Journal of Computational Mathematics
Vol.38, No.6, 2020, 879–902.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1906-m2018-0131

TWO-VARIABLE JACOBI POLYNOMIALS FOR SOLVING SOME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

Jafar Biazar[1)] and Khadijeh Sadri
Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O. Box 41335-1914, Guilan, Rasht, Iran
E-mail: biazar@guilan.ac.ir, jafar.biazar@gmail.com, kh.sadri@uma.ac.ir

Abstract

Two-variable Jacobi polynomials, as a two-dimensional basis, are applied to solve a class of temporal fractional partial differential equations. The fractional derivative operators are in the Caputo sense. The operational matrices of the integration of integer and fractional orders are presented. Using these matrices together with the Tau Jacobi method converts the main problem into the corresponding system of algebraic equations. An error bound is obtained in a two-dimensional Jacobi-weighted Sobolev space. Finally, the efficiency of the proposed method is demonstrated by implementing the algorithm to several illustrative examples. Results will be compared with those obtained from some existing methods.

Mathematics subject classification: 35R11, 65M15, 65M70.
Key words: Fractional partial differential equation, Two-variable Jacobi polynomials, Caputo derivative, Error bound.

1. Introduction

Fractional partial differential equations (FPDEs) are used as modeling tools of various phenomena in different branches of science. For example, diffusive processes associated with subdiffusion (fractional in time), super-diffusion (fractional in space), or both, advection-diffusion, and convection-diffusion processes can be modeled by FPDEs [1–5]. The advantage of these equations in compared to integer-order partial differential equations is the ability of natural simulation of physical processes and dynamical systems more accurately [6]. For instance, some phenomena in fluid and continuum mechanics [7], viscoplastic and viscoelastic flows [8], biology, and acoustics [9], describing chemical and pollute transport in heterogeneous aquifers [10–12], pricing mechanisms and heavy stochastic processes in finance [13], and describing convection process of liquid in medium [14]. Therefore, it helps mathematicians and engineers in the better understanding of the nature and behavior of physical phenomena. For this reason, FPDEs are increasingly studied, but their analytic solving is difficult. Hence, mathematicians have been attracted to solve this class of equations numerically. For example, in [14], the normalized and rational Bernstein polynomials are applied to solve a kind of time-space fractional diffusive equation. The finite difference method is used to solve the fractional reaction-subdiffusion equation in [15]. Authors in [16] propose a wavelet method to solve a class of fractional convection-diffusion equation with variable coefficients. Chen and et al. use generalized fractional-order Legendre functions to obtain numerical solutions of FPDEs with variable coefficients [17]. Ding

* Received June 25, 2018 / Revised version received February 21, 2019 / Accepted June 20, 2019 / Published online July 29, 2019 /
1) Corresponding author

introduces a general Pade approximation method for time-space fractional diffusive equations in [18]. Also, Heydari and et al. apply the Legendre wavelet method for solving the time fractional diffusion-wave equation [19]. In [20], a two-dimensional wavelets collocation method uses to solve electromagnetic waves in dielectric media.

In this paper, an operational Tau method, based on two-variable Jacobi polynomials (TVJP-s), is proposed to deal with a class of FPDEs which involves equations such as diffusion and advection-diffusion equations. The derivative operators appeared in these equations are in the Caputo sense. First, the TVJPs, on the domain $\Omega = [0,1] \times [0,1]$, are obtained as a generalization of the classic one-variable Jacobi polynomials (OVJPs) on the interval $\Omega_0 = [0,1]$. A given continuous function $u(x,t)$, defined on Ω, can be approximated in terms of the two-variable presented basis. In order to approximate the terms including the derivative operators in the equation under study, the operational matrices of the integration of fractional and integer orders are derived for the one-variable Jacobi basis, then the resultant matrices are applied to construct the two-dimensional integral operational matrices for both two independent variables x and t. Applying these matrices together with the Tau method leads to reduce the given equation to the corresponding system of the algebraic equations which is a Sylvester equation. Solving the resulting system leads to determine the vector of unknown coefficients, therefore, an approximate solution is obtained. Also, the convergence of the proposed approach is investigated in a two-dimensional Jacobi-weighted Sobolev space and an error bound is computed for an approximate solution. Finally, the suggested algorithm is implemented to several illustrative examples.

The outline of the paper is as follows: Section 2 gives some elementary definitions and concepts of the fractional calculus. In Section 3, the TVJPs are constructed with help of the OVJPs. The integral operational matrices of fractional and integer orders are derived in Section 4, which are used to construct the operational matrices corresponding to the fractional partial derivative operators. In Section 5, an error bound is given in a two-dimensional Sobolev space. The applicability and efficiency of the proposed approach are demonstrated by implementing the method on several illustrative examples in Section 6. Finally, a conclusion is presented in Section 7.

2. Elementary Definitions of Fractional Calculus

The two most used fractional operators are the Caputo derivative and the Riemann-Liouville integral operators.

Definition 2.1. *If $\gamma \in \mathbb{R}$ and $n = \lceil \gamma \rceil$, the Caputo derivative operator is defined as,*

$$\begin{aligned} D^{\gamma} u(t) &= \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1}\, u(s)\, ds, \qquad t \in \Omega_0, \\ D^0 u(t) &= u(t). \end{aligned} \tag{2.1}$$

Definition 2.2. *If $\nu \in \mathbb{R}$, the Riemann-Liouville integral operator is defined as,*

$$\begin{aligned} J^{\nu} u(t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)\, ds, \qquad t \in \Omega_0, \\ J^0 u(t) &= u(t). \end{aligned} \tag{2.2}$$

These two operators satisfy the following properties.

1. $J^{\nu_1} J^{\nu_2} u(t) = J^{\nu_1+\nu_2} u(t)$,
2. $J^{\nu}(\lambda_1\ u_1(t) + \lambda_2\ u_2(t)) = \lambda_1\ J^{\nu} u_1(t) + \lambda_2\ J^{\nu} u_2(t)$,
3. $J^{\nu} t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\nu+\gamma+1)} t^{\nu+\gamma}, \qquad \gamma > -1$,
4. $D^{\nu} J^{\nu} u(t) = u(t)$,
5. $J^{\nu} D^{\nu} u(t) = u(t) - \displaystyle\sum_{k=0}^{m-1} \frac{u^{(k)}(0^+)}{k!} t^k, \quad \lceil \nu \rceil = m.$

3. Shifted Jacobi Polynomials

In this section, the shifted Jacobi polynomials, over the interval $\Omega_0 = [0,1]$, are introduced. Then, the two-variable Jacobi polynomials are derived from the extension of them.

3.1. One-variable shifted Jacobi polynomials

These polynomials can be obtained from the following recursive relation,

$$\begin{aligned} P_{i+1}^{(\alpha,\beta)}(t) =& A(\alpha,\beta,i)\ P_i^{(\alpha,\beta)}(t) + (2t-1)\ B(\alpha,\beta,i)\ P_i^{(\alpha,\beta)}(t) \\ & - E(\alpha,\beta,i)\ P_{i-1}^{(\alpha,\beta)}(t), \qquad i = 1,2,\cdots, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} A(\alpha,\beta,i) &= \frac{(2i+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}, \\ B(\alpha,\beta,i) &= \frac{(2i+\alpha+\beta+2)(2i+\alpha+\beta+1)}{2(i+1)(i+\alpha+\beta+1)}, \\ E(\alpha,\beta,i) &= \frac{(i+\alpha)(i+\beta)(2i+\alpha+\beta+2)}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}. \end{aligned}$$

The initial values are as,

$$P_0^{(\alpha,\beta)}(t) = 1, \qquad P_1^{(\alpha,\beta)}(t) = \frac{\alpha+\beta+2}{2}(2t-1) + \frac{\alpha-\beta}{2}.$$

These polynomials are orthogonal related to the weight function $w^{(\alpha,\beta)}(t) = t^{\beta}(1-t)^{\alpha}$, that is,

$$\int_0^1 P_i^{(\alpha,\beta)}(t)\ P_j^{(\alpha,\beta)}(t)\ w^{(\alpha,\beta)}(t)\ dt = h_i^{\alpha,\beta}\ \delta_{ij}, \qquad i = 0,1,2,\cdots,$$

where

$$h_i^{\alpha,\beta} = \frac{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\ i!\ \Gamma(i+\alpha+\beta+1)}, \tag{3.2}$$

and δ_{ij} denotes the Kronecker function. The shifted Jacobi polynomials can also be obtained from the following series,

$$P_i^{(\alpha,\beta)}(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)\ t^k}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\ (i-k)!\ k!}, \qquad i = 0,1,2,\cdots. \tag{3.3}$$

The representation (3.3) is used to achieve integral operational matrices. We will refer to it in Sections 3 and 4. A square integrable function $y(t)$ with respect to $w^{(\alpha,\beta)}(t)$, in the interval $\Omega_0 = [0,1]$, can be expanded in terms of the shifted Jacobi polynomials as follows.

$$y(t) = \sum_{j=0}^{\infty} c_j \; P_j^{(\alpha,\beta)}(t), \tag{3.4}$$

where the coefficients c_j are given by,

$$c_j = \frac{1}{h_j^{\alpha,\beta}} \int_0^1 y(t) \; P_j^{(\alpha,\beta)}(t) \; w^{(\alpha,\beta)}(t) \; dt, \qquad j = 0,1,2,\cdots .$$

Indeed, only the first $(N+1)-$terms of the shifted Jacobi polynomials are applied to expand a given continuous function. On the other hand, one has,

$$y_N(t) = \sum_{j=0}^{N} c_j \; P_j^{(\alpha,\beta)}(t) = \Phi^T(t) \; \tilde{\mathbf{c}} = \tilde{\mathbf{c}}^T \; \Phi(t), \tag{3.5}$$

where the vectors $\tilde{\mathbf{c}}$ and $\Phi(t)$ are given by,

$$\tilde{\mathbf{c}} = [c_0, c_1, \cdots, c_N]^T, \qquad \Phi(t) = [P_0^{(\alpha,\beta)}(t), P_1^{(\alpha,\beta)}(t), \cdots, P_N^{(\alpha,\beta)}(t)]^T. \tag{3.6}$$

The shifted Jacobi polynomials satisfy the following relations.

$$(i) \; P_i^{(\alpha,\beta)}(0) = (-1)^i \begin{pmatrix} i+\beta \\ i \end{pmatrix}, \tag{3.7a}$$

$$(ii) \; \frac{d^i P_n^{(\alpha,\beta)}(t)}{dt^i} = \frac{\Gamma(n+i+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-i}^{(\alpha+i,\beta+i)}(t), \quad i = 0,1,\cdots . \tag{3.7b}$$

3.2. Two-variable shifted Jacobi polynomials

Two-variable shifted Jacobi polynomials, $P_{i,j}^{(\alpha,\beta)}(x,t)$, are defined on the domain $\Omega = [0,1] \times [0,1]$ as follows,

$$P_{i,j}^{(\alpha,\beta)}(x,t) = P_i^{(\alpha,\beta)}(x) \; P_j^{(\alpha,\beta)}(t), \quad i,j = 0,1,2,\cdots, \quad (x,t) \in \Omega.$$

It is easily seen that these polynomials are orthogonal with weight function $W^{(\alpha,\beta)}(x,t) = w^{(\alpha,\beta)}(x) \; w^{(\alpha,\beta)}(t)$ on Ω, [21]. That is,

$$\begin{aligned} &\int_0^1 \int_0^1 P_{i,j}^{(\alpha,\beta)}(x,t) \; P_{k,l}^{(\alpha,\beta)}(x,t) \; W^{(\alpha,\beta)}(x,t) \; dx \; dt \\ &= \int_0^1 P_i^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) \; dx \times \int_0^1 P_j^{(\alpha,\beta)}(t) P_l^{(\alpha,\beta)}(t) \; w^{(\alpha,\beta)}(t) \; dt \\ &= \begin{cases} h_i^{\alpha,\beta} \; h_j^{\alpha,\beta}, & (i,j) = (k,l), \\ 0, & i \neq k \;\; or \;\; j \neq l. \end{cases} \end{aligned}$$

A two variables continuous function $y(x,t)$, defined over Ω, may be expanded by the TVJPs as

$$y(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{ij} \; P_{i,j}^{(\alpha,\beta)}(x,t), \tag{3.8}$$

where the coefficients C_{ij} are computed as follows,

$$C_{ij} = \frac{1}{h_i^{\alpha,\beta}\ h_j^{\alpha,\beta}} \int_0^1 \int_0^1 y(x,t)\ P_{i,j}^{(\alpha,\beta)}(x,t)\ W^{(\alpha,\beta)}(x,t)\ dx\ dt.$$

A finite presentation of $y(x,t)$, base on the infinite series (3.8), can be presented as the following,

$$y(x,t) \simeq y_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} C_{ij}\ P_{i,j}^{(\alpha,\beta)}(x,t) = \mathbf{\Phi}^T(x,t)\ C, \tag{3.9}$$

where C and $\mathbf{\Phi}(x,t)$ are the Jacobi coefficients and the TVJPs vectors, respectively,

$$C = [C_{00}, C_{01}, \cdots, C_{0M}, C_{10}, C_{11}, \cdots, C_{1M}, \cdots, C_{N0}, C_{N1}, ..., C_{NM}]^T, \tag{3.10a}$$

$$\begin{aligned}\mathbf{\Phi}(x,t) = [&P_{0,0}^{(\alpha,\beta)}(x,t), P_{0,1}^{(\alpha,\beta)}(x,t), \cdots, P_{0,M}^{(\alpha,\beta)}(x,t), P_{1,0}^{(\alpha,\beta)}(x,t), P_{1,1}^{(\alpha,\beta)}(x,t), \cdots,\\ &P_{1,M}^{(\alpha,\beta)}(x,t), \cdots, P_{N,0}^{(\alpha,\beta)}(x,t), P_{N,1}^{(\alpha,\beta)}(x,t), \cdots, P_{N,M}^{(\alpha,\beta)}(x,t)]^T.\end{aligned} \tag{3.10b}$$

4. Jacobi Operational Matrices of Integration

In this section, first, the operational matrix of the integration of order ν is derived. Then, the operational matrices of the integration of fractional and integer orders, corresponding to both two independent variables x and t, are constructed.

Lemma 4.1. *If $i \in \mathbb{N}$ and $l \geq i$, then one has,*

$$\begin{aligned}&\int_0^1 t^l\ P_i^{(\alpha,\beta)}(t)\ w^{(\alpha,\beta)}(t)\ dt\\ =&\sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ \Gamma(l+k+\beta+1)\ \Gamma(\alpha+1)}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ \Gamma(l+k+\alpha+\beta+2)\ (i-k)!\ k!}.\end{aligned}$$

See [21] for proof (Lemma 2.2).

Theorem 4.1. *Let $\Phi(t)$ be the Jacobi basis vector in Eq.* (3.8) *and $\nu \in \mathbb{R}$. The fractional integral of order ν of $\Phi(t)$ can be expressed as,*

$$J^\nu \Phi(t) \simeq \mathbf{P}^{(\nu)}\ \Phi(t),$$

where J^ν is the Riemann-Liouville fractional integral operator of order ν and $\mathbf{P}^{(\nu)}$ is the $(M+1)\times(M+1)$ fractional operational matrix of the integration and is defined by,

$$\mathbf{P}^{(\nu)} = \begin{bmatrix} \theta(0,0) & \theta(0,1) & \dots & \theta(0,M)\\ \theta(1,0) & \theta(1,1) & \dots & \theta(1,M)\\ \vdots & \vdots & \ddots & \vdots\\ \theta(M,0) & \theta(M,1) & \dots & \theta(M,M)\end{bmatrix},$$

where

$$\theta(i,j) = \sum_{k=0}^{i} \omega'_{ijk}, \qquad i = 0,1,\cdots,M,\ \ j = 1,\cdots,M, \tag{4.1}$$

and ω'_{ijk} is given by, for $0 \le i, j \le M$,

$$\omega'_{ijk} = \frac{(-1)^{i-k}\ \Gamma(j+\beta+1)\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ \Gamma(\alpha+1)}{h_j\ \Gamma(j+\alpha+\beta+1)\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ \Gamma(k+\nu+1)\ (i-k)!} \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\ \Gamma(j+l+\alpha+\beta+1)\ \Gamma(l+k+\nu+\beta+1)}{\Gamma(l+\beta+1)\ \Gamma(l+k+\nu+\alpha+\beta+2)\ l!(j-l)!}.$$

Proof. Applying fractional integral operator (2.2) to series (3.3) leads to

$$J^{\nu} P_i^{(\alpha,\beta)}(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ t^{k+\nu}}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ \Gamma(k+\nu+1)\ (i-k)!}. \tag{4.2}$$

$t^{k+\nu}$ can be approximated in terms of the shifted Jacobi polynomials as the following,

$$t^{k+\nu} \simeq \sum_{j=0}^{N} \rho_{k,j}\ P_j^{(\alpha,\beta)}(t),$$

where

$$\rho_{k,j} = \frac{1}{h_j} \int_0^1 t^{k+\nu}\ P_j^{(\alpha,\beta)}(t)\ w^{(\alpha,\beta)}(t)\ dt.$$

According to Lemma 4.1, relation (4.2) can be rewritten as

$$\begin{aligned} J^{\nu} P_i^{(\alpha,\beta)}(t) &\simeq \sum_{j=0}^{N} \left\{ \sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(j+\beta+1)\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ \Gamma(\alpha+1)}{h_j\ \Gamma(j+\alpha+\beta+1)\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ \Gamma(k+\nu+1)\ (i-k)!} \right. \\ &\quad \left. \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\ \Gamma(j+l+\alpha+\beta+1)\ \Gamma(l+k+\nu+\beta+1)}{\Gamma(l+\beta+1)\ \Gamma(l+k+\nu+\alpha+\beta+2)\ l!(j-l)!} \right\} P_j^{(\alpha,\beta)}(t) \\ &= \sum_{j=0}^{N} \theta(i,j)\ P_j^{(\alpha,\beta)}(t), \end{aligned}$$

where $\theta(i,j)$ is given in (4.1). This leads to the desired result. □

Corollary 4.1. *For $\nu = 1$ in Theorem 4.1, the operational matrix of the integration of the integer order is achieved which is denoted by $\mathbf{P}$ throughout this paper.*

Definition 4.1. *The Kronecker product of the given matrices $A = (a)_{ij}$ and $B = (b)_{kl}$ is defined as follows:*

$$(A \otimes B)_{ij} = a_{ij}\ B, \qquad i = 1, \cdots, m, \quad j = 1, \cdots, n.$$

Theorem 4.2. *If $M = N$, $\mathbf{P}^{(\nu)}$ and $\mathbf{\Phi}(x,t)$ are the operational matrix of the integration and the two-variable basis vector, introduced by Theorem 4.1 and Eq. (3.10), respectively, then the operational matrices of the integration with respect to x and t are defined as follows,*

$$J_x^{\nu} \mathbf{\Phi}(x,t) \simeq \mathbf{P}_{(\mathbf{x})}^{(\nu)} \mathbf{\Phi}(x,t) = (\mathbf{P}^{(\nu)} \otimes I) \mathbf{\Phi}(x,t), \tag{4.3a}$$

$$J_t^{\nu} \mathbf{\Phi}(x,t) \simeq \mathbf{P}_{(\mathbf{t})}^{(\nu)} \mathbf{\Phi}(x,t) = (I \otimes \mathbf{P}^{(\nu)}) \mathbf{\Phi}(x,t), \tag{4.3b}$$

where $\mathbf{P}^{(\nu)}_{(\mathbf{x})}$ *and* $\mathbf{P}^{(\nu)}_{(\mathbf{t})}$ *are the* $(N+1)^2 \times (N+1)^2$ *operational matrices of the integraion of the fractional order corresponding to the variables* x *and* t *respectively,* I *is the* $(N+1)\times(N+1)$ *identity matrix, and the fractional integral operators* J^{ν}_x *and* J^{ν}_t *are as,*

$$\begin{aligned} J^{\nu}_x y(x,t) &= \frac{1}{\Gamma(\nu)}\int_0^x (x-\xi)^{\nu-1} y(\xi,t)\, d\xi,\\ J^{\nu}_t y(x,t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} y(x,\tau)\, d\tau. \end{aligned} \tag{4.4}$$

Proof. See [21] (Theorem 3.4). □

Corollary 4.2. *It is clear that the operational matrices of the integration of integer orders,* $\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}$ *and* $\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}$, *in the two-variable case are obtained for* $\nu = 1$ *in Theorem* 4.2.

Now, consider the following fractional-time PDE,

$$\begin{aligned} &\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} + \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = f(x,t), \qquad 0<\gamma\le 1, \quad (x,t)\in\Omega,\\ &u(x,0)=g(x), \quad u(0,t)=h_1(t), \quad u_x(0,t)=h_2(t), \quad u_{xx}(0,t)=h_3(t). \end{aligned} \tag{4.5}$$

In order to compute an approximate solution, $u_{N,N}(x,t)$, for Eq. (4.5) first consider the following approximation,

$$\frac{\partial^4 u(x,t)}{\partial t\, \partial x^3} \cong \mathbf{\Phi}^T(x,t)\; C. \tag{4.6}$$

Integrating of approximation (4.6) from 0 to t leads to the following approximation,

$$\frac{\partial^3 u(x,t)}{\partial x^3} \cong \mathbf{\Phi}^T(x,t)\; {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\; C + g^{'''}(x). \tag{4.7}$$

By the consecutive integrating of (4.7), the following approximations are obtained.

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &\cong \mathbf{\Phi}^T(x,t)\; {\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T\; {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\; C + (g^{''}(x) - g^{''}(0)) + h_3(t),\\ \frac{\partial u(x,t)}{\partial x} &\cong \mathbf{\Phi}^T(x,t)\; ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^2\; {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\; C + (g^{'}(x) - g^{'}(0)) + (h_3(t) - g^{''}(0))x + h_2(t),\\ u(x,t) &\cong \mathbf{\Phi}^T(x,t)\; ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^3\; {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\; C + (g(x) - g(0)) + g^{'}(x) + (h_2(t) - g^{'}(0))x\\ &\quad + (h_3(t) - g^{''}(0))\frac{x^2}{2} + h_1(t). \end{aligned} \tag{4.8}$$

By the consecutive integrating of (4.7) from 0 to x, one has,

$$\begin{aligned} \frac{\partial^3 u(x,t)}{\partial t\, \partial x^2} &\cong \mathbf{\Phi}^T(x,t)\; {\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T\; C + h_3^{'}(t),\\ \frac{\partial^2 u(x,t)}{\partial t\, \partial x} &\cong \mathbf{\Phi}^T(x,t)\; ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^2\; C + h_3^{'}(t)x + h_2^{'}(t),\\ \frac{\partial u(x,t)}{\partial t} &\cong \mathbf{\Phi}^T(x,t)\; ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^3\; C + h_3^{'}(t)\frac{x^2}{2} + h_2^{'}(t)x + h_1^{'}(t). \end{aligned} \tag{4.9}$$

For approximating the function $\partial^{\gamma} u(x,t)/\partial t^{\gamma}$, approximation (4.9) is utilized as follows:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1-\gamma}\, \partial^{\gamma} u(x,t)}{\partial t^{1-\gamma}\, \partial t^{\gamma}} \cong \mathbf{\Phi}^T(x,t)\; ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^3\; C + h_3^{'}(t)\frac{x^2}{2} + h_2^{'}(t)x + h_1^{'}(t). \tag{4.10}$$

Applying the Riemann-Liouville integral operator of the order $(1-\gamma)$ to (4.10) yields the following approximation,

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} \simeq \mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\gamma)^T}\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^3\ C + G(x,t) + \frac{\partial^\gamma u(x,0)}{\partial t^\gamma},$$

where

$$G(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-\tau)^{-\gamma}\ G_0(x,\tau)\ d\tau, \qquad G_0(x,t) = h_3^{'}(t)\frac{x^2}{2} + h_2^{'}(t)x + h_1^{'}(t).$$

By using Eq. (4.10), an approximation for $u(x,t)$ is obtained as

$$u(x,t) \simeq \mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^3\ C + (h_3(t)-h_3(0))\frac{x^2}{2} + (h_2(t)-h_2(0))x + (h_1(t)-h_1(0)) + g(x),$$

$$\frac{\partial^\gamma u(x,0)}{\partial t^\gamma} \simeq \left(\mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^3\ C + \frac{d^\gamma h_3(t)}{dt^\gamma}\frac{x^2}{2} + \frac{d^\gamma h_2(t)}{dt^\gamma}x + \frac{d^\gamma h_1(t)}{dt^\gamma}\right)\bigg|_{t=0}$$
$$= \frac{d^\gamma h_3(0)}{dt^\gamma}\frac{x^2}{2} + \frac{d^\gamma h_2(0)}{dt^\gamma}x + \frac{d^\gamma h_1(0)}{dt^\gamma}.$$

The last line is because all components of the vector $\mathbf{\Phi}^T(x,t)$ are polynomials in terms of x, t. Therefore, one has

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} \simeq \mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\gamma)^T}\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^3\ C + G(x,t) + V(x),$$

where

$$V(x) = \frac{d^\gamma h_3(0)}{dt^\gamma}\frac{x^2}{2} + \frac{d^\gamma h_2(0)}{dt^\gamma}x + \frac{d^\gamma h_1(0)}{dt^\gamma}.$$

By substituting the above approximations into Eq. (4.5), one obtains the following matrix equation,

$$\mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\gamma)^T}\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^3\ C + \mathbf{\Phi}^T(x,t)\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^2\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ C + \mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ C - \mathbf{\Phi}^T(x,t)\ F \approx 0,$$

where F is a vector and its components are calculated as,

$$F_i = \frac{1}{(h_i^{\alpha,\beta})^2}\int_0^1 \int_0^1 f_0(x,t)\ P_{i,i}^{(\alpha,\beta)}(x,t)\ W^{(\alpha,\beta)}(x,t)\ dx\ dt,$$

with

$$f_0(x,t) = G(x,t) + V(x) + (g^{'}(x) - g^{'}(0)) + (h_3(t) - g^{''}(0))x + h_2(t) + g^{'''}(x) - f(x,t).$$

According to the Tau method, $(N+1)^2$ linear algebraic equations are generated including the unknown coefficients, C_{ij}, $i, j = 0, 1, \cdots, N$,

$$\mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\gamma)^T}\ (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^3\ C + (\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T})^2\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ C + \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ C \approx F. \qquad (4.11)$$

System (4.11) is a Sylvester equation and by solving this equation, an approximate solution can be computed by (4.8).

5. Error Bound

An error bound for an integer derivative of an approximate solution is computed in a Jacobi-weighted Sobolev space, then the resultant bound will be extended to fractional-order derivatives.

First, suppose that $\mathbb{P}_{N,M} = span\{P_{i,j}^{(\alpha,\beta)}(x,t), i = 0,1,\cdots,N,\ j = 0,1,\cdots,M\}$. Let us assume that $u_{N,M}(x,t) \in \mathbb{P}_{N,M}$ be the best approximation of $u(x,t) \in L^2([0,1]\times[0,1])$, on the other hand,

$$||u(x,t) - u_{N,m}(x,t)|| = inf_{v(x,t)\in\mathbb{P}_{N,M}} ||u(x,t) - v(x,t)||,$$

where

$$u_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} C_{ij}\ P_{i,j}^{(\alpha,\beta)}(x,t).$$

Definition 5.1. *The two-dimensional Sobolev space $W^{m,2}(\Omega)$, $\Omega = [0,1]\times[0,1]$, $m \in \mathbb{N}$, is defined as,*

$$W^{m,2}(\Omega) = \left\{ u(x,t) \in L^2(\Omega) \;\middle|\; \frac{\partial^{i+j} u}{\partial x^i\ \partial t^j} \in L^2(\Omega),\ 0 \le i+j \le m \right\}.$$

If $\sigma = (\sigma_1, \sigma_2)$ such that $\sigma_i \in \mathbb{Z}^+$, $i = 1,2$, and $|\sigma| = \sigma_1 + \sigma_2$, the norm of the space $W^{m,2}(\Omega)$ is defined as,

$$||u||_{m,W^{(\alpha,\beta)}} = \left(\sum_{|\sigma|\le m} ||D^\sigma u||^2_{W^{(\alpha,\beta)}_{|\sigma|}} \right)^{\frac{1}{2}} = \left(\sum_{|\sigma|\le m} \left|\left| \frac{\partial^{|\sigma|} u}{\partial x^{\sigma_1}\ \partial t^{\sigma_2}} \right|\right|^2_{W^{(\alpha,\beta)}_{|\sigma|}} \right)^{\frac{1}{2}},$$

where $W^{(\alpha,\beta)}_{|\sigma|}(x,t) = w^{(\alpha+\sigma_1,\beta+\sigma_1)}(x) w^{(\alpha+\sigma_2,\beta+\sigma_2)}(t)$. Meanwhile, $W^{m,2}(\Omega)$ can be denoted by $H^m(\Omega)$ which is called a Hilbert space.

Definition 5.2 (Stirling's formula). *The following formula is applied to calculate approximate values of the Gamma function for large values of n,*

$$\Gamma(n+1) = n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}\ S(n),$$

where

$$S(n) = 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + O(n^{-5}).$$

Theorem 5.1. *Let $0 \le l \le m < N+1$. Suppose that $u \in W^{m,2}(\Omega)$ and $u_{N,M}(x,t)$ be the Jacobi approximation to u. A bound for the integer derivative of the order l of the error function, relative to x, can be computed as follows,*

$$\left|\left| \frac{\partial^l (u - u_{N,M})}{\partial x^l} \right|\right|_{W_l^{(\alpha,\beta)}} \le \sqrt{3}\ C_0 (N(N+\alpha+\beta))^{\frac{l-m}{2}} ||u||_{W^{m,2}(\Omega)}, \tag{5.1}$$

where C_0 is a constant.

Proof. Suppose that $u_{N,M}(x,t)$ be the Jacobi approximation of $u(x,t)$, which is obtained from the proposed algorithm. We have,

$$u(x,t) - u_{N,M}(x,t) = \sum_{i=N+1}^{\infty}\sum_{j=M+1}^{\infty} C_{ij} P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(t)$$
$$+ \sum_{i=0}^{N}\sum_{j=M+1}^{\infty} C_{ij} P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(t) + \sum_{i=N+1}^{\infty}\sum_{j=0}^{M} C_{ij} P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(t).$$

Based on the orthogonality of the TVJPs and the second relation in (3.7), we have,

$$\left|\left|\frac{\partial^k u}{\partial x^k}\right|\right|^2_{W_k^{(\alpha,\beta)}} = \sum_{i=k}^{\infty}\sum_{j=0}^{\infty} C_{ij}^2\ (d_{i,k}^{\alpha,\beta})^2\ h_{i-k}^{\alpha+k,\beta+k} h_j^{\alpha,\beta}, \qquad k<m,$$

where

$$d_{i,k}^{\alpha,\beta} = \frac{\Gamma(i+k+\alpha+\beta+1)}{\Gamma(i+\alpha+\beta+1)}, \qquad W_k^{(\alpha,\beta)}(x,t) = w^{(\alpha+k,\beta+k)}(x)\ w^{(\alpha,\beta)}(t). \tag{5.2}$$

So, we have,

$$\begin{aligned}
\left|\left|\frac{\partial^l (u-u_{N,M})}{\partial x^l}\right|\right|^2_{W_l^{(\alpha,\beta)}} &\le \sum_{i=N+1}^{\infty}\sum_{j=M+1}^{\infty} C_{ij}^2 (d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l} h_j^{\alpha,\beta}\\
&+\sum_{i=l}^{N}\sum_{j=M+1}^{\infty} C_{ij}^2 (d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l} h_j^{\alpha,\beta} + \sum_{i=N+1}^{\infty}\sum_{j=0}^{M} C_{ij}^2 (d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l} h_j^{\alpha,\beta}\\
&= \sum_{i=N+1}^{\infty}\sum_{j=M+1}^{\infty} \frac{(d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l}}{(d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m}} C_{ij}^2 (d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m} h_j^{\alpha,\beta}\\
&+\sum_{i=l}^{N}\sum_{j=M+1}^{\infty} \frac{(d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l}}{(d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m}} C_{ij}^2 (d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m} h_j^{\alpha,\beta}\\
&+\sum_{i=N+1}^{\infty}\sum_{j=0}^{M} \frac{(d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l}}{(d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m}} C_{ij}^2 (d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m} h_j^{\alpha,\beta}\\
&\le \sup_{i\ge N+1}\left\{\frac{(d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l}}{(d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m}}\right\}\sum_{i=N+1}^{\infty}\sum_{j=M+1}^{\infty} C_{ij}^2 (d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m} h_j^{\alpha,\beta}\\
&+\max_{l\le i\le N}\left\{\frac{(d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l}}{(d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m}}\right\}\sum_{i=l}^{N}\sum_{j=M+1}^{\infty} C_{ij}^2 (d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m} h_j^{\alpha,\beta}\\
&+\sup_{i\ge N+1}\left\{\frac{(d_{i,l}^{\alpha,\beta})^2\ h_{i-l}^{\alpha+l,\beta+l}}{(d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m}}\right\}\sum_{i=N+1}^{\infty}\sum_{j=0}^{M} C_{ij}^2 (d_{i,m}^{\alpha,\beta})^2\ h_{i-m}^{\alpha+m,\beta+m} h_j^{\alpha,\beta}\\
&\le\left\{\frac{2\ (d_{N+1,l}^{\alpha,\beta})^2\ h_{N-l+1}^{\alpha+l,\beta+l}}{(d_{N+1,m}^{\alpha,\beta})^2\ h_{N-m+1}^{\alpha+m,\beta+m}} + \frac{(d_{N,l}^{\alpha,\beta})^2\ h_{N-l}^{\alpha+l,\beta+l}}{(d_{N,m}^{\alpha,\beta})^2\ h_{N-m}^{\alpha+m,\beta+m}}\right\}\left|\left|\frac{\partial^m u}{\partial x^m}\right|\right|^2_{W_m^{(\alpha,\beta)}}.
\end{aligned}$$

By (3.2) and (5.2), we can obtain,

$$\frac{(d_{N+1,l}^{\alpha,\beta})^2\ h_{N-l+1}^{\alpha+l,\beta+l}}{(d_{N+1,m}^{\alpha,\beta})^2\ h_{N-m+1}^{\alpha+m,\beta+m}} = \frac{\Gamma(N+l+\alpha+\beta+2)(N-m+1)!}{\Gamma(N+m+\alpha+\beta+2)(N-l+1)!}\ .$$

Using Stirling's formula, Definition 5.3, yields the following inequality,

$$\frac{\Gamma(N+l+\alpha+\beta+2)(N-m+1)!}{\Gamma(N+m+\alpha+\beta+2)(N-l+1)!} \le C'(N+\alpha+\beta)^{l-m}N^{l-m}.$$

A combination of the above estimates leads to the desired result in (5.1).

Corollary 5.1. *According to Theorem* 5.1*, if* $0 \le k \le m < M+1$*, for any* $u \in W^{m,2}(\Omega)$*, a bound for the integer derivative of the order* k *of the error function, relative to* t*, can similarly be computed as follows,*

$$\left|\left|\frac{\partial^k (u-u_{N,M})}{\partial t^k}\right|\right|_{W_k^{(\alpha,\beta)}} \le \sqrt{3}\; C_1 (M(M+\alpha+\beta))^{\frac{k-m}{2}} ||u||_{W^{m,2}(\Omega)}, \tag{5.3}$$

where C_1 *is a constant and* $W_k^{(\alpha,\beta)}(x,t) = w^{(\alpha,\beta)}(x) w^{(\alpha+k,\beta+k)}(t)$*.*

Corollary 5.2. *Suppose that* $u \in W^{m,2}(\Omega)$ *and* $u_{N,M}(x,t)$ *be the Jacobi approximation to* $u(x,t)$*, then, using Theorem* 5.1*, an error bound for the approximate solution can be computed as follows,*

$$\left|\left|u-u_{N,M}\right|\right|_{W^{(\alpha,\beta)}} \le \sqrt{3}\; C_2 (\tilde{N}(\tilde{N}+\alpha+\beta))^{-\frac{m}{2}} ||u||_{W^{m,2}(\Omega)}, \tag{5.4}$$

where C_2 *is a constant and* $\tilde{N} = \min\{N, M\}$*.*

Theorem 5.2. *Let* $u \in W^{m,2}(\Omega)$*. If* $n = \lceil \gamma \rceil$*,* $n < m < N+1$*, then,*

$$\left|\left|\frac{\partial^\gamma (u-u_{N,M})}{\partial x^\gamma}\right|\right|_{W_n^{(\alpha,\beta)}} \le \frac{\sqrt{3}}{\Gamma(n-\gamma+1)}\; C_0 (N(N+\alpha+\beta))^{\frac{n-m}{2}} ||u||_{W^{m,2}(\Omega)}. \tag{5.5}$$

Similarly, for the fractional-time derivative, the following inequality is achieved,

$$\left|\left|\frac{\partial^\nu (u-u_{N,M})}{\partial t^\nu}\right|\right|_{W_{n'}^{(\alpha,\beta)}} \le \frac{\sqrt{3}}{\Gamma(n'-\nu+1)}\; C_1 (M(M+\alpha+\beta))^{\frac{n'-m}{2}} ||u||_{W^{m,2}(\Omega)}, \tag{5.6}$$

where $n' = \lceil \nu \rceil$ *and* $n' < m < N+1$*.*

Proof. By using the Riemann-Liouville integral operator, the Caputo derivative operator is rewritten as follows,

$$\frac{\partial^\gamma (u-u_{N,M})}{\partial x^\gamma} = J_x^{n-\gamma} \frac{\partial^n (u-u_{N,M})}{\partial x^n}.$$

By using the inequality $||f * g||_p \le ||f||_1\; ||g||_p$ that the star symbol denotes the convolution of two given functions f and g, one has,

$$\begin{aligned}
\left|\left|\frac{\partial^\gamma (u-u_{N,M})}{\partial x^\gamma}\right|\right|^2_{W_n^{(\alpha,\beta)}} &= \left|\left|J_x^{n-\gamma}\frac{\partial^n (u-u_{N,M})}{\partial x^n}\right|\right|^2_{W_n^{(\alpha,\beta)}} \\
&= \left(\frac{1}{\Gamma(n-\gamma)}\right)^2 \left|\left|\frac{1}{x^{1+\gamma-n}} * \frac{\partial^n (u-u_{N,M})}{\partial x^n}\right|\right|^2_{W_n^{(\alpha,\beta)}} \\
&\le \left(\frac{1}{\Gamma(n-\gamma)}\right)^2 \left|\left|\frac{1}{x^{1+\gamma-n}}\right|\right|^2_{L^1(\Omega)} \left|\left|\frac{\partial^n (u-u_{N,M})}{\partial x^n}\right|\right|^2_{W_n^{(\alpha,\beta)}} \\
&= \left(\frac{B(n+\beta-\gamma,\alpha+1)}{\Gamma(n-\gamma)}\right)^2 \left|\left|\frac{\partial^n (u-u_{N,M})}{\partial x^n}\right|\right|^2_{W_n^{(\alpha,\beta)}},
\end{aligned}$$

where $B(m,n)$ is the Beta function. Using Theorem 5.1 leads to the desired result. □

Lemma 5.1. *For any two functions $f, g \in L^2(\Omega), \Omega = [0,1]\times[0,1]$, one has*

$$\langle f, g\rangle_{L^2(\Omega)} \le \frac{1}{2}\Big(||f||^2_{L^2(\Omega)} + ||g||^2_{L^2(\Omega)}\Big),$$

where $\langle .,.\rangle_{L^2(\Omega)}$ and $||.||_{L^2(\Omega)}$ indicate the inner product and the norm in the space $L^2(\Omega)$, respectively $(Interestedreadersarereferredto[22])$.

Now, consider the following FPDE with the constant coefficients.

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + \frac{\partial^\nu u(x,t)}{\partial t^\nu} + u(x,t) = f(x,t), \tag{5.7}$$

such that $n = \lceil\gamma\rceil$, $n' = \lceil\nu\rceil$. Suppose that $u_{N,M}(x,t)$ be the approximate solution obtained from the proposed algorithm. Therefore, $u_{N,M}(x,t)$ will be the solution of the following equation,

$$\frac{\partial^\gamma u_{N,M}(x,t)}{\partial x^\gamma} + \frac{\partial^\nu u_{N,M}(x,t)}{\partial t^\nu} + u_{N,M}(x,t) = f(x,t) + H_{N,M}(x,t), \tag{5.8}$$

$$\frac{\partial^\gamma e_{N,M}(x,t)}{\partial x^\gamma} + \frac{\partial^\nu e_{N,M}(x,t)}{\partial t^\nu} + e_{N,M}(x,t) = -H_{N,M}(x,t), \tag{5.9}$$

$$\begin{aligned}\langle -H_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)} &= \langle\frac{\partial^\gamma e_{N,M}(x,t)}{\partial x^\gamma}, e_{N,M}(x,t)\rangle_{L^2(\Omega)} \\ &+ \langle\frac{\partial^\nu e_{N,M}(x,t)}{\partial t^\nu}, e_{N,M}(x,t)\rangle_{L^2(\Omega)} + \langle e_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)} \\ &\le \frac{1}{2}\Bigg(\bigg|\bigg|\frac{\partial^\gamma e_{N,M}(x,t)}{\partial x^\gamma}\bigg|\bigg|^2_{L^2(\Omega)} + \bigg|\bigg|\frac{\partial^\nu e_{N,M}(x,t)}{\partial t^\nu}\bigg|\bigg|^2_{L^2(\Omega)} + 4||e_{N,M}(x,t)||^2_{L^2(\Omega)}\Bigg).\end{aligned} \tag{5.10}$$

Noting that $||y||_{L^2(\Omega)} \le ||y||_{W^{m,2}(\Omega)}$, one has

$$\begin{aligned}&||H_{N,M}(x,t)||^2_{W^{m,2}(\Omega)} \\ \le&\bigg|\bigg|\frac{\partial^\gamma e_{N,M}(x,t)}{\partial x^\gamma}\bigg|\bigg|^2_{W^{m,2}(\Omega)} + \bigg|\bigg|\frac{\partial^\nu e_{N,M}(x,t)}{\partial t^\nu}\bigg|\bigg|^2_{W^{m,2}(\Omega)} + 3||e_{N,M}(x,t)||^2_{W^{m,2}(\Omega)} \\ \le&\Bigg(\frac{3\ C_0^2 B^2(n+\beta-\gamma,\alpha+1)}{\Gamma^2(n-\gamma)}(N(N+\alpha+\beta))^{n-m} \\ &+ \frac{3\ C_1^2 B^2(n+\beta-\nu,\alpha+1)}{\Gamma^2(n'-\nu)}(M(M+\alpha+\beta))^{n'-m} \\ &+ 9\ C_2^2(\tilde{N}(\tilde{N}+\alpha+\beta))^{-m}\Bigg)||u||^2_{W^{m,2}(\Omega)}.\end{aligned} \tag{5.11}$$

Since $u(x,t)$ is a bounded function, $||H_{N,M}||_{W^{m,2}(\Omega)} \to 0$ as $N, M \to \infty$. By referring to Eq.

(5.10), a bound can be obtained for the error of the method in a similar way,

$$\begin{aligned}&\langle -e_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)} = \langle \frac{\partial^{\gamma} e_{N,M}(x,t)}{\partial x^{\gamma}}, e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\ &\quad + \langle \frac{\partial^{\nu} e_{N,M}(x,t)}{\partial t^{\nu}}, e_{N,M}(x,t)\rangle_{L^2(\Omega)} + \langle H_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\ &\leq \frac{1}{2}\Bigg(\Bigg|\Bigg|\frac{\partial^{\gamma} e_{N,M}(x,t)}{\partial x^{\gamma}}\Bigg|\Bigg|^2_{L^2(\Omega)} + \Bigg|\Bigg|\frac{\partial^{\nu} e_{N,M}(x,t)}{\partial t^{\nu}}\Bigg|\Bigg|^2_{L^2(\Omega)}\\ &\quad + 3||e_{N,M}(x,t)||^2_{L^2(\Omega)} + ||H_{N,M}(x,t)||^2_{L^2(\Omega)}\Bigg).\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}&||e_{N,M}(x,t)||^2_{W^{m,2}(\Omega)} \leq \frac{1}{2}\Bigg(\Bigg|\Bigg|\frac{\partial^{\gamma} e_{N,M}(x,t)}{\partial x^{\gamma}}\Bigg|\Bigg|^2_{W^{m,2}(\Omega)} + \Bigg|\Bigg|\frac{\partial^{\nu} e_{N,M}(x,t)}{\partial t^{\nu}}\Bigg|\Bigg|^2_{W^{m,2}(\Omega)}\\ &\quad + 3||e_{N,M}(x,t)||^2_{W^{m,2}(\Omega)} + ||H_{N,M}(x,t)||^2_{W^{m,2}(\Omega)}\Bigg)\\ &\leq \frac{1}{2}\Bigg(\frac{3\ C_0^2 B^2(n+\beta-\gamma,\alpha+1)}{\Gamma^2(n-\gamma)}(N(N+\alpha+\beta))^{n-m}\\ &\quad + \frac{3\ C_1^2 B^2(n+\beta-\nu,\alpha+1)}{\Gamma^2(n'-\nu)}(M(M+\alpha+\beta))^{n'-m}\\ &\quad + 9\ C_2^2(\tilde{N}(\tilde{N}+\alpha+\beta))^{-m}\Bigg)||u||^2_{W^{m,2}(\Omega)} + \frac{1}{2}||H_{N,M}(x,t)||^2_{W^{m,2}(\Omega)}.\end{aligned}$$

Based on Corollary 5.2, Theorem 5.2, and the bound in the inequality (5.1) can be concluded $||e_{N,M}||_{W^{m,2}(\Omega)} \to 0$ when N, M tend to infinity.

Now, consider the following fractional time-space PDE with the variable coefficients.

$$\frac{\partial^{\eta} u(x,t)}{\partial t^{\eta}} = d(x,t)\frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}} + b(x,t)\frac{\partial^{\nu} u(x,t)}{\partial t^{\nu}} + f(x,t), \tag{5.12}$$

such that $r = \lceil \eta \rceil$. If $u_{N,M}(x,t)$ is the Jacobi approximation to $u(x,t)$, then it is a solution for the following equation

$$\frac{\partial^{\eta} u_{N,M}(x,t)}{\partial t^{\eta}} = d(x,t)\frac{\partial^{\gamma} u_{N,M}(x,t)}{\partial x^{\gamma}} + b(x,t)\frac{\partial^{\nu} u_{N,M}(x,t)}{\partial t^{\nu}} + f(x,t) + H_{N,M}(x,t). \tag{5.13}$$

Subtracting Eq. (5.13) from Eq. (5.12) leads to the following error equation,

$$H_{N,M}(x,t) = d(x,t)\frac{\partial^{\gamma} e_{N,M}(x,t)}{\partial x^{\gamma}} + b(x,t)\frac{\partial^{\nu} e_{N,M}(x,t)}{\partial t^{\nu}} - \frac{\partial^{\eta} e_{N,M}(x,t)}{\partial t^{\eta}}. \tag{5.14}$$

Suppose that $||d(x,t)||_{L^2(\Omega)} = M_1$ and $||b(x,t)||_{L^2(\Omega)} = M_2$. Using the Cauchy-Schwartz inequality $||fg||_{L^2(\Omega)} \leq ||f||_{L^2(\Omega)}||g||_{L^2(\Omega)}$ for Eq. (5.13) leads to the following inequality:

$$\begin{aligned}
&\langle H_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)} = \langle d(x,t)\frac{\partial^\gamma u_{N,M}(x,t)}{\partial x^\gamma}, e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\
&\quad + \langle b(x,t)\frac{\partial^\nu e_{N,M}(x,t)}{\partial t^\nu}, e_{N,M}(x,t)\rangle_{L^2(\Omega)} - \langle \frac{\partial^\eta e_{N,M}(x,t)}{\partial t^\eta}, e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\
&\leq \frac{1}{2}\Big(||d||^2_{L^2(\Omega)}\Big|\Big|\frac{\partial^\gamma u_{N,M}}{\partial x^\gamma}\Big|\Big|^2_{L^2(\Omega)} + ||b||^2_{L^2(\Omega)}\Big|\Big|\frac{\partial^\nu e_{N,M}}{\partial t^\nu}\Big|\Big|^2_{L^2(\Omega)} + \Big|\Big|\frac{\partial^\eta e_{N,M}}{\partial t^\eta}\Big|\Big|^2_{L^2(\Omega)} + 3||e_{N,M}||^2_{L^2(\Omega)}\Big)\\
&\leq \frac{1}{2}\Big(M_1^2\Big|\Big|\frac{\partial^\gamma u_{N,M}}{\partial x^\gamma}\Big|\Big|^2_{L^2(\Omega)} + M_2\Big|\Big|\frac{\partial^\nu e_{N,M}}{\partial t^\nu}\Big|\Big|^2_{L^2(\Omega)} + \Big|\Big|\frac{\partial^\eta e_{N,M}}{\partial t^\eta}\Big|\Big|^2_{L^2(\Omega)} + 3||e_{N,M}||^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Therefore, the following inequality can be achieved

$$\begin{aligned}
&||H_{N,M}(x,t)||^2_{W^{m,2}(\Omega)}\\
&\leq M_1^2\Big|\Big|\frac{\partial^\gamma u_{N,M}}{\partial x^\gamma}\Big|\Big|^2_{w^{m,2}(\Omega)} + M_2^2\Big|\Big|\frac{\partial^\nu e_{N,M}}{\partial t^\nu}\Big|\Big|^2_{W^{m,2}(\Omega)} + \Big|\Big|\frac{\partial^\eta e_{N,M}}{\partial t^\eta}\Big|\Big|^2_{W^{m,2}(\Omega)} + 2||e_{N,M}||^2_{W^{m,2}(\Omega)}.
\end{aligned}$$

Pursuing the same above argument, it can be concluded that $||H_{N,M}||_{W^{m,2}(\Omega)} \to 0$ as $N, M \to \infty$. By using the Riemann-Liouville integral operator, Eq. (5.14) can be rewritten as follows,

$$\begin{aligned}
e_{N,M}(x,t) &= \frac{1}{\Gamma(r-\eta)}\int_0^t (t-s)^{r-\eta+1} d(x,s)\ \frac{\partial^\gamma e_{N,M}(x,s)}{x^\gamma}\ ds\\
&\quad + \frac{1}{\Gamma(r-\eta)}\int_0^t (t-s)^{r-\eta+1} b(x,s)\ \frac{\partial^\nu e_{N,M}(x,s)}{x^\nu}\ ds\\
&\quad - \frac{1}{\Gamma(r-\eta)}\int_0^t (t-s)^{r-\eta+1} H_{N,M}(x,s)\ ds\\
&= \frac{1}{\Gamma(r-\eta)} t^{r-\eta+1} * d(x,t)\ \frac{\partial^\gamma e_{N,M}(x,t)}{x^\gamma}\\
&\quad + \frac{1}{\Gamma(r-\eta)} t^{r-\eta+1} * b(x,t)\ \frac{\partial^\nu e_{N,M}(x,t)}{x^\nu} - \frac{1}{\Gamma(r-\eta)} t^{r-\eta+1} * H_{N,M}(x,t).
\end{aligned}$$

$$\begin{aligned}
&\langle e_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\
&\quad = \frac{1}{\Gamma(r-\eta)}\langle t^{r-\eta+1} * d(x,t)\ \frac{\partial^\gamma e_{N,M}(x,t)}{x^\gamma}, e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\
&\qquad + \frac{1}{\Gamma(r-\eta)}\langle t^{r-\eta+1} * b(x,t)\ \frac{\partial^\nu e_{N,M}(x,t)}{x^\nu}\rangle_{L^2(\Omega)}\\
&\qquad - \frac{1}{\Gamma(r-\eta)}\langle t^{r-\eta+1} * H_{N,M}(x,t), e_{N,M}(x,t)\rangle_{L^2(\Omega)}\\
&\quad \leq \frac{1}{2\Gamma(r-\eta)}\Big(||t^{r-\eta+1}||^2_{L^1(\Omega)}||d||^2_{L^2(\Omega)}\Big|\Big|\frac{\partial^\gamma e_{N,M}(x,t)}{x^\gamma}\Big|\Big|^2_{L^2(\Omega)}\\
&\qquad + ||t^{r-\eta+1}||^2_{L^1(\Omega)}||b||^2_{L^2(\Omega)}\Big|\Big|\frac{\partial^\nu e_{N,M}(x,t)}{x^\nu}\Big|\Big|^2_{L^2(\Omega)}\\
&\qquad + ||t^{r-\eta+1}||^2_{L^1(\Omega)}||H_{N,M}(x,t)||^2_{L^2(\Omega)} + 3||e_{N,M}(x,t)||^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&||e_{N,M}||^2_{W^{m,2}(\Omega)}\\
\leq&\frac{B^2(r+\beta-\eta+2,\alpha+1)}{\Gamma(r-\eta)}\Bigg(M_1^2\Bigg|\Bigg|\frac{\partial^\gamma e_{N,M}}{x^\gamma}\Bigg|\Bigg|^2_{W^{m,2}(\Omega)}+M_2^2\Bigg|\Bigg|\frac{\partial^\nu e_{N,M}}{x^\nu}\Bigg|\Bigg|^2_{W^{m,2}(\Omega)}\\
&+||H_{N,M}||^2_{W^{m,2}(\Omega)}+3||e_{N,M}||^2_{W^{m,2}(\Omega)}\Bigg),
\end{aligned}
$$

where $B(m,n)$ is the Beta function. When N and M tend to infinity, the right-hand side of the above inequality also tends to zero. Therefore, $e_{N,M}(x,t)\to 0$.

6. Illustrative Examples

In this section, the proposed numerical method is carried out for several time- and time-space fractional equations. All the test are done in Maple 13. The obtained results are compared to those obtained from some existing methods such as Haar wavelet, Variational iteration, and finite difference methods. Meanwhile, for all of the examples will be set $N=M$.

Example 6.1. As the first example, consider the following fourth-order fractional diffusion-wave equation ([23, 24]):

$$
\frac{\partial^\nu u(x,t)}{\partial t^\nu}+\frac{\partial^4 u(x,t)}{\partial x^4}=\exp(x)\bigg(\frac{\Gamma(\nu+4)}{6}t^3+t^{\nu+3}+t\bigg),\qquad 0<x,t\leq 1,\quad 1<\nu\leq 2,\tag{6.1}
$$

with the initial and boundary conditions, respectively,

$$
\begin{aligned}
&u(x,0)=0,\qquad u_t(x,0)=\exp(x),\\
&u(0,t)=u_x(0,t)=u_{xx}(0,t)=u_{xxx}(0,t)=t^{\nu+3}+t.
\end{aligned}\tag{6.2}
$$

The exact solution is $u(x,t)=\exp(x)(t^{\nu+3}+t)$. Pursuing the procedure described in the previous section and substituting the appropriate approximations in Eq. (6.1), the following equation is obtained.

$$
\mathbf{\Phi}^T(x,t)\ \mathbf{P}^{(\mathbf{2}-\nu)^T}_{(\mathbf{t})}\ (\mathbf{P}^{\mathbf{1}\ T}_{(\mathbf{x})})^4\ C+\mathbf{\Phi}^T(x,t)\ (\mathbf{P}^{\mathbf{1}\ T}_{(\mathbf{t})})^2\ C\approx\mathbf{\Phi}^T(x,t)\ F,\tag{6.3}
$$

where

$$
\begin{aligned}
&\bigg(\frac{\Gamma(\nu+4)}{6}t^3+t^{\nu+3}+t\bigg)e^x-(\nu+3)t^{\nu+2}\bigg(\frac{x^3}{3!}+\frac{x^2}{2!}+x+1\bigg)-e^x\\
&\qquad-(\nu+3)(\nu+2)t^{\nu+1}(x+1)\simeq\mathbf{\Phi}^T(x,t)\ F.
\end{aligned}
$$

The left-hand side of the above approximation is obtained from initial and boundary conditions (6.1). Using the Tau method for Eq. (6.3) leads to the following system.

$$
\mathbf{P}^{(\mathbf{2}-\nu)^T}_{(\mathbf{t})}\ (\mathbf{P}^{\mathbf{1}\ T}_{(\mathbf{x})})^4\ C+(\mathbf{P}^{\mathbf{1}\ T}_{(\mathbf{t})})^2\ C-F\approx 0.\tag{6.4}
$$

Eq. (6.1) is solved in [23] by means of a spectral Tau method, which introduces approximations to another form. Authors set $N=M=9$ in [23]. By choosing $N=M=9$ and considering various values of the parameters α and β, approximate solutions are computed. The maximum absolute errors of the resultant solutions are seen in Table 6.1 for the values of $\nu=1.5$ and

Table 6.1: Example 6.1: Maximum errors for $N = M = 9$, $\nu = 1.5$, and different values of α and β.

(α, β)	Error	(α, β)	Error	(α, β)	Error
$(0, 0)$	1.5617×10^{-8}	$(1, 1)$	4.3487×10^{-8}	$(2, 2)$	2.3774×10^{-6}
$(1, 2)$	1.0356×10^{-7}	$(2, 1)$	2.6729×10^{-8}	$(0, 1)$	6.0150×10^{-8}
$(1, 0)$	6.1472×10^{-8}	$(1, 3)$	2.0409×10^{-5}	$(3, 2)$	6.7800×10^{-7}

Table 6.2: Example 6.1: Errors at $t = 1$ with $N = M = 9$, $\alpha = \beta = 0$, and different values of ν.

x_i	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.5$	$\nu = 1.7$	$\nu = 1.8$
0.1	2.5215×10^{-13}	1.4662×10^{-13}	1.5636×10^{-13}	3.5438×10^{-13}	4.8383×10^{-13}
0.2	8.5130×10^{-12}	1.0232×10^{-11}	1.0061×10^{-11}	6.8115×10^{-12}	4.6960×10^{-12}
0.3	4.0776×10^{-11}	4.9694×10^{-11}	4.8882×10^{-11}	3.2125×10^{-11}	2.1175×10^{-11}
0.4	1.3233×10^{-10}	1.6139×10^{-11}	1.5934×10^{-10}	1.0556×10^{-10}	7.0075×10^{-11}
0.5	3.3350×10^{-10}	4.0734×10^{-10}	4.0524×10^{-10}	2.7268×10^{-10}	1.8342×10^{-10}
0.6	7.1266×10^{-10}	8.7443×10^{-10}	8.8138×10^{-10}	6.0642×10^{-10}	4.1440×10^{-10}
0.7	1.3732×10^{-9}	1.6958×10^{-9}	1.7439×10^{-9}	1.2413×10^{-9}	8.6882×10^{-10}
0.8	2.4582×10^{-9}	3.0636×10^{-9}	3.2391×10^{-9}	2.4095×10^{-9}	1.7362×10^{-9}
0.9	4.1703×10^{-9}	5.2620×10^{-9}	5.7630×10^{-9}	4.5120×10^{-9}	3.3538×10^{-9}

$N = M = 9$. The data show that the approximate solutions are in good agreement with the exact solution. In Fig. 6.1, the plots of the exact and approximate solutions and the absolute error function are depicted in parts of (a)-(c) for $N = M = 9$, $\alpha = 3$, $\beta = 2$, and $\nu = 1.5$. The absolute errors for the values of $\nu = 1.2, 1.3, 1.5, 1.7, 1.8$, $\alpha = \beta = 0$, at the points $x_i = 0.1i, i = 1, \cdots, 9$, and $t = 1$ are listed in Table 6.2. In Table 6.3, the values of the absolute errors are seen for $N = M = 4, 6$, $\alpha = \beta = 0$, and $\nu = 1.2, 1.3, 1.5, 1.7, 1.8$. The resultant errors are compared to those obtained from Refs. [23] and [24]. As it is observed the achieved results are more accurate than those reported by [23, 24]. Also, the plots of the exact and the approximate solutions are seen in Fig. 6.2 at $t = 0.5$ for $N = M = 9$, $\alpha = 3$, $\beta = 2$, and $\nu = 1.5$ The figures demonstrate good agreement between the numerical solution with the exact solution.

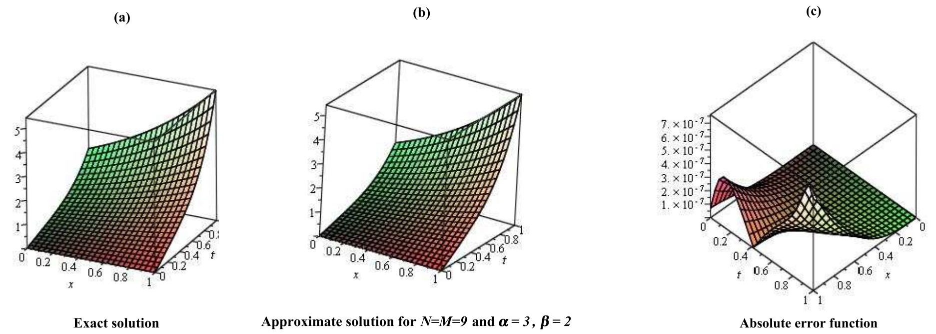

Fig. 6.1. Example 6.1: (a) Exact solution, (b) Approximate solution for $N = M = 9$, $\alpha = 3$, $\beta = 2$, and $\nu = 1.5$, (c) Absoulet error function.

Example 6.2. Consider the following linear time-fractional equation ([23]):

$$\frac{\partial^\nu u(x,t)}{\partial t^\nu} + \frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{2t^{2-\nu}}{\Gamma(2-\nu)} + 2x - 2, \quad 0 < x, t \le 1, \quad 0 < \nu \le 1, \tag{6.5}$$

with the initial and boundary conditions,

$$u(x,0) = x^2, \quad u(0,t) = t^2, \quad u_x(0,t) = 0.$$

The exact solution is $u(x,t) = x^2 + t^2$. Pursuing the procedure described in the previous section and substituting the appropriate approximations in Eq. (6.5), the following equation is obtained.

$$\mathbf{\Phi}^T(x,t)\,\mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\nu)^T}\,(\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\;T})^2\,C + \mathbf{\Phi}^T(x,t)\,\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\;T}\,\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\;T}\,C - \mathbf{\Phi}^T(x,t)\,\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\;T}\,C \approx \mathbf{\Phi}^T(x,t)\,F, \tag{6.6}$$

where

$$\frac{2t^{2-\nu}}{\Gamma(2-\nu)} - \frac{2t^{2-\nu}}{\Gamma(3-\nu)} \simeq \mathbf{\Phi}^T(x,t)\,F.$$

Using the Tau method for Eq. (43) leads to the following system.

$$\mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\nu)^T}\,(\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\;T})^2\,C + \mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\;T}\,\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\;T}\,C - \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\;T}\,CF \approx 0, \tag{6.7}$$

Table 6.3: Example 6.2: Maximum errors with $N = M = 4, 6$, $\alpha = \beta = 0$, and different values of ν.

	Jacobi	operational	method		
$N = M$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.5$	$\nu = 1.7$	$\nu = 1.8$
4	7.6270×10^{-5}	9.0340×10^{-5}	1.2603×10^{-4}	1.7205×10^{-4}	1.9892×10^{-4}
6	4.3855×10^{-7}	5.7996×10^{-7}	6.7874×10^{-7}	5.3830×10^{-7}	3.9252×10^{-7}
	Method	in	Ref. [23]		
$N = M$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.5$	$\nu = 1.7$	$\nu = 1.8$
4	1.15×10^{-3}	1.80×10^{-3}	3.55×10^{-3}	5.97×10^{-3}	7.41×10^{-3}
6	1.48×10^{-5}	1.48×10^{-5}	2.41×10^{-5}	1.87×10^{-5}	1.36×10^{-5}
	Method	in	Ref. [24]		
τ	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.5$	$\nu = 1.7$	$\nu = 1.8$
1/5	1.20×10^{-3}	7.21×10^{-3}	1.72×10^{-2}	1.16×10^{-4}	5.73×10^{-2}
1/10	3.03×10^{-4}	2.25×10^{-3}	6.21×10^{-3}	2.94×10^{-4}	2.57×10^{-2}

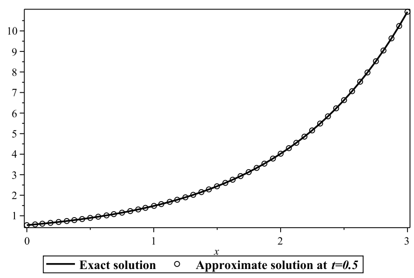

Fig. 6.2. Example 6.1: Exact and approximate solutions for $N = M = 9$, $\alpha = 3$, $\beta = 2$, $\nu = 1.5$, and $t = 0.5$.

where is a Sylvester equation. Eq. (6.7) is solved in [23] by means of a Tau Jacobi method, which applied shifted fractional-order Jacobi orthogonal functions. By choosing $N = M = 4$ and considering various values of the parameters α and β, approximate solutions are computed. The maximum absolute errors of the resultant solutions are seen in Table 6.4 for the values of $\nu = 0.5$ and $N = M = 4$. The data show that the approximate solutions are in good agreement with the exact solution. The absolute errors of the approximate solution obtained for $\alpha = \beta = 0.5$, $N = M = 4$, and $\nu = 0.5$ are computed at the points $x_i = 0.1i, 1 \leq i \leq 9$, and $t = 0.1, 0.5, 1$ and are listed in Table 6.5. The data of Table 6.5 are compared to the results reported by [23]. It is observed that the numerical solutions are in good agreement with the exact solutions and the results of the proposed method are more accurate than the results obtained from the methods in [23]. The plots of the approximate solutions and their error functions are depicted in Fig. 6.3.

Table 6.4: Example 6.2: Maximum errors for $N = M = 4$, $\nu = 0.5$, and different values of α and β.

(α,β)	Error	(α,β)	Error	(α,β)	Error
$(0,0)$	7.7483×10^{-17}	$(1,1)$	2.2082×10^{-17}	$(2,2)$	2.4223×10^{-16}
$(0.5,0.5)$	1.2673×10^{-18}	$(-0.5,0.5)$	2.3023×10^{-17}	$(0.5,-0.5)$	2.5065×10^{-18}
$(-1/3,-1/3)$	8.9134×10^{-19}	$(1/4,1/3)$	5.1893×10^{-19}	$(1/4,-1/3)$	6.8270×10^{-19}

Table 6.5: Example 6.2: Errors at $t = 0.1, 0.5, 1$ with $N = M = 4$, $\alpha = \beta = 0.5$, $\nu = 0.5$.

	Proposed method			Method in [23]		
x_i	$t = 0.1$	$t = 0.5$	$t = 1$	$t = 0.1$	$t = 0.5$	$t = 1$
0.1	9.30×10^{-20}	1.00×10^{-19}	1.00×10^{-19}	2.08×10^{-17}	1.04×10^{-16}	1.60×10^{-16}
0.2	1.55×10^{-19}	1.80×10^{-19}	3.00×10^{-19}	3.19×10^{-16}	1.94×10^{-16}	1.94×10^{-16}
0.3	1.99×10^{-19}	2.10×10^{-19}	3.00×10^{-19}	5.13×10^{-16}	3.61×10^{-16}	9.16×10^{-16}
0.4	2.50×10^{-19}	2.60×10^{-19}	4.00×10^{-19}	8.33×10^{-16}	5.97×10^{-16}	1.73×10^{-15}
0.5	3.10×10^{-19}	3.20×10^{-19}	5.00×10^{-19}	1.05×10^{-15}	6.94×10^{-16}	2.64×10^{-15}
0.6	4.20×10^{-19}	4.10×10^{-19}	7.00×10^{-19}	1.17×10^{-15}	6.94×10^{-16}	3.37×10^{-15}
0.7	5.60×10^{-19}	5.50×10^{-19}	1.00×10^{-18}	1.21×10^{-15}	6.80×10^{-16}	3.66×10^{-15}
0.8	7.40×10^{-19}	7.40×10^{-19}	1.40×10^{-18}	1.26×10^{-15}	7.77×10^{-16}	3.18×10^{-15}
0.9	9.50×10^{-19}	9.00×10^{-19}	1.80×10^{-18}	9.30×10^{-15}	7.01×10^{-16}	2.03×10^{-15}

Example 6.3. Consider the following linear time-space fractional convection-diffusion equation.

$$\frac{\partial^\nu u(x,t)}{\partial t^\nu} + \frac{\partial u(x,t)}{\partial x} - \frac{\Gamma(2.8)}{2}\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + u(x,t) = f(x,t), \qquad 0 < x, t \leq 1, \tag{6.8}$$

where $0 < \nu \leq 1$ and $1 < \gamma \leq 2$.The initial and boundary conditions are

$$u(x,0) = u(0,t) = u_x(0,t) = 0,$$

and the exact solution is $u(x,t) = x^2(1-x)t^2$. Substituting the suitable approximations in the Eq. (6.8) leads to the following equation.

$$\begin{aligned} &\mathbf{\Phi}^T(x,t)\ \mathbf{P}^{(\mathbf{1}-\nu)^T}_{(\mathbf{t})}\ ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^2\ C + \mathbf{\Phi}^T(x,t)\ {\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T\ {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\ C \\ &\qquad - \frac{\Gamma(2.8)}{2}\mathbf{\Phi}^T(x,t)\ \mathbf{P}^{(\mathbf{2}-\gamma)^T}_{(\mathbf{x})}\ {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\ C \approx \mathbf{\Phi}^T(x,t)\ F. \end{aligned} \tag{6.9}$$

Using the Tau method for Eq. (46) leads to the following system.

$$\mathbf{P}^{(\mathbf{1}-\nu)^T}_{(\mathbf{t})}\ ({\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T)^2\ C + {\mathbf{P}^{\mathbf{1}}_{(\mathbf{x})}}^T\ {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\ C - \frac{\Gamma(2.8)}{2}\mathbf{P}^{(\mathbf{2}-\gamma)^T}_{(\mathbf{x})}\ {\mathbf{P}^{\mathbf{1}}_{(\mathbf{t})}}^T\ C - F \approx 0. \tag{6.10}$$

By choosing $N = M = 4$ and considering various values of the parameters α and β, approximate solutions are computed. The maximum absolute errors of the resultant solutions are seen in

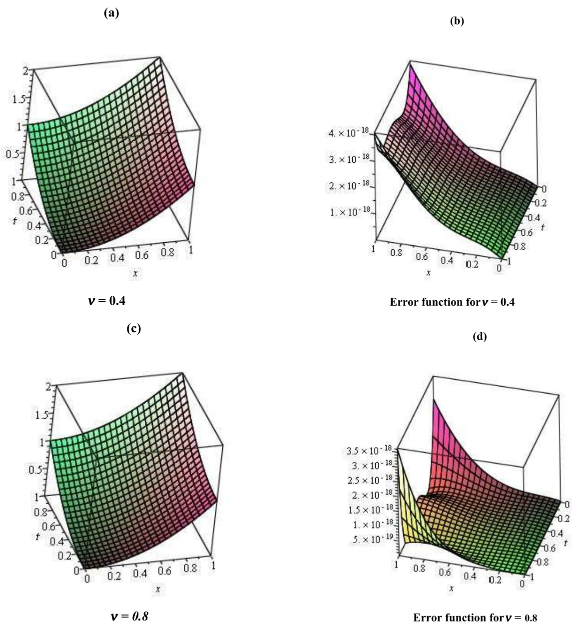

Fig. 6.3. Example 6.2: (a) Approximate solution for $\nu = 0.4$, (b) Error function, (c) Approximate solution for $\nu = 0.8$, (d) Error function for $N = M = 4$, $\alpha = \beta = 0.5$.

Table 6.6 for the values of $\nu = 0.5$, $\gamma = 1.5$, and $N = M = 4$. The data show that the approximate solutions are in good agreement with the exact solution. The absolute errors of the approximate solution obtained for $\alpha = \beta = 1$, $N = M = 4$, and various values of ν and γ are computed at the points $x_i = 0.1i, 1 \le i \le 9$, and $t = 0.2$ and are listed in Table 6.7. The plots of the approximate solution and the error function are depicted in parts (a)-(d) of Fig. 6.4 for $\alpha = 1/5$ and $\beta = -1/4$.

Table 6.6: Example 6.3: Maximum errors for $N = M = 4$, $\nu = 0.5$, $\gamma = 1.5$, and different values of α and β .

(α,β)	Error	(α,β)	Error	(α,β)	Error
$(0,0)$	6.7682×10^{-15}	$(1,1)$	1.1157×10^{-14}	$(2,2)$	3.4359×10^{-14}
$(0.5,0.5)$	3.8231×10^{-16}	$(-0.5,0.5)$	8.4200×10^{-16}	$(0.5,-0.5)$	1.8660×10^{-17}
$(1/3,1/3)$	6.3632×10^{-16}	$(-1/3,1/3)$	1.2548×10^{-15}	$(-1/4,-1/3)$	1.3944×10^{-16}

Example 6.4. Consider the linear inhomogeneous fractional KdV equation [25].

$$\frac{\partial^\nu u(x,t)}{\partial t^\nu} + \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = \frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos(t), \qquad 0 < x,t \le 1, \;\; 0 < \nu \le 1. \tag{6.11}$$

The initial and boundary conditions are

$$u(x,0) = 0, \;\; u(0,t) = 0, \;\; u_x(0,t) = 0, \;\; u_{xx}(0,t) = 0,$$

and the exact solution is $u(x,t) = t^2\cos(x)$. Substituting the suitable approximations in the

Table 6.7: Example 6.3: Absolute errors at $t = 0.2$ with $N = M = 4$, $\alpha = \beta = 1$, and various values of ν and γ.

	(ν, γ)	(ν, γ)	(ν, γ)
x_i	$(0.1, 1.1)$	$(0.8, 1.8)$	$(0.2.1.7)$
0.1	5.8159×10^{-18}	8.6733×10^{-18}	7.4638×10^{-18}
0.2	8.5252×10^{-18}	1.4166×10^{-17}	1.1295×10^{-17}
0.3	1.5173×10^{-18}	1.7868×10^{-17}	1.5506×10^{-17}
0.4	7.3162×10^{-18}	2.1556×10^{-17}	2.1133×10^{-17}
0.5	8.0745×10^{-18}	2.5375×10^{-17}	2.9036×10^{-17}
0.6	4.8050×10^{-19}	2.7836×10^{-17}	3.9893×10^{-17}
0.7	6.1202×10^{-19}	2.5819×10^{-17}	5.4207×10^{-17}
0.8	7.2428×10^{-19}	1.4570×10^{-17}	7.2301×10^{-17}
0.9	6.9161×10^{-19}	1.2294×10^{-17}	9.4318×10^{-17}

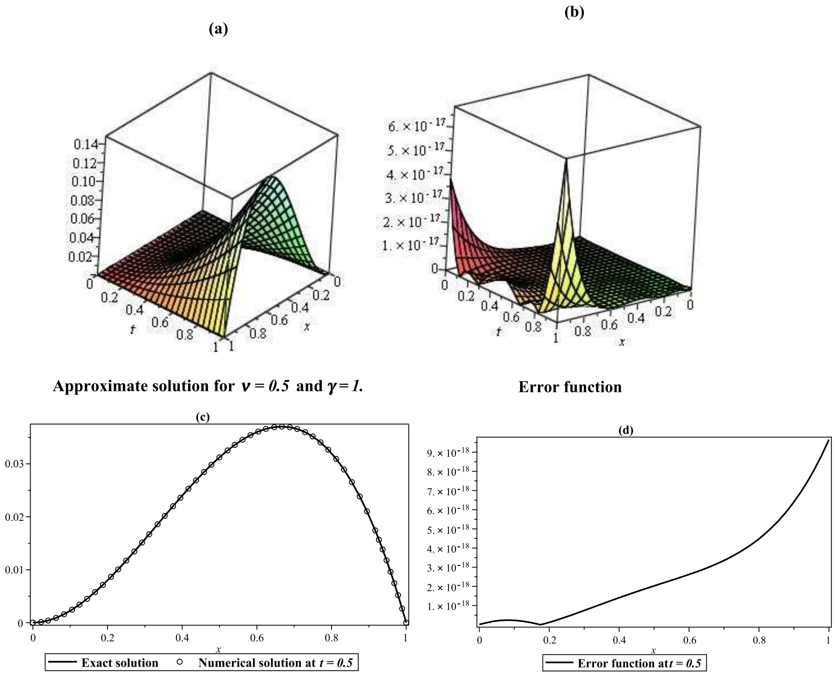

Fig. 6.4. Example 6.3: (a) Approximate solution, (b) Absolute error function, (c) Approximate solution at $t = 0.5$, (d) Absolute error function for $N = M = 4$, $\alpha = 1/5$, $\beta = -1/4$, $\nu = 0.5$, and $\gamma = 1.5$.

Eq. (6.11) leads to the following equation.

$$\begin{aligned}&\mathbf{\Phi}^T(x,t)\, \mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\nu)^T}\, ({\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}}}^T)^3\, C + \mathbf{\Phi}^T(x,t)\, ({\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}}}^T)^2\, {\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}}}^T\, C + \mathbf{\Phi}^T(x,t)\, {\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}}}^T\, C \\ \approx &\mathbf{\Phi}^T(x,t)\, F. \end{aligned} \tag{6.12}$$

Using the Tau method for Eq. (6.12) leads to the following Sylvester system.

$$\mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\nu)^T}\, ({\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}}}^T)^3\, C + ({\mathbf{P}_{(\mathbf{x})}^{\mathbf{1}}}^T)^2\, {\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}}}^T\, C + {\mathbf{P}_{(\mathbf{t})}^{\mathbf{1}}}^T\, C - F \approx 0.$$

By choosing $N = M = 7$ and considering various values of the parameters α and β, approximate solutions are computed. The maximum absolute errors of the resultant solutions are seen in Table 6.8 for the value of $\nu = 0.5$. The data show that the approximate solutions are in good agreement with the exact solution. Also, the maximum absolute errors are computed for different values of N and M ($N = M$). It is observed in Table 6.9 when N (M) increases the absolute error decreases. Decreasing the absolute error is also observed in Fig. 6.5 as well. The absolute errors of the approximate solution obtained for $\alpha = 1/2$, $\beta = -1/2$, $N = M = 7$, and various values of ν are computed at the points $x_i = t_i = 0.2i, i = 1, ..., 4$ and are listed in Table 6.10. The errors at the equally spaced points are almost constant while the values of ν are changing.

Table 6.8: Example 6.4: Maximum errors for $N = M = 7$, $\nu = 0.5$, and different values of α and β.

(α, β)	Error	(α, β)	Error
$(0, 0)$	1.6771×10^{-9}	$(1, 1)$	4.3165×10^{-9}
$(2, 2)$	7.7293×10^{-9}	$(0.5, 0.5)$	5.6746×10^{-9}
$(-0.5, 0.5)$	5.5008×10^{-9}	$(0.5, -0.5)$	5.6746×10^{-9}

Table 6.9: Example 6.4: Maximum errors for different values of N, M, $\nu = 0.5$, and $\alpha = \beta = 1$.

$N = M$	3	4	5	6	7
Error	8.1617×10^{-4}	3.14×10^{-5}	2.7392×10^{-6}	6.7861×10^{-8}	4.13165×10^{-9}

Table 6.10: Example 6.4: Errors for $N = M = 7$, $\alpha = 1/2$, $\beta = -1/2$, and varous values of ν.

$x_i = t_i$	$\nu = 0.2$	$\nu = 0.4$	$\nu = 0.6$	$\nu = 0.8$
0.2	7.2315×10^{-13}	7.2333×10^{-13}	7.2361×10^{-13}	7.2403×10^{-13}
0.4	4.3590×10^{-11}	4.3588×10^{-11}	4.3585×10^{-11}	4.3583×10^{-11}
0.6	1.5524×10^{-10}	1.5525×10^{-10}	1.5526×10^{-10}	1.5528×10^{-10}
0.8	4.8155×10^{-10}	4.8153×10^{-10}	4.8149×10^{-10}	4.8146×10^{-10}

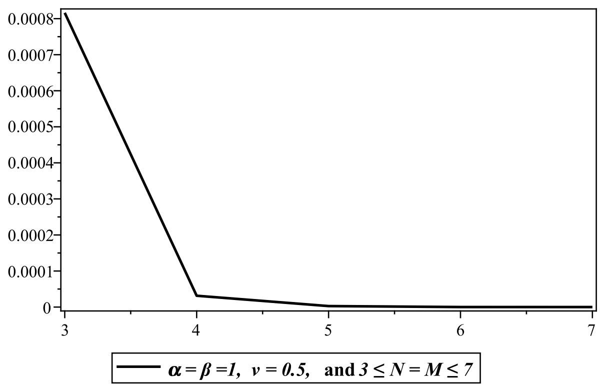

Fig. 6.5. Example 6.4: Maximum absolute errors for $\alpha = \beta = 1$, $\nu = 0.5$, and $N = M = 3 : 1 : 7$.

Example 6.5. Consider the time-space fractional differential equation [26].

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + \frac{\partial^\nu u(x,t)}{\partial t^\nu} = \frac{2x^{2-\gamma}(t^2+1)}{\Gamma(3-\gamma)} + \frac{2(x^2+1)t^{2-\nu}}{\Gamma(3-\nu)}, \quad 0 < x,t \leq 1, \ \ 0 < \nu, \gamma \leq 1, \quad (6.13)$$

such that $u(x,0) = x^2+1, u(0,t) = t^2+1$, and the exact solution is $u(x,t) = (x^2+1)(t^2+1)$. Substituting the suitable approximations in the Eq. (6.13), the following equation is obtained.

$$\mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\nu)^T}\ \mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T}\ C + \mathbf{\Phi}^T(x,t)\ \mathbf{P}_{(\mathbf{x})}^{(\mathbf{1}-\gamma)^T}\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ C \approx \mathbf{\Phi}^T(x,t)\ F. \quad (6.14)$$

Using the Tau method for Eq. (6.14) leads to the following Sylvester system.

$$\mathbf{P}_{(\mathbf{t})}^{(\mathbf{1}-\nu)^T}\ \mathbf{P}_{(\mathbf{x})}^{\mathbf{1}\ T}\ C + \mathbf{P}_{(\mathbf{x})}^{(\mathbf{1}-\gamma)^T}\ \mathbf{P}_{(\mathbf{t})}^{\mathbf{1}\ T}\ C - F \approx 0. \quad (6.15)$$

Table 6.11: Example 6.5: Maximum errors for N=M=4, $\nu = 1/3$, $\gamma = 1/2$, and different values of α and β.

(α,β)	Error	(α,β)	Error
$(0,0)$	2.5000×10^{-19}	$(1,1)$	3.0000×10^{-19}
$(2,2)$	3.0000×10^{-19}	$(0.5,0.5)$	4.0000×10^{-19}
$(-0.5,0.5)$	4.1000×10^{-19}	$(0.5,-0.5)$	5.2000×10^{-19}
$(-1/4,-1/4)$	9.3500×10^{-18}	$(1/3,1/2)$	1.2140×10^{-17}

Table 6.12: Example 6.5: Errors for $N = M = 4$, $\alpha = -1/2$, $\beta = 1$, $\nu = 1/3$, and $\gamma = 1/2$.

	Jacobi	method	Method in [26]
(x,t)	$M = N = 4$	$M = N = 5$	$m = 8$
$(0,0)$	0.0000	0.0000	5.415112×10^{-6}
$(1/8,1/8)$	0.0000	0.0000	2.774770×10^{-5}
$(2/8,2/8)$	1.0000×10^{-19}	1.0000×10^{-19}	5.600267×10^{-5}
$(3/8,3/8)$	0.0000	1.0000×10^{-19}	4.902546×10^{-5}
$(4/8,4/8)$	0.0000	0.0000	7.292534×10^{-5}
$(5/8,5/8)$	1.0000×10^{-19}	1.0000×10^{-19}	3.242363×10^{-5}
$(6/8,6/8)$	1.0000×10^{-19}	1.0000×10^{-19}	6.928689×10^{-5}
$(7/8,7/8)$	0.0000	1.0000×10^{-19}	7.414284×10^{-5}

By choosing $N = M = 4$, 25 algebraic equations are generated by Eq. (6.15). Determining various values of the parameters α and β leads to approximate solutions. The maximum absolute errors of the resultant solutions are seen in Table 6.11 for the values of $\nu = 1/3$ and $\gamma = 1/2$. The data show that the approximate solutions are in good agreement with the exact solution. The absolute errors of the approximate solution obtained for $\alpha = -1/2$, $\beta = 1$, $N = M = 4$, and the values of $\nu = 1/3$ and $\gamma = 1/2$ are computed at the points $x_i = t_i = i/8, 0 \leq i \leq 7$ and are listed in Table 6.12. Eq. (6.13) is solved in [26] by means of the Haar wavelet method. The last column of Table 6.12 displays the results reported by [26]. It is seen that the proposed method provides more precise solutions in comparison with those obtained from the Haar wavelet method. The plots of the exact and approximate solutions are depicted in Fig. 6.6 at $t = 0.25$.

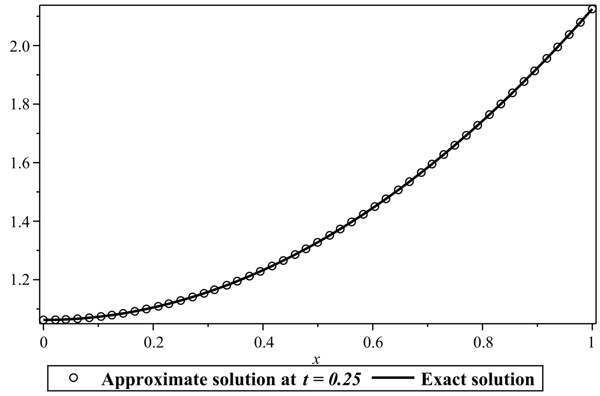

Fig. 6.6. Example 6.5: Exact and approximate solutions for $\alpha = -1/2$, $\beta = 1$, $\nu = 2/3$, $\gamma = 3/4$, and $N = M = 4$.

7. Conclusion

The concern of this paper was the solution of a class of fractional partial differential equations. It was done by presenting a scheme which was a combination of the Tau spectral method and the Jacobi polynomials based on the operational matrices. First, the operational matrices of the integration of the fractional and integer orders derived for the one-dimensional shifted Jacobi polynomials, then the obtained matrices were generalized to the two-dimensional case. The introduced approximations were different from those presented in [23]. The proposed method applied to solve several time- and time-space fractional partial differential equations to demonstrate the efficiency of the approach. The results was compared to those reported in [23-26]. As observed from Tables 6.3, 6.5, and 6.12, the obtained results from the suggested algorithm possess more accurate than those obtained from spectral Tau, Variational iteration, and Haar wavelet and methods, [23, 25, 26]. The authors intend to apply the proposed method to fractional integro-partial differential equations in their future works.

Acknowledgements. The authors are grateful to the anonymous referee for careful checking of the details and for helpful comments that improved this paper. This Postdoctoral research is financially supported by the Iran National Science Foundation: INFS under Grant No. 95009788 and is also under supplementary support of The University of Guilan, Iran.

References

[1] R. Gorenflo and F. Mainardi, Random walk models for space-fractional diffusion processes, *Fract. Cal. Appl. Anal.*, **1** (1998), 167–191.
[2] M. Giona and H.E. Roman, Fractional diffusion equation for transport phenomena in random media, *Phys A*, **185** (1992), 87–97.
[3] R. Metzler and J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77.
[4] K. Seki, M. Wojcik and M Tachiya, Fractional reaction diffusion equation, *J. Chem. Phys.*, **119** (2003), 2165–2174.
[5] B.I. Henry and S.L. Wearne, Fractional reaction diffusion, *Phys A*, **276** (2000), 448-455.

[6] G.E. Farin, J. Hoschek and M.S. Kim, Handbook of Computer Aided Geometric Design. Elsevier, Amsterdam, 2002.

[7] F. Mainardi, Fractals and Fractional Calculus Continuum Mechanics, Springer Verlag, (1997), 291–348.

[8] K. Diethelm and A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity, in: Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer Verlag, Heidelberg, (1999), 217–224.

[9] W.M. Ahmad and El-Khazali R, Fractional-order dynamical models of love, *Chaos. Soliton. Fract.*, **33** (2007), 1367–1375.

[10] E.E. Adams and L.W. Gelhar, Field study of dispersion in a heterogeneous aquifer: 2. Spatial moment analysis. *Water Resour. Res.*, **28** (1992), 3293–3307.

[11] D.A. Benson, S. Wheatcraft and M.M. Meerschaert, Application of a fractional advection-dispersion equation. *Water Resour. Res.*, **36** (2000), 1403–1412.

[12] M.M. Meerschaert, D.A. Benson and S.W. Wheatcraft, Subordinated advection-dispersion equation for contaminant transport. *Water Resour. Res.*, **37** (2001), 1543–1550.

[13] E. Scalas, R. Gorenflo and F. Mainardi, Fractional calculus and continuous time finance. *Phys. A*, **284** (2000), 376–384.

[14] A. Baseri, E. Babolian and S. Abbasbandy, Normalized Bernstein polynomials in solving space-time fractional diffusion equation, Adv. Differ. Equ., 346 (2017), 1-25. DOI: 10.1186/s13662-017-1401-1.

[15] C.M. Chen, F. Liu and K. Burrage, Finite difference methods and a fourier analysis for the fractional reactionsubdiffusion equation, *App. Math. Comput*, **198** (2008), 754–769.

[16] Y. Chen, Y. Wu, Y. Cui, Z. Wang and D. Jin, Wavelet method for a class of fractional convection-diffusion equation with variable coefficients, *J. Comput. Sci.*, **1** (2010), 146–149.

[17] Y. Chen, Y. Yannan Sun and L. Liu, Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order Legendre functions, *App. Math. Comput.*, **244** (2014), 847–858.

[18] H. Ding, General Padé approximation method for time-space fractional diffusion equation, J. Comput. App. Math., **299** (2017), 221–228.

[19] Heydaria, M.H., Hooshmandasla, M.R., Maalek Ghainia, F.M. and Cattanic, C., (2015), Wavelets method for the time fractional diffusion-wave equation, *Phys. Lett. A*, **379** (2016), 71–76.

[20] V.K. Patel, S. Singh and V.K. Singh, Two-dimensional wavelets collocation method for electromagnetic waves in dielectric media, *J. Comput. App. Math.*, **317** (2017), 307–330.

[21] A. Borhanifar and K. Sadri, A generalized operational method for solving integro-partial differential equations based on Jacobi polynomials, *Hacet. J. Math. Stat.*, **45**:2 (2016), 311–335.

[22] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons. Inc., Canada, 1978.

[23] A.H. Bhrawy, E.H. Doha, D. Baleanu and S.S. Ezz-Eldien, A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations, *J. of Comput. Phys.*, **293** (2015), 142–156.

[24] X. Hu and L. Zhang, On finite difference methods for fourth-order fractional diffusion-wave and subdiffusion systems, *Appl. Math. Comput.*, **218** (2012), 5019-5034.

[25] S. Momani, Z. Odibat and A. Alawneh, Variational Iteration Method for Solving the Space- and Time-Fractional KdV Equation, *Numer. Meth. Part. D. E.*, **24**:1 (2007), 262–271.

[26] L. Wang, Y. Ma and Z. Meng, Haar wavelet method for solving fractional partial differential equations numerically, *Appl. Math. Comput.*, **227** (2014), 66–76.