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MIXED FINITE ELEMENT METHODS FOR FRACTIONAL NAVIER-STOKES EQUATIONS*

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Abstract

This paper gives the detailed numerical analysis of mixed finite element method for fractional Navier-Stokes equations. The proposed method is based on the mixed finite element method in space and a finite difference scheme in time. The stability analyses of semi-discretization scheme and fully discrete scheme are discussed in detail. Furthermore, We give the convergence analysis for both semidiscrete and fully discrete schemes and then prove that the numerical solution converges the exact one with order $O(h^2+k)$, where h and k respectively denote the space step size and the time step size. Finally, numerical examples are presented to demonstrate the effectiveness of our numerical methods.

Mathematics subject classification: 60N15, 65M60, 60N30, 75D05.
Key words: Time-fractional Navier-Stokes equations, Finite element method, Error estimates; Strong convergence.

1. Introduction

The purpose of the present paper is to study the error estimates of the mixed finite element method for the incompressible fractional Navier-Stokes equations

$$\begin{cases} u_t + \mathscr{B}^\alpha \mathscr{L} u + u\cdot\nabla u + \nabla p = f, & \text{in } \ \Omega\times[0,T], \\ \nabla\cdot u = 0, & \text{in } \ \Omega\times[0,T], \\ u(x,0) = u_0, & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega\times[0,T], \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded and connected polygonal domain, u represents the velocity field, p is the associated pressure, u_0 is the initial velocity and f is an external force, $\mathscr{L}u = -\nu\triangle u$ ($\nu>0$ is the viscosity coefficient), $\mathscr{B}^\alpha := {}^R D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative in time defined by: for $0<\alpha<1$,

$$\mathscr{B}^\alpha\varphi(t) := \frac{\partial}{\partial t}\mathscr{I}^\alpha\varphi(t) := \frac{\partial}{\partial t}\int_0^t \omega_\alpha(t-s)\varphi(s)ds \ \text{ with } \ \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{1.2}$$

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with $\mathscr{I}^{\alpha}$ being the temporal Riemann-Liouville fractional integral operator of order α.

The above-mentioned problem has many physical applications in many areas such as heterogeneous flows and materials, turbulence, viscoelasticity and electromagnetic theory. Particularly when $\alpha = 1$, the problem (1.1) reduces to the classical Navier-Stokes equations, numerical approximations of which have been studied by many authors [1–4, 8–19, 25, 32, 34, 37–44, 46, 47]. However, for the fractional Navier-Stokes equations (FNSE) which are nonlinear in character, most of them do not have exact analytical solutions. It is shown that very few cases in which the exact solution of fractional Navier-Stokes equations can be obtained, where it have to make certain assumptions about the state of the fluid and a simple configuration for the flow pattern is to be considered. Hence it is necessary to analyze and study the approximation and numerical techniques of FNSE. However, to our best knowledge, numerical analysis of such problem for fractional Navier-Stokes equations is missing except [27, 56] in the literature. Therefore, this article aims to fill the gap, study and obtain the strong convergence approximations of FNSE like (1.1).

In recent years, there have been numerous studies on fractional diffusion equation. Lin and Xu [31] have proposed the finite difference scheme in time and Legendre spectral methods in space for the time-fractional diffusion equation. Deng [6] has established the stability and error estimates for the time fractional Fokker-Planck equation and then proved that the convergent order is $O(k^{2\alpha}+h^{\mu})$, where k is the time step size and h is the space step size. Liu et al. [35] have developed a two-grid algorithm based on the mixed finite element method for a nonlinear fourth-order reaction-diffusion equation with the time-fractional derivative of Caputo-type. Jin et al. [21], by using piecewise linear functions, have studied two semidiscrete approximation schemes, i.e., Galerkin finite element method and lumped mass Galerkin finite element method, for the homogeneous time-fractional diffusion equation. Zeng et al. [51] have studied the second-order accurate schemes for the time-fractional diffusion equation with unconditional stability based on finite element method in space and the fractional linear multistep methods in time. Besides, some other interesting works in this aspect can be found in [5, 7, 20, 22–24, 30, 33, 36, 45, 52–55, 57].

In this article, our goal is to give some detailed numerical analysis of the mixed finite element method for the problem (1.1). On one hand, the discretization in space is done by the mixed finite element method. First of all, the velocity is split into two parts by introducing a linearized discrete problem with solution v_h. In particular, Motivated by the Ritz-Volterra projection, we then introduce the fractional Stokes-Volterra projection S_hu, the role of which is similar to that of a Ritz projection in treating the heat equation. Subsequently, with virtue of the property of the operator E_h as well as the standard duality arguments, the L^2-error estimate for the velocity is shown. On the other hand, firstly following the idea of Zhuang et al. [53] that has discretized the Riemann-Liouville fractional derivative $\mathscr{B}^{\alpha}$ in time, then we adopt the finite difference method and obtain the stability and convergence properties related to the time discretization. The stability analyses of semi-discretization scheme and fully discrete scheme are discussed in detail. Furthermore, We give the convergence analysis for both semidiscrete and fully discrete schemes and prove that the numerical solution converges the exact one with order $O(h^2 + k)$, where h and k respectively the space step size and the time step size.

The structure of this paper is as follows: In section 2, we introduce some preliminaries and notations, give the definition of the Mittag-Leffler function. In Section 3, we introduce the notations for finite element spatial semidiscretization, describe the semidiscrete Galerkin approximations about space and establish the error estimate for the velocity. In Section 4, we present several lemmas which play a crucial role in the proof of the error estimate of the time discretization. We firstly give the fully discrete scheme for (1.1) and then obtain the error estimate for the fully discrete scheme. Finally, in Section 5, numerical examples are presented to demonstrate the effectiveness of our numerical methods.

2. Preliminaries

Throughout the paper, we denote as C a constant that may not be of the same form from one occurrence to another, even in the same line. In this section, we introduce some notations and recall the definition of Mittag-Leffler function .

We use the standard notation $H^s(\Omega), \|\cdot\|_s, (\cdot,\cdot)_s, s\geq 0$ for the Sobolev spaces, the standard Sobolev norm and inner product, respectively. When $s=0$, $L^2(\Omega)$ is written instead of $H^0(\Omega)$, the L_2-inner product and L_2-norm are separately denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. For the mathematical setting of problem (1.1), the following spaces

$$X=H_0^1(\Omega)^2,\quad Y=L^2(\Omega)^2,\quad M=L_0^2(\Omega)=\Big\{q\in L^2(\Omega):\int_\Omega q dx=0\Big\},$$

are introduced. Next, let the closed subset V of X be given by

$$V=\Big\{v\in X,\quad \text{div } v=0\Big\},$$

and denote by H the closed subset of Y, i.e.,

$$H=\Big\{v\in Y,\quad \text{div } v=0, v\cdot n|_{\partial\Omega}=0\Big\}.$$

Moreover, the continuous bilinear form $d(\cdot,\cdot)$ on $X\times M$ and the bilinear operator $B(\cdot,\cdot)$ on $X\times X$ are separately defined by

$$\begin{aligned} d(\phi,p)&=(\text{div}\phi,p), && \forall\phi\in X,\quad \forall p\in M,\\ B(u,v)&=(u\cdot\nabla)v+\frac{1}{2}(\text{div}u)v, && \forall u,v\in X. \end{aligned}$$

At the same time, a trilinear form $b(\cdot,\cdot,\cdot)$ on $X\times X\times X$ is introduced by

$$b(u,v,w)=(u\cdot\nabla v,w),\qquad u,v,w\in X,$$

which has the following properties (cf., [13] [10]):

$$\begin{aligned} &b(u,v,w)=-b(u,w,v),\qquad b(u,v,v)=0, && \forall u,v,w\in X,\\ &\|b(u,v,w)\|\leq M\|\nabla u\|\|\nabla v\|\|\nabla w\|, && \forall u,v,w\in X. \end{aligned}\tag{2.1}$$

For the readers convenience, we give the definition of Mittag-Leffler function. We shall use extensively the Mittag-Leffler function $E_{\alpha,\beta}(z)$ [26] defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, z\in\mathbb{C},$$

where $\Gamma(.)$ is the standard Gamma function defined as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\text{d}t, \Re(z)>0.$$

3. Space Semi-Discretization

In this section, we will give the variational formulation of (1.1), describe the semidiscrete Galerkin approximations and then derive the error estimates for the velocity about space discretization. From now on, we denote by h with $0 < h < 1$ a real positive discretization parameter tending to zero.

3.1. Semidiscrete fractional Navier-Stokes equations

Let $A(\cdot,\cdot) : X \times X \to \mathbb{R}$ be the bilinear form associated with the operator $\mathscr{L} = -\nu\triangle$ which is symmetric and positive definite on X. With the notations in Section 2, the variational formulation of (1.1) is as follows: find $(u,p) \in (X,M)$ for all $t \in [0,T]$ such that for all $(\phi,q) \in (X,M)$

$$\begin{cases} (u_t,\phi) + A(\mathscr{B}^\alpha u,\phi) + b(u,u,\phi) - d(\phi,p) + d(u,q) = (f,\phi), \\ u(x,0) = u_0. \end{cases} \tag{3.1}$$

We introduce the finite element subspace (X_h, M_h) of $(X,M), Y_h \subset Y$ and define the subspace V_h of X_h given by

$$V_h = \Big\{ v_h \in X_h, \quad \mathrm{div} v_h = 0 \Big\}.$$

We assume that the couple (X_h, M_h) satisfies the discrete LBB(or named inf-sup) condition

$$\sup_{v_h \in X_h} \frac{(\varphi_h, \nabla\cdot v_h)}{\|\nabla v_h\|} \geq \beta\|\varphi_h\|, \quad \forall \varphi_h \in M_h, \tag{3.2}$$

where $\beta > 0$ is a constant.

Let $P_h : Y \to V_h$ denotes the L^2-orthogonal projection defined by

$$(P_h v, v_h) = (v, v_h), \quad v \in Y, \quad v_h \in V_h.$$

For the space M_h we assume that, for each $q \in H^1(\Omega) \cap M$, there exists an approximation $j_h q \in M_h$ such that

$$\|q - j_h q\| \leq ch\|q\|_1,$$

where c is a positive constant, independent of h.

The operator $E_h(t)$ is introduced by

$$E_h(t)v_h = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j^h t^\alpha)(v,\varphi_j^h)\varphi_j^h, \quad v_h \in X_h, \tag{3.3}$$

where $\{\lambda_j^h\}_{j=1}^N$ and $\{\varphi_j^h\}_{j=1}^N$ are respectively the eigenvalues and the eigenfunctions of the discrete Laplace operator $\mathscr{F} := -\nu\Delta_h$ defined by

$$-(\nu\Delta_h\psi,\chi) = \nu(\nabla\psi,\nabla\chi), \quad \forall\psi,\chi \in X_h.$$

For later use, we need the regularity result which is related to the operator $E_h(t)$ and collect the result in the next lemma.

Lemma 3.1 ([22]). *Let $E_h(t)$ be defined by* (3.3) *and $\psi \in X_h$. Then it holds that*

$$\|E_h(t)\psi\|_p \leq ct^{\frac{-\alpha(p-q)}{2}}\|\psi\|_q, \tag{3.4}$$

where $\alpha \in (0,1), q \leq p$ and $0 \leq p-q \leq 2$.

The discrete analogue of weak formulation (3.1) now reads as follows: find $(u_h, p_h) \in (X_h, M_h)$ such that for all $(\phi_h, q_h) \in (X_h, M_h)$,

$$(u_{ht}, \phi_h) + A(\mathscr{B}^\alpha u_h, \phi_h) + b(u_h, u_h, \phi_h) - d(\phi_h, p_h) + d(u_h, q_h) = (f, \phi_h), \tag{3.5}$$

with $u_h(0) = P_h u_0$.

For the discrete approximation, it is straightforward to verify that the trilinear term $b(u_h, v_h, w_h)$ enjoys the following properties (cf., [14]):

$$b(u_h, v_h, w_h) = -b(u_h, w_h, v_h), \qquad b(u_h, v_h, v_h) = 0, \qquad \forall u_h, v_h, w_h \in X_h, \tag{3.6}$$

$$\|b(u_h, v_h, w_h)\| \leq M\|\nabla u_h\|\|\nabla v_h\|\|\nabla w_h\|, \qquad \forall u_h, v_h, w_h \in X_h. \tag{3.7}$$

3.2. Error estimate for the velocity

With V_h as above, we now introduce an equivalent Galerkin formulation. Find $u_h \in V_h$ such that $u_h(0) = P_h u_0$ and for $t > 0$

$$(u_{ht}, \phi_h) + A(\mathscr{B}^\alpha u_h, \phi_h) + b(u_h, u_h, \phi_h) = (f, \phi_h), \qquad \forall \phi_h \in V_h. \tag{3.8}$$

For $f \in L^2(\Omega)$, problem (3.8) is uniquely solvable for all $t > 0$. Indeed, setting $\phi_h = u_h$ in (3.8) and recalling the property $b(u_h, u_h, u_h) = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + A(\mathscr{B}^\alpha u_h, u_h) = (f, u_h). \tag{3.9}$$

On integration with respect to the temporal variable t and using the following property $\mathscr{B}^\alpha$ has satisfied ([28])

$$\int_0^t A(\mathscr{B}^\alpha u_h, u_h) ds \geq c_\alpha T^{\alpha-1}\int_0^t \|\nabla u_h\|^2 ds, \tag{3.10}$$

we observe that

$$\|u_h(t)\| \leq \|u_h(0)\| + t\|f\|. \tag{3.11}$$

Thus, by classical theorems of the theory of ordinary differential equations, (3.8) has a unique global solution $u_h(t), t > 0$.

The next theorem summarizes the error bound about space discretization.

Theorem 3.1 (Error estimate for space discretization). *Let u and u_h be the solutions of* (3.1) *and* (3.8)*, respectively, then the following estimate holds*

$$\|u - u_h\| \leq Ch^2,$$

where $C = c(\|u\|_2 + \mathscr{I}^{1-\alpha}\|p\|_1 + \|u_t\|_2 + \mathscr{I}^{1-\alpha}\|p_t\|_1)$.

We firstly dissociate the non-linearity by introducing an intermediate solution v_h. Let v_h be a finite element Galerkin approximation to a linearised time-fractional Navier-Stokes equation satisfying

$$(v_{ht}, \phi_h) + A(\mathscr{B}^\alpha v_h, \phi_h) + b(u, u, \phi_h) = (f, \phi_h), \qquad \forall \phi_h \in V_h, \tag{3.12}$$

with $v_h(0) = P_h u_0$. Subsequently, the error is split as

$$e := u - u_h = (u - v_h) + (v_h - u_h) =: \xi + \eta.$$

Note that ξ is the error committed by approximating a linearized (Stokes) problem and η represents the error due to the presence of the nonlinearity in the equation.

Define the fractional Stokes-Volterra projection $S_h u \in V_h$ such that

$$A(\mathscr{B}^\alpha (u - S_h u), \phi_h) = (p, \nabla \cdot \phi_h), \qquad \phi_h \in V_h,$$

that is,

$$A((u - S_h u), \phi_h) = \mathscr{I}^{1-\alpha}(p, \nabla \cdot \phi_h), \qquad \phi_h \in V_h, \tag{3.13}$$

where $\mathscr{I}^{1-\alpha}$ denotes the temporal fractional integral operator with singular kernel $t^{-\beta}/\Gamma(1-\beta)$. The form of the above projection is motivated by the Ritz-Volterra projection, see Lin [29] for parabolic integro-differential equations and by the elliptic projection introduced by Wheeler [49] for parabolic initial and boundary value problems.

Below, We now decompose ξ as

$$\xi := (u - S_h u) + (S_h u - v_h) =: \zeta + \theta.$$

First of all, we derive the error bound for $u - S_h u$.

Lemma 3.2. *For the fractional Stokes-Volterra projection $S_h u$, we have*

$$\|u - S_h u\| + h\|\nabla(u - S_h u)\| \le ch^2\Big(\|u\|_2 + \mathscr{I}^{1-\alpha}\|p\|_1\Big), \tag{3.14}$$

$$\|(u - S_h u)_t\| + h\|\nabla(u - S_h u)_t\| \le ch^2\Big(\|u_t\|_2 + \mathscr{I}^{1-\alpha}\|p_t\|_1\Big). \tag{3.15}$$

Proof. Setting $\phi_h = P_h u - S_h u$ in (3.13) yields

$$\begin{aligned}\nu\|\nabla(u - S_h u)\|^2 &= A(u - S_h u, u - S_h u)\\ &= A(u - S_h u, u - P_h u) + \mathscr{I}^{1-\alpha}(p - j_h p, \nabla \cdot (P_h u - S_h u)),\end{aligned}$$

and hence

$$\|\nabla(u - S_h u)\| \le ch\Big(\|u\|_2 + \mathscr{I}^{1-\alpha}\|p\|_1\Big).$$

Now we introduce the following dual problem,

$$-\Delta w + \nabla z = u - S_h u,$$

then it satisfies the following regularity result

$$\|w\|_2 + \|z\|_1 \le c\|u - S_h u\|.$$

Hence we obtain

$$\begin{aligned}
&\|u-S_hu\|^2=(-\Delta w,u-S_hu)+(\nabla z,u-S_hu)\\
=&\frac{1}{\nu}A(u-S_hu,w)-(z,\nabla\cdot(u-S_hu))\\
=&\frac{1}{\nu}A(u-S_hu,w-P_hw)+\frac{1}{\nu}\mathscr{I}^{1-\alpha}(p-j_hp,\nabla\cdot P_hw)-(z,\nabla\cdot(u-S_hu))\\
=&\frac{1}{\nu}A(u-S_hu,w-P_hw)+\frac{1}{\nu}\mathscr{I}^{1-\alpha}(p-j_hp,\nabla\cdot P_hw)-(z-j_hz,\nabla\cdot(u-S_hu))\\
\le& ch\|\nabla(u-S_hu)\|(\|w\|_2+\|z\|_1)+ch^2\mathscr{I}^{1-\alpha}\|p\|_1\|w\|_2\\
\le& ch\|\nabla(u-S_hu)\|\|u-S_hu\|+ch^2\mathscr{I}^{1-\alpha}\|p\|_1\|u-S_hu\|.
\end{aligned}$$

This clearly implies (3.14). The second inequality (3.15) is proved similarly. □

Now we are in a position to estimate ξ. Since $\xi=\zeta+\theta$ and the estimates of ζ are known from the Lemma 3.2, it is sufficient to estimate θ.

Lemma 3.3. *For $\xi=u-v_h$, it satisfies the following estimate*

$$\|\xi\|\le ch^2\Big(\|u\|_2+\mathscr{I}^{1-\alpha}\|p\|_1+\|u_t\|_2+\mathscr{I}^{1-\alpha}\|p_t\|_1\Big).$$

Proof. Subtracting (3.12) from (3.1), the equation in ξ is written as

$$(\xi_t,\phi_h)+A(\mathscr{B}^\alpha\xi,\phi_h)=(p,\nabla\cdot\phi_h),\qquad \phi_h\in V_h.$$

To complete the estimate for ξ, we only need to estimate θ. The equation in θ reads as

$$(\zeta_t,\phi_h)+(\theta_t,\phi_h)+A(\mathscr{B}^\alpha\zeta,\phi_h)+A(\mathscr{B}^\alpha\theta,\phi_h)=(p,\nabla\cdot\phi_h),\qquad \forall\phi_h\in V_h.$$

Making use of the definition of S_hu, then the above equation can be simplified as

$$(\theta_t,\phi_h)+A(\mathscr{B}^\alpha\theta,\phi_h)=-(\zeta_t,\phi_h).$$

Let $\phi_h=\theta$ in the above equation, there holds

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2+A(\mathscr{B}^\alpha\theta,\theta)=-(\zeta_t,\theta).$$

On integration with respect to the temporal variable t, we obtain that

$$\|\theta\|^2+\int_0^t A(\mathscr{B}^\alpha\theta,\theta)ds\le\|\theta(0)\|^2+\int_0^t\|\zeta_t\|\|\theta\|ds.$$

With (3.10), we derive

$$\|\theta\|\le\|\theta(0)\|+\int_0^t\|\zeta_t\|ds.$$

By virtue of Lemma 3.2, we obtain

$$\|\theta\|\le ch^2\Big(\|u_t\|_2+\mathscr{I}^{1-\alpha}\|p_t\|_1\Big).$$

By the triangle inequality, we have

$$\|\xi\| \leq \|\zeta\| + \|\theta\| \leq ch^2\Big(\|u\|_2 + \mathscr{I}^{1-\alpha}\|p\|_1 + \|u_t\|_2 + \mathscr{I}^{1-\alpha}\|p_t\|_1\Big),$$

which completes the proof. □

Subsequently, we give the main result in this section. By making use of the properties of the operator E_h and the standard duality arguments, the methods of which are different from classical Navier-Stokes equations, we derive the error estimate for the velocity.

Proof of Theorem 3.1. Since $e = u - u_h = (u - v_h) + (v_h - u_h) = \xi + \eta$ and the estimate of ξ is known from Lemma 3.3, it is enough to estimate η. From (3.8) and (3.12), the equation in η becomes

$$(\eta_t, \phi_h) + A(\mathscr{B}^\alpha \eta, \phi_h) + b(u, u, \phi_h) - b(u_h, u_h, \phi_h) = 0,$$

with $\eta(0) = 0$.

Equivalently, the above equation can be recast as

$$\eta_t + \mathscr{B}^\alpha \mathscr{L}\eta + B(u, u) - B(u_h, u_h) = 0, \qquad \eta(0) = 0.$$

By Duhamel's principle(cf. [48]) and Lemma 3.1 with $p = 1, q = 0$, one can derive that

$$\begin{aligned} \|\eta\| &= \left\| \int_0^t E_h(t-s)(B(u,u) - B(u_h, u_h))ds \right\| \\ &\leq \int_0^t \|\mathscr{F}^{1/2} E_h(t-s)\mathscr{F}^{-1/2}(B(u,u) - B(u_h, u_h))\|ds \\ &\leq \int_0^t (t-s)^{-\alpha/2}\|\mathscr{F}^{-1/2}(B(u,u) - B(u_h,u_h))\|ds, \end{aligned} \tag{3.16}$$

where $\mathscr{F}$ denotes the discrete Laplace operator. Thus it is enough to estimate $\|\mathscr{F}^{-1/2}(B(u,u) - B(u_h, u_h))\|$. We proceed by the standard duality arguments, using the splitting

$$B(u,u) - B(u_h, u_h) = B(u, e) + B(e, u_h).$$

By the triangle inequality, it yields

$$\|\mathscr{F}^{-1/2}(B(u,u) - B(u_h,u_h))\| \leq \|B(u,e)\|_{-1} + \|B(e,u_h)\|_{-1}, \tag{3.17}$$

so that the proof is reduced to estimate each of the above negative norms on the right-hand side. Using the skew-symmetry property (3.6) and noticing div$u = 0$, one obtains for the first term:

$$\begin{aligned} \|B(u,e)\|_{-1} &= \sup_{\|\phi\|=1} \left| -((u\cdot\nabla)\phi, e) - \frac{1}{2}((\nabla\cdot u)\phi, e) \right| \\ &\leq \sup_{\|\phi\|=1} \Big(\|e\|\|u\|_\infty \|\phi\|_1 + \|e\|\|\nabla\cdot u\|_{L^4}\|\phi\|_{L^4} \Big) \leq C\|e\|. \end{aligned}$$

Regarding the other term in (3.17), we derive

$$\begin{aligned} \|B(e,u_h)\|_{-1} &= \sup_{\|\phi\|=1} \left| \frac{1}{2}((e\cdot\nabla)u_h, \phi) - \frac{1}{2}((e\cdot\nabla)\phi, u_h) \right| \\ &\leq \sup_{\|\phi\|=1} \Big(\|e\|\|\nabla u_h\|_{L^4}\|\phi\|_{L^4} + \|e\|\|\phi\|_1\|u_h\|_\infty \Big) \leq C\|e\|, \end{aligned}$$

where, in the last inequality, we have used the Sobolev's imbeddings $\|\phi\|_{L^4} \leq C\|\phi\|_1$. It is obvious that there holds

$$\|\mathscr{F}^{-1/2}(B(u,u) - B(u_h, u_h))\| \leq C\|e\|. \tag{3.18}$$

Substituting (3.18) in (3.16), we can show that

$$\|\eta\| \leq C \int_0^t (t-s)^{-\alpha/2} \|u(s) - u_h(s)\| ds.$$

By Gronwall's lemma, a use of the triangle inequality with Lemma 3.3 completes the rest of the proof. □

4. Full Discretization

In this section, firstly following the idea of Zhuang et al. [53] that discretize the Riemann-Liouville fractional derivative $\mathscr{B}^\alpha$ in time, then we adopt the finite difference method in time and obtain the stability and convergence properties related to the time discretization. By collecting the convergence results for the space discretization and for the time discretization, the error estimate for fully discrete scheme of (1.1) has been obtained.

We suppose $t_n = n\triangle t, n = 0, 1, \cdots, N$, in which $k := \triangle t = \frac{T}{N}$ denotes the step of time. In the subsequence, we will give several lemmas which play an important role in the time discretization.

Firstly, we will give the numerical scheme to discretize the temporal fractional integral.

Lemma 4.1 ([53]). *If $u(t) \in C^1[0,T], 0 < \alpha < 1$, then*

$$\mathscr{I}^\alpha u(t_n) = \frac{k^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} b_j^{(\alpha)} u(t_{n-j}) + r^{(1)}, \tag{4.1}$$

where $|r^{(1)}| \leq ct_n^\alpha k, b_j^{(\alpha)} = (j+1)^\alpha - j^\alpha, j = 0, 1, \cdots, N$.

The coefficients $b_n^{(\alpha)}$ possess the following properties.

Lemma 4.2 ([50]). *In (4.1), the coefficients $b_n^{(\alpha)} (n = 0, 1, \cdots, N)$ satisfy the following properties:*

(i) $b_0^{(\alpha)} = 1, b_n^{(\alpha)} > 0, n = 0, 1, \cdots, N$,

(ii) $b_n^{(\alpha)} > b_{n+1}^{(\alpha)}, n = 0, 1, \cdots, N$,

(iii) *there exists a positive constant $c > 0$ such that $k \leq cb_n^{(\alpha)} k^\alpha$.*

Based on Lemma 4.1, in the sequel we will give the approximation of the Riemann-Liouville fractional derivative $\mathscr{B}^\alpha$ about time.

Lemma 4.3 ([53]). *If $u(t) \in C^2[0,T]$, then*

$$\mathscr{B}^\alpha u(t_n) = \frac{k^{\alpha-1}}{\Gamma(\alpha+1)} \left[u(t_n) + \sum_{j=1}^{n-1} (b_j^{(\alpha)} - b_{j-1}^{(\alpha)}) u(t_{n-j}) \right] + R_1^{(n)}, \tag{4.2}$$

where $|R_1^{(n)}| \leq cb_{n-1}^{(\alpha)} k^\alpha$.

Let

$$\mathscr{L}^{\alpha}u_h(t_n) := \frac{k^{\alpha-1}}{\Gamma(\alpha+1)}\left[u_h(t_n) + \sum_{j=1}^{n-1}(b_j^{(\alpha)} - b_{j-1}^{(\alpha)})u_h(t_{n-j})\right],$$

then $\mathscr{B}^{\alpha}u_h(t_n) = \mathscr{L}^{\alpha}u_h(t_n) + R_1^{(n)}$.

In order to simplify the notations and without loss of generality, we consider the case $f(x,t) = 0$ in the scheme construction and its numerical analysis.

With the help of Lemma 4.3 and using $\mathscr{L}^{\alpha}u_h(t_n)$ as an approximation of $\mathscr{B}^{\alpha}u_h(t_n)$ in the equation (3.5), then we will get the fully discrete scheme of (1.1): find $(u_h^n, p_h^n) \in (X_h, M_h)$ such that for all $(\phi_h, q_h) \in (X_h, M_h)$,

$$\begin{aligned}(u_h^n, \phi_h) = (u_h^{n-1}, \phi_h) - d_1\left((\nabla u_h^n, \nabla\phi_h) + \sum_{j=1}^{n-1}(b_j^{(\alpha)} - b_{j-1}^{(\alpha)})(\nabla u_h^{n-j}, \nabla\phi_h)\right)\\ -kb(u_h^n, u_h^n, \phi_h) + kd(\phi_h, p_h^n) - kd(u_h^n, q_h),\end{aligned} \tag{4.3}$$

where $n = 0, 1, \cdots, N, d_1 = \frac{\nu k^{\alpha}}{\Gamma(\alpha+1)}$.

The stability analysis for the fully discrete scheme of (1.1) is given in the following lemma.

Lemma 4.4. *Let*

$$E_n = \|u_h^n\|^2 + d_1\sum_{j=0}^{n-1} b_j^{(\alpha)}\|\nabla u_h^{n-j}\|^2.$$

Then the fully discrete problem (4.3) *is unconditionally stable. Besides, it holds*

$$E_n \le E_{n-1} \le \cdots \le \|u_h^0\|^2, \tag{4.4}$$

where $n = 0, 1, \cdots, N$.

Proof. Setting $\phi_h = u_h^n \in X_h$ and $q_h = p_h^n \in M_h$ in (4.3), and making use of the property (3.6) of $b(u_h, v_h, w_h)$, we obtain

$$(u_h^n, u_h^n) = (u_h^{n-1}, u_h^n) - d_1\left((\nabla u_h^n, \nabla u_h^n) + \sum_{j=1}^{n-1}(b_j^{(\alpha)} - b_{j-1}^{(\alpha)})(\nabla u_h^{n-j}, \nabla u_h^n)\right).$$

Using the inequality $(u,v) \le \frac{1}{2}(\|u\|^2 + \|v\|^2)$ and noting that $b_{j-1}^{(\alpha)} > b_j^{(\alpha)}$, we derive

$$\|u_h^n\|^2 \le \frac{1}{2}(\|u_h^{n-1}\|^2 + \|u_h^n\|^2) - d_1\|\nabla u_h^n\|^2 + \frac{d_1}{2}\sum_{j=1}^{n-1}(b_{j-1}^{(\alpha)} - b_j^{(\alpha)})(\|\nabla u_h^{n-j}\|^2 + \|\nabla u_h^n\|^2).$$

Further, with Lemma 4.2, the above expression can be organized into

$$\begin{aligned}\frac{1}{2}\|u_h^n\|^2 &\le \frac{1}{2}\|u_h^{n-1}\|^2 - d_1\|\nabla u_h^n\|^2 + \frac{d_1}{2}\sum_{j=1}^{n-1}(b_{j-1}^{(\alpha)} - b_j^{(\alpha)})\|\nabla u_h^{n-j}\|^2 + \frac{d_1(1-b_{n-1}^{(\alpha)})}{2}\|\nabla u_h^n\|^2\\ &\le \frac{1}{2}\|u_h^{n-1}\|^2 - \frac{d_1(1+b_{n-1}^{(\alpha)})}{2}\|\nabla u_h^n\|^2 + \frac{d_1}{2}\sum_{j=1}^{n-1}(b_{j-1}^{(\alpha)} - b_j^{(\alpha)})\|\nabla u_h^{n-j}\|^2\\ &\le \frac{1}{2}\|u_h^{n-1}\|^2 - \frac{d_1}{2}\|\nabla u_h^n\|^2 + \frac{d_1}{2}\sum_{j=1}^{n-1}b_{j-1}^{(\alpha)}\|\nabla u_h^{n-j}\|^2 - \frac{d_1}{2}\sum_{j=1}^{n-1}b_j^{(\alpha)}\|\nabla u_h^{n-j}\|^2,\end{aligned}$$

which can be simplified as

$$\|u_h^n\|^2 + d_1 \sum_{j=0}^{n-1} b_j^{(\alpha)} \|\nabla u_h^{n-j}\|^2 \le \|u_h^{n-1}\|^2 + d_1 \sum_{j=0}^{n-2} b_j^{(\alpha)} \|\nabla u_h^{n-1-j}\|^2.$$

In other words, we obtained the desired asseat (4.4). □

The error analysis for the solution of the semi-discrete problem about time is discussed in the following theorem.

Theorem 4.1 (Error estimate for time discretization). *Let $u_h(t_n)$ be the solution of (3.5), $\{u_h^n\}_{n=1}^N$ be the time-discrete solution of (4.3). Under the assumptions of a small initial data and a small time step size, then we have the following error estimate*

$$\|u_h(t_n) - u_h^n\| \le ck.$$

Proof. Let $e^n = u_h(t_n) - u_h^n, \tilde{e}^n = p_h(t_n) - p_h^n$. Subtracting (4.3) from (3.5), the error equation can be written as

$$\begin{aligned}(e^n, \phi_h) = (e^{n-1}, \phi_h) - d_1 &\left((\nabla e^n, \nabla \phi_h) + \sum_{j=1}^{n-1} (b_j^{(\alpha)} - b_{j-1}^{(\alpha)})(\nabla e^{n-j}, \nabla \phi_h) \right) \\ &-kb(e^n, u_h(t_n), e^n) + kd(\phi_h, \tilde{e}^n) - kd(e^n, q_h) + k(R_1^n, \phi_h).\end{aligned} \tag{4.5}$$

Taking $\phi_h = e^n, q_h = \tilde{e}^n$ in (4.5), there holds

$$\begin{aligned}(e^n, e^n) = (e^{n-1}, e^n) - d_1 &\left((\nabla e^n, \nabla e^n) + \sum_{j=1}^{n-1} (b_j^{(\alpha)} - b_{j-1}^{(\alpha)})(\nabla e^{n-j}, \nabla e^n) \right) \\ &-kb(e^n, u_h(t_n), e^n) + k(R_1^n, e^n).\end{aligned}$$

Making use of the inequality $(u, v) \le \frac{1}{2}(\|u\|^2 + \|v\|^2)$ and noting that $b_{j-1}^{(\alpha)} > b_j^{(\alpha)}$, we observe that

$$\begin{aligned}\|e^n\|^2 \le &\frac{1}{2}(\|e^{n-1}\|^2 + \|e^n\|^2) - d_1 \|\nabla e^n\|^2 \\ &+ \frac{d_1}{2} \sum_{j=1}^{n-1} (b_{j-1}^{(\alpha)} - b_j^{(\alpha)})(\|\nabla e^{n-j}\|^2 + \|\nabla e^n\|^2) - kb(e^n, u_h(t_n), e^n) + k(R_1^n, e^n),\end{aligned}$$

which can be simplified as

$$\begin{aligned}\|e^n\|^2 \le \|e^{n-1}\|^2 &- d_1 \sum_{j=0}^{n-1} b_j^{(\alpha)} \|\nabla e^{n-j}\|^2 - d_1 b_{n-1}^{(\alpha)} \|\nabla e^n\|^2 \\ &+ d_1 \sum_{j=0}^{n-2} b_j^{(\alpha)} \|\nabla e^{n-1-j}\|^2 - 2kb(e^n, u_h(t_n), e^n) + 2k(R_1^n, e^n).\end{aligned}$$

Let $Y_n = \|e^n\|^2 + d_1 \sum\limits_{j=0}^{n-1} b_j^{(\alpha)} \|\nabla e^{n-j}\|^2$, then the above expression can be read as

$$Y_n \le Y_{n-1} - d_1 b_{n-1}^{(\alpha)} \|\nabla e^n\|^2 - 2kb(e^n, u_h(t_n), e^n) + 2k(R_1^n, e^n). \tag{4.6}$$

Applying the property (2.1) of $b(u, v, w)$, we have

$$-2kb(e^n, u_h, e^n) \le 2kM\|e^n\|_1^2\|u_h\|,$$

note that the right hand side of the above inequality can be sufficiently small which can be derived from (3.11) only if it requires a small initial data and a small time step size. Making use of the Sobolev's imbeddings $\|e^n\| \le \gamma\|\nabla e^n\|$, the inequality (4.6) can be written as

$$Y_n \le Y_{n-1} - \frac{d_1 b_{n-1}^{(\alpha)}}{\gamma^2}\|e^n\|^2 + 2k(R_1^n, e^n). \tag{4.7}$$

Note that

$$(\alpha(x), \beta(x)) \le \lambda\|\alpha(x)\|^2 + \frac{1}{4\lambda}\|\beta(x)\|^2, \quad (\lambda > 0).$$

Hence,

$$2k(R_1^n, e^n) \le \frac{d_1 b_{n-1}^{(\alpha)}}{\gamma^2}\|e^n\|^2 + \frac{\gamma^2}{4d_1 b_{n-1}^{(\alpha)}}\|2kR_1^n\|^2. \tag{4.8}$$

Substituting (4.8) into (4.7), it holds

$$Y_n \le Y_{n-1} + \frac{\gamma^2 k^2}{d_1 b_{n-1}^{(\alpha)}}\|R_1^n\|^2 \le Y_{n-1} + \frac{\gamma^2}{\nu}\Gamma(\alpha+1)b_{n-1}^{(\alpha)}k^{\alpha+2},$$

the last inequality of which can be obtained from $d_1 = \frac{\nu k^\alpha}{\Gamma(\alpha+1)}$ and $|R_1^{(n)}| \le cb_{n-1}^{(\alpha)}k^\alpha$. Since $Y_0 = e^0 = 0$, it derives that

$$\begin{aligned} Y_n &\le \frac{\gamma^2}{\nu}\Gamma(\alpha+1)k^{\alpha+2}\sum_{j=1}^n b_{j-1}^{(\alpha)} \\ &= \frac{\gamma^2}{\nu}\Gamma(\alpha+1)k^{\alpha+2}\sum_{j=1}^n (j^\alpha - (j-1)^\alpha) \\ &\le \frac{\gamma^2}{\nu}\Gamma(\alpha+1)T^\alpha k^2 \le ck^2, \end{aligned}$$

where $c = \frac{\gamma^2}{\nu}\Gamma(\alpha+1)T^\alpha$. The proof is completed. □

Next we will give the error estimate for the fully discrete scheme by collecting the convergence results for the space discretization and for the time discretization.

Theorem 4.2 (Error estimate for fully discrete scheme). *Let $\{u(t_n)\}$ be the solution of* (1.1) *and let $\{u_h^n\}_{n=1}^N$ be the solution of the scheme* (4.3) *with $T = N\triangle t$. Under the assumptions of a small initial data and a small time step size, then there is $c > 0$ such that*

$$\|u(t_n) - u_h^n\| \le c\left(h^2 + k\right).$$

Proof. The proof follows from Theorems 3.1 and 4.1 by the triangle inequality. We are no longer to repeat here. □

5. Numerical Experiment

In order to demonstrate the effectiveness of our numerical methods, numerical examples are presented. The main purpose is to check the convergence behavior of the numerical solution with respect to the time step Δt and the space step h used in the calculation.

We consider the following fractional equation

$$\begin{aligned} &u_t - \mathscr{B}^\alpha u_{xx} + uu_x = f(x,t), && x \in \Omega, \quad t \in [0,1], \\ &u(0,t) = u(1,t) = 0, && t \in [0,1], \\ &u(x,0) = 0, && x \in \Omega, \end{aligned}$$

where $\Omega = [0,1]$.

We compute the errors in L^2 discrete norm. And all the numerical results in the tables below are evaluated at T=1. The spatial and temporal meshes are taken uniform. The finite element method using piecewise-linear polynomials is used for the space and the scheme for time described in previous sections is used in the examples.

Example 5.1. The source term f is chosen as

$$f(x,t) = 2t\sin(\pi x) + \frac{2(\pi)^2}{\Gamma(2+\alpha)} t^{1+\alpha}\sin(\pi x) + \pi t^4 \sin(\pi x)\cos(\pi x).$$

Then the exact solution is $u(x,t) = t^2 \sin(\pi x)$.

For the $0 < \alpha < 1$ case, the theoretical convergence order is $O(h^2 + k)$. To examine the rate of convergence for this method, in Table 1, for a fixed time step $\Delta t = 1/800$ and some different spatial meshes, we can see the orders of convergence for u in L^2-norms are close to 2 which are accord with the spatial convergence order $O(h^2)$. In Table 2, for a fixed spatial step $h = 1/1000$, it shows that the orders of convergence for u in L^2-norms are close to 1 which are accord with the time convergence order $O(k)$. The numerical results are consistent with our theoretical results in Theorem 4.2.

Table 5.1: The errors and space convergence rates for u with fixed time step $\Delta t = 1/800$.

h	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$\|u(T) - u_h^N\|$	cv.rate	$\|u(T) - u_h^N\|$	cv.rate	$\|u(T) - u_h^N\|$	cv.rate
1/4	3.5673E-2	-	3.4926E-2	-	3.4058E-2	-
1/8	8.8397E-3	2.013	8.6821E-3	2.008	8.4523E-3	2.011
1/16	2.1018E-3	2.072	2.1059E-3	2.044	2.0528E-3	2.042
1/32	4.2061E-4	2.321	4.6254E-4	2.187	4.5466E-4	2.174

Example 5.2. We choose the exact solution $u(x,t) = t^{1/2}\sin(\pi x)$, the smoothing property of which is less. Then the source term can be arrived at

$$f(x,t) = \frac{1}{2} t^{-\frac{1}{2}} \sin(\pi x) + \frac{\Gamma(3/2)\pi^2}{\Gamma(1/2+\alpha)} t^{\alpha-1/2}\sin(\pi x) + \pi t \sin(\pi x)\cos(\pi x),$$

where $f(x,t)$ is nonsmooth in time.

In Table 3, for a fixed time step $\Delta t = 1/5000$ and some different spatial meshes, we can see the orders of convergence for u in L^2-norms are close to 2 which are in agreement with the spatial

Table 5.2: The errors and time convergence rates for u with fixed space step $h = 1/1000$.

Δt	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$\|u(T) - u_h^N\|$	cv.rate	$\|u(T) - u_h^N\|$	cv.rate	$\|u(T) - u_h^N\|$	cv.rate
1/8	3.7512E-2	-	1.7726E-2	-	1.0798E-2	-
1/16	1.6424E-2	1.192	7.5884E-3	1.224	5.0085E-3	1.108
1/32	7.1783E-3	1.194	3.3325E-3	1.187	2.4005E-3	1.061
1/64	3.1469E-3	1.189	1.5045E-3	1.147	1.1738E-3	1.032

Table 5.3: The errors and space convergence rates.

α	Δt	h	$\|u(T) - u_h^N\|$	cv.rate
0.1	1/5000	1/2	1.4513E-1	-
	1/5000	1/4	3.9126E-2	1.891
	1/5000	1/8	1.1972E-2	1.708
0.3	1/5000	1/2	1.4382E-1	-
	1/5000	1/4	3.7275E-2	1.948
	1/5000	1/8	9.9406E-3	1.907
0.9	1/5000	1/2	1.4572E-1	-
	1/5000	1/4	3.7357E-2	1.967
	1/5000	1/8	9.4572E-3	1.982

Table 5.4: The errors and time convergence rates.

α	h	Δt	$\|u(T) - u_h^N\|$	cv.rate
0.3	1/1000	1/10	5.3429E-2	-
	1/1000	1/20	3.3131E-2	0.703
	1/1000	1/40	2.0364E-2	0.702
0.6	1/1000	1/10	1.8203E-2	-
	1/1000	1/20	1.0321E-2	0.819
	1/1000	1/40	6.1121E-3	0.756
0.9	1/1000	1/10	4.2612E-3	-
	1/1000	1/20	2.3878E-3	0.836
	1/1000	1/40	1.4430E-3	0.727

convergence order $O(h^2)$. Table 4 exhibits the numerical errors in temporal direction with different α for a fixed sufficiently small spatial step. Because the source term f is nonsmooth in time, the convergence for small fractional order α suffers some loss.

All in all, based on the above numerical examples, one can find that our numerical method is effective.

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