{e'}^2}(z) \right| + \left| \frac{\partial^2 u}{\partial \mathbf{t}_e^2}(z) \right| + \left| \frac{\partial^2 u}{\partial \mathbf{t}_{e'} \partial \mathbf{n}_{e'}}(z) \right| + \left| \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(z) \right| \right) \|v_h\|_{L^\infty(\Omega)} \\ &\lesssim h^2 (\#\mathcal{N}_{2,h}^B + \#\mathcal{N}_h^V) \left(\|u\|_{H^3(\Gamma)} + \|g\|_{H^2(\Gamma)} + \|q\|_{W^{2,\infty}(\Gamma)} \|u\|_{H^2(\Gamma)} \right) \cdot |\log h|^{\frac{1}{2}} \|v_h\|_{H^1(\Omega)} \\ &\lesssim h^{2-2\mu} |\log h|^{\frac{1}{2}} \|v_h\|_1 \left(\|u\|_{H^3(\Gamma)} + \|g\|_{H^2(\Gamma)} + \|q\|_{W^{2,\infty}(\Gamma)} \|u\|_{H^2(\Gamma)} \right). \end{aligned} \quad (3.22)$$

Using (3.14), (3.17), (3.18)–(3.22), we complete the proof of the lemma. \square

We can also obtain the supercloseness between u_h and u_I from Lemma 3.2.

Theorem 3.1. *Assume that \mathcal{M}_h satisfies the condition (α, σ, μ) . We have*

$$\|u_h - u_I\|_1 \lesssim \left(h^{1+\min(\alpha, 1-\sigma)} + \min(h^{2-2\mu} |\log h|^{\frac{1}{2}}, h^{\frac{3}{2}}) \right) C_{u,g,q}. \quad (3.23)$$

Proof. Denote $v_h = u_h - u_I$. By the Galerkin orthogonality, we have

$$\begin{aligned} \|u_h - u_I\|_1^2 &\lesssim a(u_h - u_I, v_h) \\ &= (\nabla(u - u_I), \nabla v_h) + (c(u - u_I), v_h) + \langle q(u - u_I), v_h \rangle. \end{aligned} \quad (3.24)$$

By the trace inequality,

$$\begin{aligned} |\langle q(u - u_I), v_h \rangle| &\leq \|q\|_{L^\infty(\Gamma)} \|u - u_I\|_{L^2(\Gamma)} \|v_h\|_{L^2(\Gamma)} \\ &\lesssim h^2 \|q\|_{L^\infty(\Gamma)} \|u\|_{H^2(\Gamma)} \cdot \|v_h\|_{L^2(\Gamma)}. \end{aligned} \quad (3.25)$$

By combining Lemma 3.2, (3.24), (3.25), and the fact that $\|u - u_I\|_0 \lesssim h^2 \|u\|_2$, we complete the proof. \square

Remark 3.1. Our estimate of supercloseness holds on mildly structured grids up to the boundary. In particular, under the same conditions as [3, Theorem 3.1] (i.e. $q = 0, g = 0, \mu = 0, \alpha = 1$) our Theorem 3.1 implies that

$$\|u_h - u_I\|_1 \lesssim (h^{2-\sigma} + h^2 |\log h|^{\frac{1}{2}}) (\|u\|_3 + |u|_{W^{2,\infty}(\Omega)} + |u|_{H^3(\Gamma)}).$$

When $\sigma > 0$, our convergence order is $O(h^{2-\sigma})$ since $h^\sigma |\log h|^{\frac{1}{2}} \rightarrow 0$ as $h \rightarrow 0^+$, while the order given by [3] is $O(h^{2-\sigma} |\log h|^{\frac{1}{2}})$. On the other hand, our regularity requirement on u is weaker than that of [3].

4. Superconvergence and the a Posteriori Error Estimator

In this section, we introduce and analyze the polynomial preserving recovery method (PPR), which is applied to improve the gradient of the finite element solution.

We first recall the polynomial preserving method (PPR) [13]. Define $G_h : C(\bar{\Omega}) \mapsto V_h \times V_h$ as the gradient recovery operator on the finite element space in the following way: Given a node $z \in \mathcal{N}_h$, select an element patch ω_z . We then denote all nodes on ω_z (including z) as $z_j, j = 1, 2, \dots, n (\geq 6)$ and use values of $w \in C(\bar{\Omega})$ at these sampling points to fit a quadratic polynomial in the least squares sense

$$\sum_{j=1}^n (p_2 - w)^2(z_j) = \min_{q \in P_2} \sum_{j=1}^n (q - w)^2(z_j). \quad (4.1)$$

Here $P_2(\omega_z)$ is the set of continuous, piecewise quadratic polynomial functions defined on ω_z . The value of the recovered gradient of w at z is then defined as

$$G_h w(z) = (\nabla p_2)(z). \quad (4.2)$$

The reader is referred to [13, Theorem 2.3] for a practical sufficient condition so that the above least squares fitting procedure has a unique solution. We shall use the PPR method to improve the numerical solutions.

Some properties of the PPR operator should be shown:

- (i) $\|G_h v_h\|_0 \lesssim \|\nabla v_h\|_0 \quad \forall v_h \in V_h$.
- (ii) For any nodal point z , $(G_h p)(z) = \nabla p(z)$ if $p \in P_j(\omega_z)$, $j = 1, 2$.
- (iii) $G_h w = G_h I_h^j w \quad \forall w \in C(\bar{\Omega}), j = 1, 2$.

Here $I_h^1 w$ and $I_h^2 w$ are the linear nodal value interpolant and quadratic nodal value interpolant of w , respectively. The reader is referred to [13, 16, 21, 23] for more details of these properties.

We decompose the error into

$$\nabla u - G_h u_h = \nabla u - G_h u + G_h(u_I - u_h). \quad (4.3)$$

Note that $G_h u = G_h u_I$ since $u_I = u$ at all vertices and the recovery operator G_h is completely determined by nodal values of u . From (i), we have $\|G_h(u_I - u_h)\|_0 \lesssim \|\nabla(u_I - u_h)\|_0$. It remains to estimate $\|\nabla u - G_h u\|_0$.

Lemma 4.1. *For any element $\tau \in \mathcal{M}_h$ and any function $w \in H^3(\tilde{\tau})$,*

$$\|G_h w - \nabla w\|_{L^2(\tau)} \lesssim h^2 \|w\|_{H^3(\tilde{\tau})}, \quad (4.4)$$

where $\tilde{\tau} = \bigcup \{\omega_z : z \in \mathcal{N}_h \cap \tau\}$.

Proof. The property (iii) implies

$$\|G_h w - \nabla w\|_{L^2(\tau)} = \|G_h I_h^2 w - \nabla w\|_{L^2(\tau)}. \quad (4.5)$$

For any $\eta \in P_2(\tilde{\tau})$, it is clear that $G_h \eta = \nabla \eta$ in τ by the property (ii) and the fact that $G_h \eta \in V_h \times V_h$ and $\nabla \eta \in P_1(\tilde{\tau}) \times P_1(\tilde{\tau})$. Then we have

$$\begin{aligned} \|G_h I_h^2 w - \nabla w\|_{L^2(\tau)} &= \|G_h(I_h^2 w - \eta) - \nabla(w - \eta)\|_{L^2(\tau)} \\ &\leq \|G_h(I_h^2 w - \eta)\|_{L^2(\tau)} + \|\nabla(w - \eta)\|_{L^2(\tau)}. \end{aligned} \quad (4.6)$$

By the definition and properties of G_h ,

$$\begin{aligned} &\|G_h(I_h^2 w - \eta)\|_{L^2(\tau)} \\ &\lesssim h \max_{z \in \mathcal{N}_h \cap \tau} |G_h(I_h^2 w - \eta)(z)| \lesssim h \|\nabla(I_h^2 w - \eta)\|_{L^\infty(\tilde{\tau})} \\ &\lesssim \|\nabla(I_h^2 w - \eta)\|_{L^2(\tilde{\tau})} \lesssim \|\nabla(I_h^2 w - w)\|_{L^2(\tilde{\tau})} + \|\nabla(w - \eta)\|_{L^2(\tilde{\tau})}. \end{aligned} \quad (4.7)$$

By combining (4.5)–(4.7), we have

$$\|G_h I_h^2 w - \nabla w\|_{L^2(\tau)} \lesssim \inf_{\eta \in P_2(\tilde{\tau})} \|\nabla(w - \eta)\|_{L^2(\tilde{\tau})} + \|\nabla(I_h^2 w - w)\|_{L^2(\tilde{\tau})}. \quad (4.8)$$

Using the Bramble–Hilbert lemma and the scaling argument, we have

$$\inf_{\eta \in P_2(\tilde{\tau})} \|\nabla(w - \eta)\|_{L^2(\tilde{\tau})} \lesssim h^2 \|w\|_{H^3(\tilde{\tau})}, \quad (4.9)$$

and from the approximation theory

$$\|\nabla(I_h^2 w - w)\|_{L^2(\tilde{\tau})} \lesssim h^2 \|w\|_{H^3(\tilde{\tau})}. \quad (4.10)$$

The proof is completed by combining (4.5) with (4.8)–(4.10). \square

We remark that similar results as the above lemma are proved in [16, 18]. Using Lemma 4.1, we obtain the following superconvergence property of G_h in $H^3(\Omega)$ space for the linear element.

Theorem 4.1. *We have the following estimate:*

$$\|G_h u - \nabla u\|_0 \lesssim h^2 \|u\|_3. \quad (4.11)$$

Proof. From Lemma 4.1, we have

$$\|G_h u - \nabla u\|_0 = \left(\sum_{\tau \in \mathcal{M}_h} \|G_h u - \nabla u\|_{L^2(\tau)}^2 \right)^{\frac{1}{2}} \lesssim h^2 \left(\sum_{\tau \in \mathcal{M}_h} \|u\|_{H^3(\tau)}^2 \right)^{\frac{1}{2}} \lesssim h^2 \|u\|_3.$$

We obtain the superconvergence property of the linear FE solution post-processed by the PPR operator, which can be proved by combining (4.3), Theorems 3.1 and 4.1. \square

Theorem 4.2. *Assume that \mathcal{M}_h satisfies condition (α, σ, μ) . Let u and u_h be the solutions to (1.1) and (1.2), respectively. We have*

$$\|G_h u_h - \nabla u\|_0 \lesssim \left(h^{1+\min(\alpha, 1-\sigma)} + \min(h^{2-2\mu} |\log h|^{\frac{1}{2}}, h^{\frac{3}{2}}) \right) C_{u,g,q}. \tag{4.12}$$

With preparation from the previous work, it is now straightforward to prove the asymptotic exactness of an error estimator based on the recovery operator G_h . The global error estimator is naturally defined by

$$\eta_h = \|G_h u_h - \nabla u_h\|_0. \tag{4.13}$$

Theorem 4.3. *Let u_h be the linear finite element approximation of u . Assume that \mathcal{M}_h satisfies Condition (α, σ, μ) . Furthermore, assume that $h \lesssim \|\nabla(u - u_h)\|_0$. Then*

$$\left| \frac{\eta_h}{\|\nabla(u - u_h)\|_0} - 1 \right| \lesssim h^\rho + \min(h^{1-2\mu} |\log h|^{\frac{1}{2}}, h^{\frac{1}{2}}), \quad \rho = \min(\alpha, 1 - \sigma). \tag{4.14}$$

5. Numerical Examples

In this section we present experiments to study the supercloseness and superconvergence rates on different meshes in H^1 -seminorms as well as the asymptotically exactness of the a posteriori error estimator based on the recovery operator G_h .

we solve $-\Delta u + u = f$ with the Robin boundary condition $\frac{\partial u}{\partial \mathbf{n}} + u = g$, where f and g are chosen such that the solution is $u = \sin(\pi(x + y))$.

We consider the supercloseness property on the domain $\bar{\Omega} = [0, 1] \times [0, 1]$. We first consider discretizations performed on meshes consisting of $O(h^{1+\alpha})$ approximate parallelograms. We divide $\bar{\Omega}$ into $\frac{n}{2} \times n$ equivalent rectangles which containing four types of triangles (cf. Fig. 5.1), where $h = 1/n$ and $2\tilde{h} + \tilde{h}^{1+\alpha} = 2h$, for even integer n . Clearly, $\frac{2}{3}h \leq \tilde{h} \leq h$. Fig. 5.2

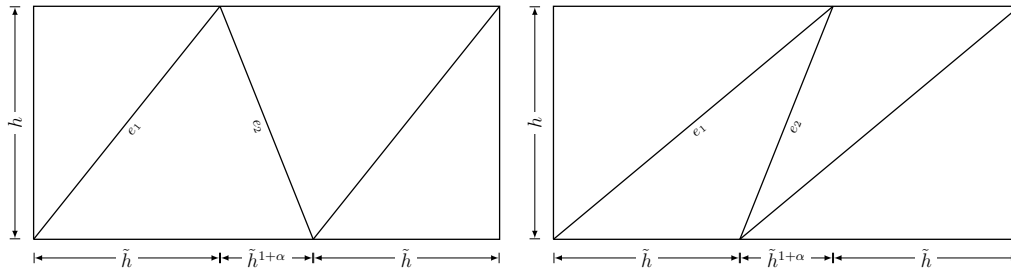


Fig. 5.1. The cells constituting the finite element grids.

plots the meshes for $n = 4, 32$ and $\alpha = 0, \frac{1}{2}$, respectively. For any $e \in \mathcal{E}_h^I$, denote by $\epsilon_e = |h_{e-1} - h'_{e-1}| + |h_{e+1} - h'_{e+1}|$. For e_1 in Fig. 5.1, we have

$$\begin{aligned} \left(\frac{2}{3}h\right)^{1+\alpha} &\leq \epsilon_{e_1} = \tilde{h}^{1+\alpha} + (h^2 + \tilde{h}^{2+2\alpha})^{\frac{1}{2}} - h = \tilde{h}^{1+\alpha} + \frac{\tilde{h}^{2+2\alpha}}{(h^2 + \tilde{h}^{2+2\alpha})^{\frac{1}{2}} + h} \\ &\leq \tilde{h}^{1+\alpha} + \frac{\tilde{h}^{2+2\alpha}}{2h} \leq \frac{3}{2}\tilde{h}^{1+\alpha} \leq \frac{3}{2}h^{1+\alpha}. \end{aligned}$$

Moreover, it is clear that $\epsilon_{e_2} = 0$ for e_2 in Fig. 5.1. As a matter of fact, it can be shown that for every $e \in \mathcal{E}_h^I$, Ω_e is an $O(h^{1+\alpha})$ approximate parallelogram. Similarly, for any $z \in \mathcal{N}_h^B \setminus \mathcal{N}_h^V$ shared by $e, e' \in \mathcal{E}_h^B$, the triangles τ_e and $\tau_{e'}$ are $O(h^{1+\alpha})$ approximate congruent triangles. That is, the meshes satisfy Condition $(\alpha, 0, 0)$.

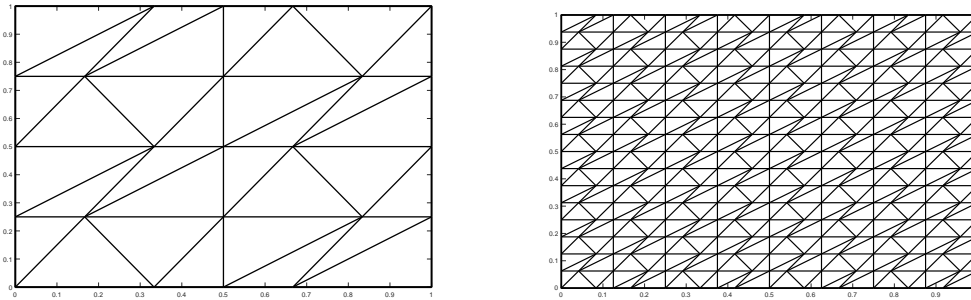


Fig. 5.2. The meshes of $\Omega = [0, 1] \times [0, 1]$ consisting of $O(h)$ approximate parallelograms for $n = 4$ (left graph) and $n = 16$ (right graph).

We plot the errors between u_h and u_I on H^1 -seminorms for $\alpha = 0, 1/4, 1/2$ and $\alpha = 1/3, 2/3, 1$ in Fig. 5.3. The convergence rates of $\|\nabla u_h - \nabla u_I\|_0$ behave much as expected. The convergence rates are almost equal to the corresponding $1 + \alpha$. Such behavior shows the necessity of the condition (i) in Definition 2.1 of Condition (α, σ, μ) .

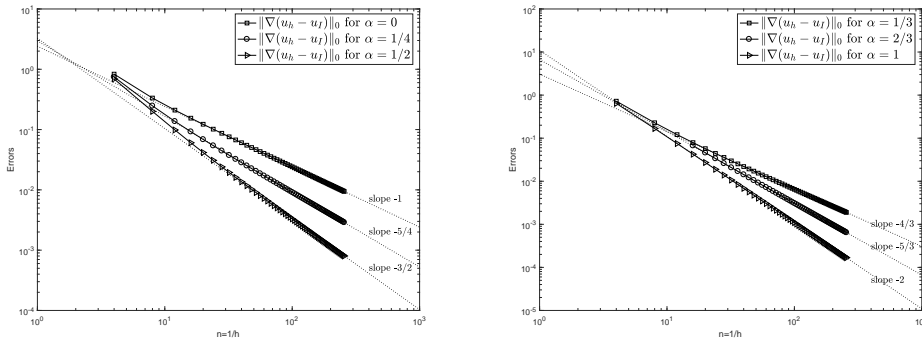


Fig. 5.3. The errors of $\|\nabla u_h - \nabla u_I\|_0$ for $\alpha = 0, 1/4, 1/2$ (left graph) and $\alpha = 1/3, 2/3, 1$ (right graph). The dotted lines indicate reference slopes.

Next, we consider the meshes containing ‘bad’ edges, that is $O(h^{1+\alpha})$ approximate parallelogram property is satisfied for pairs of adjacent triangles in most parts except for a region

of size $h^{2\sigma}$. For given $0 \leq \sigma < 1$ and non-negative integer n , we divide $\bar{\Omega}$ into two parts, $\Omega_1 = [0, mh] \times [0, mh]$ and $\Omega_2 = \Omega \setminus \Omega_1$, where $h = 1/n$ and the integer m is chosen such that the area of Ω_1 is about $h^{2\sigma}$. We discrete Ω_1 as the Criss-cross pattern uniform mesh and discrete Ω_2 as the regular pattern uniform mesh. Fig. 5.4 plots the meshes by setting $\sigma = 1/2$ for $n = 8$ and $n = 16$. Clearly $\alpha \geq 1$. It is well known that the Criss-cross pattern uniform mesh is very bad for the supercloseness property. Therefore, the convergence order $h^{2-\sigma}$ of $\|\nabla u_h - \nabla u_I\|_0$ can be expected. This is verified by Fig. 5.5 which plots the errors in H^1 -seminorms for $\sigma = 0, 1/2, 1$ and $\sigma = 1/3, 2/3$.

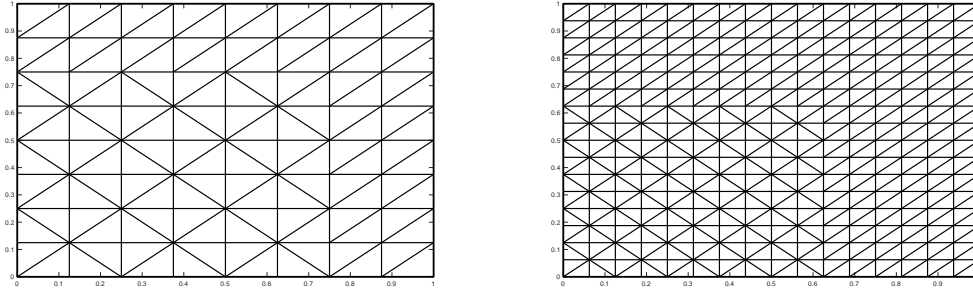


Fig. 5.4. The meshes of $\Omega = [0, 1] \times [0, 1]$ constructed by setting $\sigma = 1/2$ for $n = 8$ (left graph) and $n = 16$ (right graph).

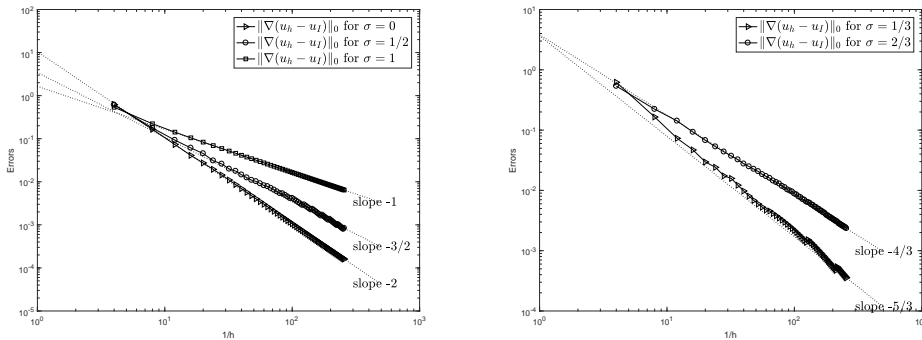


Fig. 5.5. The errors of $\|\nabla u_h - \nabla u_I\|_0$ for $\sigma = 0, 1/2, 1$ (left graph) and $\sigma = 1/3, 2/3$ (right graph). The dotted lines indicate reference slopes.

Then we construct meshes by perturbing the boundary elements of the regular uniform meshes such that two boundary triangles τ_e and $\tau_{e'}$ sharing a boundary node are $O(h)$ approximate congruent triangles (cf. (2.2)) and $\#\mathcal{N}_{2,h}^B = O(h^{-2\mu})$. Clearly, $\sigma = \mu$. Fig. 5.6 plots the perturbed mesh for $n = 15$ and $\mu = 1/4$ in the left graph and the errors of $\|\nabla u_h - \nabla u_I\|_0$ for $\mu = 1/2, 1/3, 1/4, 0$ in the right graph. The convergence order is about $O(h^{2-\mu})$ which is better than our theoretical result in Theorem 3.1.

Then we consider the superconvergence of the numerical solution based on the Polynomial Preserving Recovery technique. Fig. 5.7 plots the errors of $\|\nabla u - \nabla u_h\|_0$, $\|\nabla u_h - \nabla u_I\|_0$ and

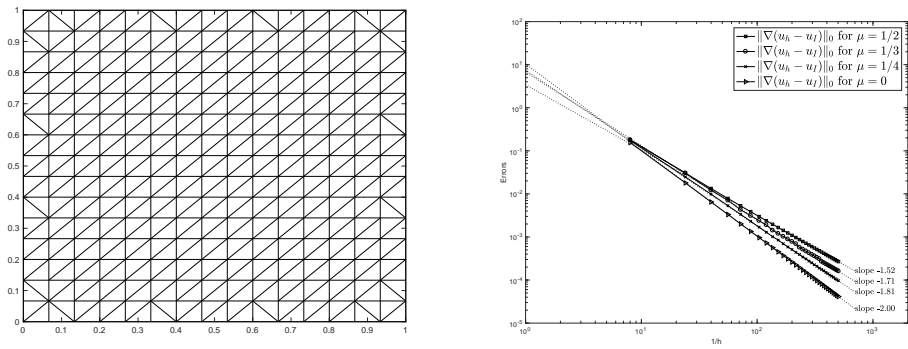


Fig. 5.6. The mesh by perturbing the regular uniform mesh for $n = 15$ and $\mu = 1/4$ (left graph) and the errors of $\|\nabla u_h - \nabla u_I\|_0$ for $\mu = 1/2, 1/3, 1/4, 0$ (right graph). The dotted lines indicate reference slopes.

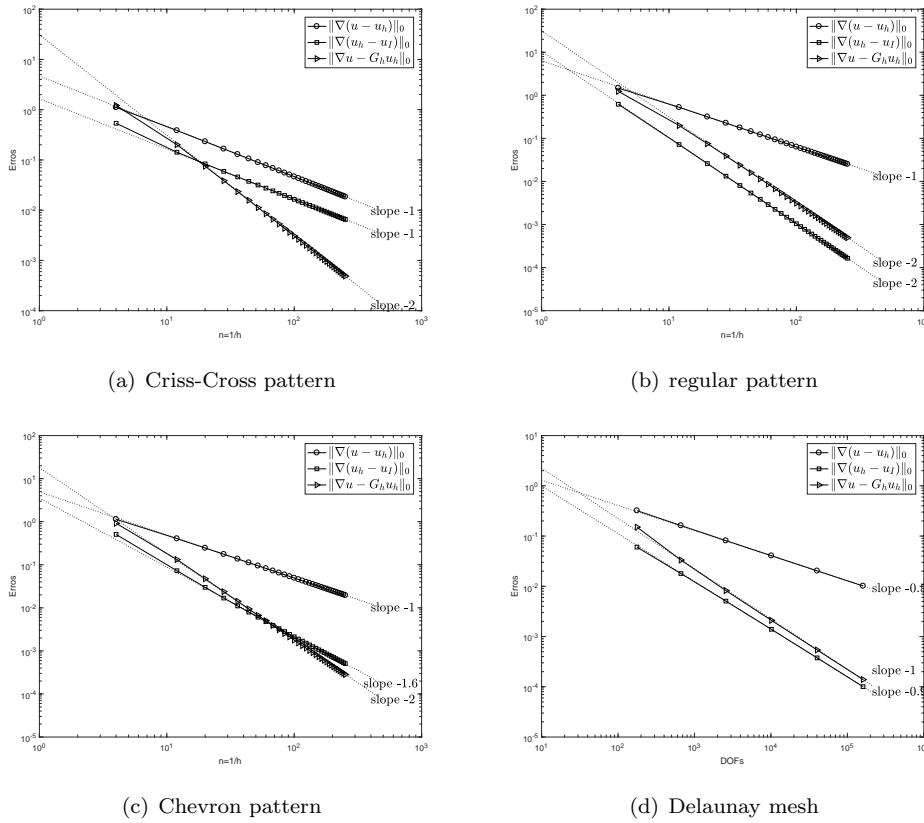


Fig. 5.7. The errors of $\|\nabla u - \nabla u_h\|_0$, $\|\nabla u_h - \nabla u_I\|_0$ and $\|\nabla u - G_h u_h\|_0$ for Criss-cross pattern uniform mesh (a), regular pattern uniform mesh (b), Chevron pattern uniform mesh (c) and the Delaunay mesh (d).

$\|\nabla u - G_h u_h\|_0$ for Criss-cross pattern uniform mesh, regular pattern uniform mesh, Chevron pattern uniform mesh and the Delaunay mesh. We see that the convergence order of $\|\nabla u - G_h u_h\|_0$ is almost $O(h^2)$ although the convergence order of $\|\nabla u_h - \nabla u_I\|_0$ is smaller except for the regular pattern uniform mesh, which shows that the PPR is a powerful tool to improve the gradients of the finite element solutions.

Finally, we verify the asymptotical exactness of the a posteriori error estimator η_h (cf. (4.13)). Table 5.1 shows the errors of u_h in H^1 -seminorms and the a posteriori error estimators on three kinds of meshes for different mesh sizes. We watch the a posteriori error estimators converge to the corresponding $\|\nabla u - \nabla u_h\|_0$ very quickly.

Table 5.1: The errors of $\|\nabla u - \nabla u_h\|_0$ and the a posteriori error estimators on Criss-cross pattern uniform mesh, regular pattern uniform mesh, Chevron pattern uniform mesh for different mesh sizes.

1/h	regular pattern		Chevron pattern		Criss-cross pattern	
	$ u - u_h _1$	η_h	$ u - u_h _1$	η_h	$ u - u_h _1$	η_h
4	1.47E + 00	1.28E + 00	1.14E + 00	1.10E + 00	1.10E + 00	1.28E + 00
12	5.26E - 01	5.24E - 01	4.05E - 01	4.06E - 01	3.86E - 01	4.09E - 01
20	3.18E - 01	3.17E - 01	2.45E - 01	2.45E - 01	2.32E - 01	2.38E - 01
28	2.27E - 01	2.27E - 01	1.76E - 01	1.75E - 01	1.66E - 01	1.68E - 01
36	1.77E - 01	1.77E - 01	1.37E - 01	1.37E - 01	1.29E - 01	1.30E - 01
44	1.44E - 01	1.44E - 01	1.12E - 01	1.12E - 01	1.06E - 01	1.06E - 01
52	1.23E - 01	1.23E - 01	9.47E - 02	9.47E - 02	8.95E - 02	8.98E - 02
60	1.06E - 01	1.06E - 01	8.21E - 02	8.21E - 02	7.75E - 02	7.77E - 02
68	9.37E - 02	9.37E - 02	7.25E - 02	7.25E - 02	6.84E - 02	6.85E - 02
76	8.38E - 02	8.38E - 02	6.49E - 02	6.48E - 02	6.12E - 02	6.13E - 02
84	7.58E - 02	7.58E - 02	5.87E - 02	5.87E - 02	5.54E - 02	5.55E - 02
92	6.92E - 02	6.92E - 02	5.36E - 02	5.36E - 02	5.06E - 02	5.06E - 02

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