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# EFFICIENT AND ACCURATE CHEBYSHEV DUAL-PETROV-GALERKIN METHODS FOR ODD-ORDER DIFFERENTIAL EQUATIONS*

Xuhong Yu, Lusha Jin and Zhongqing Wang[1)]
*School of Science, University of Shanghai for Science and Technology, Shanghai 200093, China*
*Email: xhyu@usst.edu.cn, ls.jin@foxmail.com, zqwang@usst.edu.cn*

**Abstract**

Efficient and accurate Chebyshev dual-Petrov-Galerkin methods for solving first-order equation, third-order equation, third-order KdV equation and fifth-order Kawahara equation are proposed. Some Sobolev bi-orthogonal basis functions are constructed which lead to the diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions are expanded as an infinite and truncated Fourier-like series, respectively. Numerical experiments illustrate the effectiveness of the suggested approaches.

*Mathematics subject classification:* 65N35, 33C45, 35J58.
*Key words:* Chebyshev dual-Petrov-Galerkin method, Sobolev bi-orthogonal polynomials, Odd-order differential equations, Numerical results.

## 1. Introduction

Spectral methods are based on orthogonal polynomial/function approximations, which possess the high-order accuracy and have gained more and more popularity during the past few decades, see [2, 4, 5, 8, 10, 24, 25] and the references therein. The Fourier trigonometric polynomials $e^{ikx}$, $k \in \mathbb{Z}$ are the most desirable basis, which are orthogonal with respect to each other under certain Sobolev inner product involving derivatives, thus the corresponding algebraic system is diagonal. This fact together with the availability of the fast Fourier transform (FFT) makes the Fourier spectral method be an ideal approximation approach for differential equations with periodic boundary conditions. If a Fourier method is applied to a non-periodic problem, it inevitably induces the so-called Gibbs phenomenon, and reduces the global convergence rate to $O(N^{-1})$. Consequently, one should not apply a Fourier method to problems with non-periodic boundary conditions. Instead, the Chebyshev spectral methods [4, 11–13, 17] are of our greatest interests due to the FFT for Chebyshev polynomials.

Standard Chebyshev spectral methods have been extensively investigated for solving second-order and fourth-order differential equations (see, e.g., [22]). For the one-dimensional fourth-order linear equation, Shen [22] presented a basis

$$\varphi_k(x) = T_k(x) - \frac{2(k+2)}{k+3}T_{k+2}(x) + \frac{k+1}{k+3}T_{k+4}(x), \qquad 0 \le k \le N-4,$$

with $T_k(x)$ being the $k$th Chebyshev polynomial. Note that the matrix with the term $(\partial_x^2\varphi_k, \partial_x^2(\varphi_l\omega))$ in the resulting linear system is not sparse but possesses special structure, where $\omega(x)$

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1) Corresponding author.

is the Chebyshev weight function. Benefiting from these special matrix structures, Shen [22] further derived some efficient algorithms. However, there is only a limited body of literature on spectral methods for odd-order equations. This is partly due to the fact that: (i) for the classical Chebyshev-Galerkin spectral methods, the discrete systems are not sparse and the condition numbers increase like $O(N^k)$ for the $k$-order boundary problem, where $N$ is the number of modes; (ii) for the direct collocation methods, the discrete systems are also not sparse and the condition numbers increase like $O(N^{2k})$ for $k$-order boundary problems, and often exhibit unstable modes if the collocation points are not properly chosen (see, e.g., [14,21]).

Since the main differential operators in odd-order differential equations are not symmetric, it is reasonable to use the Petrov-Galerkin spectral method. Recently, Ma and Sun [18,19] developed an efficient Legendre-Petrov-Galerkin and Chebyshev collocation method for the third-order differential equations. By choosing appropriate basis functions, the resulting linear system is sparse. Shen [23] proposed a Legendre dual-Petrov-Galerkin spectral method for the third and higher odd-order equations, and obtained linear systems which are compactly sparse. Moreover, Shen and Wang [26] presented Legendre and Chebyshev dual-Petrov-Galerkin spectral methods for the first-order hyperbolic equations, which are always stable without any restriction on the coefficients, the resulting linear systems are also compactly sparse for problems with constant coefficients and well conditioned for problems with variable coefficients by a preconditioning method.

In this paper, we consider the first and third order differential equations by using Chebyshev dual-Petrov-Galerkin method. As pointed out in [22], it is very important to choose an appropriate basis such that the resulting linear system is as simple as possible. Motivated by the success work in [1,16,27], the main purpose of this paper is to construct new Fourier-like Sobolev bi-orthogonal basis functions [7,20], such that the resulting linear systems are diagonal.

The main advantages of the suggested algorithms include:

- the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier-like series, respectively;
- the condition numbers of the resulting algebraic systems are always equal to one;
- the computational cost is much less than that of the classical Chebyshev dual-Petrov-Galerkin method.

The remainder of this paper is organized as follows. In Section 2, we introduce the Chebyshev polynomials and their basic properties. In Section 3, we construct two kinds of Sobolev bi-orthogonal Chebyshev polynomials corresponding to the odd-order differential equations, and propose new Chebyshev dual-Petrov-Galerkin methods. Some numerical results are presented in Section 4 to demonstrate the effectiveness and accuracy.

## 2. Notations and Preliminaries

Let $I$ be a certain interval and $\omega(x)$ be a weight function in the usual sense. For integer $r \geq 0$, we define the weighted Sobolev spaces $H^r_\omega(I)$ as usual, with the inner product $(u,v)_{r,\omega}$, the semi-norm $|v|_{r,\omega}$ and the norm $\|v\|_{r,\omega}$. We omit the subscript $\omega(x)$ whenever $\omega(x) \equiv 1$.

We first recall the Chebyshev polynomials. Let $I = (-1,1)$ and $T_k(x)$ be the Chebyshev polynomial of degree $k$, which is the eigenfunction of the singular Sturm-Liouville problem (cf. [25]):

$$(1-x^2)\partial_x^2 T_k(x) - x\partial_x T_k(x) + k^2 T_k(x) = 0, \qquad k \geq 0. \tag{2.1}$$

Denote $L^2_\omega(I) = H^0_\omega(I)$. The set of all Chebyshev polynomials forms a complete $L^2_\omega(I)$-orthogonal system with the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, namely,

$$\int_I T_k(x)T_l(x)\omega(x)dx = \frac{c_k\pi}{2}\delta_{k,l}, \qquad k,l \geq 0, \tag{2.2}$$

where $\delta_{k,l}$ is the Kronecker symbol, $c_0 = 2$ and $c_k = 1$ for $k \geq 1$. Thus, for any $v \in L^2_\omega(I)$,

$$v(x) = \sum_{k=0}^{\infty} \hat{v}_k T_k(x), \qquad \hat{v}_k = \frac{2}{c_k\pi}\int_I v(x)T_k(x)\omega(x)dx.$$

By virtue of (2.1) and (2.2), we have

$$\int_I \partial_x T_k(x)\partial_x T_l(x)\sqrt{1-x^2}dx = \frac{c_k k^2\pi}{2}\delta_{k,l}, \qquad k,l \geq 0. \tag{2.3}$$

Moreover, the following recurrence relations are satisfied with $T_0(x) = 1$ and $T_1(x) = x$ (cf. [25]),

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \qquad k \geq 1, \tag{2.4a}$$

$$2T_k(x) = \frac{1}{k+1}\partial_x T_{k+1}(x) - \frac{1}{k-1}\partial_x T_{k-1}(x), \qquad k \geq 2, \tag{2.4b}$$

$$(1-x^2)\partial_x T_k(x) = \frac{k}{2}(T_{k-1}(x) - T_{k+1}(x)), \qquad k \geq 1, \tag{2.4c}$$

$$\partial_x T_k(x) = 2k \sum_{\substack{i=0 \\ i+k \text{ odd}}}^{k-1} \frac{1}{c_i} T_i(x). \tag{2.4d}$$

Particularly, $T_k(-x) = (-1)^k T_k(x)$, $T_k(\pm 1) = (\pm 1)^k$ and $\partial_x T_k(\pm 1) = (\pm 1)^{k-1}k^2$ for $k \geq 0$.

In order to construct new Chebyshev dual-Petrov-Galerkin spectral method for the first-order problems defined on $I$, we need to consider the following two kinds of polynomials,

$$s_k(x) = (1+x)T_k(x), \qquad t_k(x) = (1-x)T_k(x), \qquad k \geq 0. \tag{2.5}$$

**Lemma 2.1.** *For any $k \geq 0$, we have*

$$s_k(x) = \frac{1+\delta_{k,0}}{2}T_{k+1}(x) + T_k(x) + \frac{1}{2}T_{k-1}(x), \tag{2.6}$$

$$t_k(x) = -\frac{1+\delta_{k,0}}{2}T_{k+1}(x) + T_k(x) - \frac{1}{2}T_{k-1}(x), \tag{2.7}$$

$$\partial_x s_k(x) = (k+1)T_k(x) + 2k\sum_{i=1}^{k-1} T_i(x) + kT_0(x), \tag{2.8}$$

*where $T_k(x) \equiv 0$ for any $k < 0$.*

*Proof.* By (2.4) and a direct computation, we can verify easily the result of (2.6)–(2.8) for $k = 0$. Next, by (2.4) we deduce that for $k \geq 1$,

$$s_k(x) = T_k(x) + xT_k(x) = \frac{1}{2}T_{k+1}(x) + T_k(x) + \frac{1}{2}T_{k-1}(x). \tag{2.9}$$

This leads to (2.6). Similarly, we can derive the result of (2.7). Moreover, by (2.9) and (2.4d), we have for $k \geq 1$,

$$\begin{aligned}
&\partial_x s_k(x) = \frac{1}{2}\partial_x T_{k+1}(x) + \partial_x T_k(x) + \frac{1}{2}\partial_x T_{k-1}(x)\\
=&(k+1)\sum_{\substack{i=0\\ i+k \text{ even}}}^{k} \frac{1}{c_i}T_i(x) + 2k\sum_{\substack{i=0\\ i+k \text{ odd}}}^{k-1} \frac{1}{c_i}T_i(x) + (k-1)\sum_{\substack{i=0\\ i+k \text{ even}}}^{k-2} \frac{1}{c_i}T_i(x)\\
=&(k+1)T_k(x) + 2k\sum_{\substack{i=0\\ i+k \text{ even}}}^{k-2} \frac{1}{c_i}T_i(x) + 2k\sum_{\substack{i=0\\ i+k \text{ odd}}}^{k-1} \frac{1}{c_i}T_i(x)\\
=&(k+1)T_k(x) + 2k\sum_{i=1}^{k-1} T_i(x) + kT_0(x).
\end{aligned} \tag{2.10}$$

This ends the proof. □

Further, in order to design new Chebyshev dual-Petrov-Galerkin spectral method for the third-order problems, we also need to consider the following two kinds of polynomials,

$$p_k(x) = (1-x)^2(1+x)T_k(x), \quad q_k(x) = (1-x)(1+x)^2 T_k(x), \qquad k \geq 0. \tag{2.11}$$

Clearly, $p_k(\pm 1) = \partial_x p_k(1) = 0$ and $q_k(\pm 1) = \partial_x q_k(-1) = 0$ for $k \geq 0$.

**Lemma 2.2.** *For any $k \geq 0$, we have*

$$\begin{aligned}
p_k(x) =& \frac{1}{8}(1+\delta_{k,0})T_{k+3}(x) - \frac{1}{4}(1+\delta_{k,0})T_{k+2}(x) - \frac{1}{8}(1+\delta_{k,0}-\delta_{k,1})T_{k+1}(x)\\
&+ \frac{1}{2(1+\delta_{k,1})}T_k(x) - \frac{1}{8}(1-\delta_{k,2})T_{k-1}(x) - \frac{1}{4}T_{k-2}(x) + \frac{1}{8}T_{k-3}(x),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
q_k(x) =& -\frac{1}{8}(1+\delta_{k,0})T_{k+3}(x) - \frac{1}{4}(1+\delta_{k,0})T_{k+2}(x) + \frac{1}{8}(1+\delta_{k,0}-\delta_{k,1})T_{k+1}(x)\\
&+ \frac{1}{2(1+\delta_{k,1})}T_k(x) + \frac{1}{8}(1-\delta_{k,2})T_{k-1}(x) - \frac{1}{4}T_{k-2}(x) - \frac{1}{8}T_{k-3}(x).
\end{aligned} \tag{2.13}$$

*Proof.* By (2.4) and a direct computation, we can verify easily the results of (2.12) and (2.13) for $k = 0, 1, 2$. Next, by (2.4b), (2.4c) and (2.4a) we deduce that for $k \geq 3$,

$$\begin{aligned}
&p_k(x) = (1-x)^2(1+x)T_k(x)\\
=&\frac{(1-x)^2(1+x)}{2}\left(\frac{1}{k+1}\partial_x T_{k+1}(x) - \frac{1}{k-1}\partial_x T_{k-1}(x)\right)\\
=&\frac{1-x}{4}(-T_{k+2}(x) + 2T_k(x) - T_{k-2}(x))\\
=&\frac{1}{4}(-T_{k+2}(x) + 2T_k(x) - T_{k-2}(x))\\
&- \frac{1}{4}\left(-\frac{T_{k+3}(x) + T_{k+1}(x)}{2} + T_{k+1}(x) + T_{k-1}(x) - \frac{T_{k-1}(x) + T_{k-3}(x)}{2}\right)\\
=&\frac{1}{8}T_{k+3}(x) - \frac{1}{4}T_{k+2}(x) - \frac{1}{8}T_{k+1}(x) + \frac{1}{2}T_k(x) - \frac{1}{8}T_{k-1}(x)\\
&- \frac{1}{4}T_{k-2}(x) + \frac{1}{8}T_{k-3}(x).
\end{aligned} \tag{2.14}$$

This leads to the result of (2.12). Similarly, we obtain the result of (2.13). □

We denote by $\mathbb{A} = (a_{k,l})_{0\le k,l\le N}$ the symmetric matrix with the element $a_{k,l} := (p_k, q_l)_\omega$. By the Lemma 2.2 and (2.2), we derive readily the nonzero elements $a_{k,l}$ $(k \le l)$ of $\mathbb{A}$ as following,

$$\begin{aligned} a_{k,k} =& \frac{(1+3\delta_{k,0})(2+\delta_{k,1})}{128}\pi + \frac{1}{8(1+3\delta_{k,1})}c_k\pi - \frac{(1-\delta_{k,2})}{128}c_{k-1}\pi \\ &+ \frac{1}{32}c_{k-2}\pi - \frac{1}{128}c_{k-3}\pi, \\ a_{k,k+2} =& -\frac{7(1+\delta_{k,0})+(1-\delta_{k,0})(1-\delta_{k,1})}{128}\pi - \frac{1}{16(1+\delta_{k,1})}c_k\pi + \frac{1-\delta_{k,2}}{128}c_{k-1}\pi, \\ a_{k,k+4} =& \frac{(1+\delta_{k,0})(6-\delta_{k,1})}{128}\pi, \qquad a_{k,k+6} = -\frac{(1+\delta_{k,0})}{128}\pi. \end{aligned} \tag{2.15}$$

**Lemma 2.3.** *For any $k \ge 0$, we have*

$$\begin{aligned} \partial_x p_k(x) =& \frac{(k+3)(1+\delta_{k,0})}{4}T_{k+2}(x) - \frac{(k+2)(1+\delta_{k,0})}{2}T_{k+1}(x) \\ &+ \frac{1+\delta_{k,1}}{2}T_k(x) + \frac{k-2}{2}T_{k-1}(x) - \frac{k-3}{4}T_{k-2}(x), \end{aligned} \tag{2.16}$$

$$\begin{aligned} \partial_x(q_k(x)\omega(x)) =& \Big( -\frac{(k+2)(1+\delta_{k,0})}{4}T_{k+2}(x) - \frac{(k+1)(1+\delta_{k,0})}{2}T_{k+1}(x) \\ &- \frac{\delta_{k,1}}{4}T_k(x) + \frac{k-1}{2}T_{k-1}(x) + \frac{k-2}{4}T_{k-2}(x)\Big)\omega(x). \end{aligned} \tag{2.17}$$

*Proof.* By (2.4) and a direct computation, we can verify easily the results of (2.16) and (2.17) for $k = 0, 1$. Next, by (2.4c) and (2.4a), we deduce that for $k \ge 2$,

$$\begin{aligned} &\partial_x p_k(x) = -2(1-x^2)T_k(x) + (1-x)^2 T_k(x) + (1-x)^2(1+x)\partial_x T_k(x) \\ =&(1-x)\left(-3xT_k(x) - T_k(x) + \frac{k}{2}\big(T_{k-1}(x) - T_{k+1}(x)\big)\right) \\ =&(1-x)\left(\frac{k-3}{2}T_{k-1}(x) - T_k(x) - \frac{k+3}{2}T_{k+1}(x)\right) \\ =&\frac{k+3}{4}T_{k+2}(x) - \frac{k+2}{2}T_{k+1}(x) + \frac{1}{2}T_k(x) + \frac{k-2}{2}T_{k-1}(x) - \frac{k-3}{4}T_{k-2}(x), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\partial_x(q_k(x)\omega(x)) = (1+x)\big(-2xT_k(x) + T_k(x) + (1-x^2)\partial_x T_k(x)\big)\omega(x) \\ =&(1+x)\left(-\frac{k+2}{2}T_{k+1}(x) + T_k(x) + \frac{k-2}{2}T_{k-1}(x)\right)\omega(x) \\ =&\left(-\frac{k+2}{4}T_{k+2}(x) - \frac{k+1}{2}T_{k+1}(x) + \frac{k-1}{2}T_{k-1}(x) + \frac{k-2}{4}T_{k-2}(x)\right)\omega(x). \end{aligned} \tag{2.19}$$

This ends the proof. □

We denote by $\mathbb{B} = (b_{k,l})_{0\le k,l\le N}$ and $\mathbb{C} = (c_{k,l})_{0\le k,l\le N}$ the matrixes with the elements $b_{k,l} := (p_k, \partial_x(q_l\omega))$ and $c_{k,l} := (\partial_x p_k, \partial_x(q_l\omega))$, respectively. By using (2.12), (2.17) and (2.2), we can derive the nonzero elements of $\mathbb{B}$ as following,

$$b_{kk} = \frac{(1+3\delta_{k,0})(2k+5-(k+2)\delta_{k,1})}{32}\pi - \frac{1}{8}c_k\pi - \frac{k-2}{32}c_{k-1}\pi - \frac{k-3}{32}c_{k-2}\pi,$$

$$
\begin{aligned}
b_{k,k+1} &= \frac{\pi}{64}\Big(8(k+2)-(k+3)(1-\delta_{k,0})\Big)-\frac{\pi}{32}(1-\delta_{k,1})c_k+\frac{\pi}{16}(k-2)c_{k-1}-\frac{\pi}{64}(k-3)c_{k-2},\\
b_{k+1,k} &= -\frac{\pi}{64}\Big((3k+8)(1+\delta_{k,0})+2(1+3\delta_{k,0})(1-\delta_{k,1})\Big)-\frac{(k-1)c_k\pi}{8(1+\delta_{k,1})}+\frac{\pi}{64}(k-2)(1-\delta_{k,2})c_{k-1},\\
b_{k,k+2} &= -\frac{\pi}{32}(1+\delta_{k,0})(k+4+(k+2)\delta_{k,0})+\frac{\pi}{16}(1+\delta_{k,1})c_k+\frac{\pi}{32}(k-2)c_{k-1},\\
b_{k+2,k} &= -\frac{\pi}{32}(1+\delta_{k,0})(2k+2-k\delta_{k,1})+\frac{(k-1)c_k\pi}{16(1+\delta_{k,1})},\\
b_{k,k+3} &= -\frac{\pi}{64}(1+\delta_{k,0})(5k+11)+\frac{\pi}{32}(1+\delta_{k,1})c_k,\\
b_{k+3,k} &= \frac{\pi}{64}(1+\delta_{k,0})(6+5k-k\delta_{k,1}),\qquad b_{k,k+4}=b_{k+4,k}=\frac{\pi}{32}(1+\delta_{k,0}),\\
b_{k,k+5} &= \frac{\pi}{64}(k+3)(1+\delta_{k,0}),\qquad b_{k+5,k}=-\frac{\pi}{64}(k+2)(1+\delta_{k,0}).
\end{aligned}
\tag{2.20}
$$

Similarly, by (2.16), (2.17) and (2.2), we have

$$
\begin{aligned}
c_{kk} &= \frac{\pi}{32}\Big((k+2)(3k+1)(1+3\delta_{k,0})-4\delta_{k,1}\Big)+\frac{\pi}{8}(k-1)(k-2)c_{k-1}-\frac{\pi}{32}(k-2)(k-3)c_{k-2},\\
c_{k,k+1} &= \frac{\pi}{16}\Big(2k(1+\delta_{k,1})-(k+2)(k+3+(k+1)\delta_{k,0})\Big)+\frac{\pi}{16}(k-1)(k-2)c_{k-1},\\
c_{k+1,k} &= \frac{\pi}{16}\Big((k+2)(k+3)(1+\delta_{k,0})-2(k+1)(1+3\delta_{k,0})\Big)-\frac{\pi}{16}(k-1)(k-2)c_{k-1},\\
c_{k,k+2} &= \frac{\pi}{16}\Big(k(1+\delta_{k,1})-2(k+1)(k+2)(1+\delta_{k,0})\Big),\\
c_{k+2,k} &= -\frac{\pi}{16}(2k^2+3k+2)(1+\delta_{k,0}),\\
c_{k,k+3} &= \frac{\pi}{8}(k+2)(1+\delta_{k,0}),\qquad c_{k+3,k}=-\frac{\pi}{8}(k+1)(1+\delta_{k,0}),\\
c_{k,k+4} &= \frac{\pi}{32}(k+2)(k+3)(1+\delta_{k,0}),\qquad c_{k+4,k}=\frac{\pi}{32}(k+1)(k+2)(1+\delta_{k,0}).
\end{aligned}
\tag{2.21}
$$

**Lemma 2.4.** *For any $k\ge 0$, we have*

$$
\begin{aligned}
\partial_x^2 p_k(x) &= \frac{(k+2)(k+3)(1+\delta_{k,0})}{2}T_{k+1}(x)-(k+1)(k+2)T_k(x)+\frac{(k+1)(k+6)}{2}T_{k-1}(x)\\
&\quad+6k\sum_{i=1}^{k-2}(-1)^{i+k+1}T_i(x)+(-1)^{k+1+\delta_{k,1}}3kT_0(x).
\end{aligned}
\tag{2.22}
$$

*Proof.* By (2.4) and a direct computation, we can derive the result of (2.22) for $k=0,1$ easily. Next, by (2.4d) we deduce that for $k\ge 2$,

$$
\begin{aligned}
\frac{\partial_x T_k(x)}{2k}-\frac{\partial_x T_{k-1}(x)}{2(k-1)} &= \sum_{i=0\, i+k \text{ odd}}^{k-1}\frac{1}{c_i}T_i(x)-\sum_{i=0\, i+k \text{ even}}^{k-2}\frac{1}{c_i}T_i(x)\\
&= \sum_{i=1}^{k-1}(-1)^{i+k+1}T_i(x)+(-1)^{k+1}\frac{1}{2}T_0(x).
\end{aligned}
\tag{2.23}
$$

Therefore, by (2.16), (2.4b), (2.4d) and (2.23), we have for $k\ge 2$,

$$
\partial_x^2 p_k(x)=\frac{k+3}{4}\left(2(k+2)T_{k+1}(x)+\frac{k+2}{k}\partial_x T_k(x)\right)
$$

$$
\begin{aligned}
&-\frac{k+2}{2}\left(2(k+1)T_k(x)+\frac{k+1}{k-1}\partial_x T_{k-1}(x)\right)+\frac{1}{2}\partial_x T_k(x)\\
&+\frac{k-2}{2}\partial_x T_{k-1}(x)-\frac{k-3}{4}\left(\frac{k-2}{k}\partial_x T_k(x)-2(k-2)T_{k-1}(x)\right)\\
=&\frac{1}{2}(k+2)(k+3)T_{k+1}(x)-(k+1)(k+2)T_k(x)\\
&+\frac{1}{2}(k-2)(k-3)T_{k-1}(x)+3k\left(\frac{\partial_x T_k(x)}{k}-\frac{\partial_x T_{k-1}(x)}{k-1}\right)\\
=&\frac{1}{2}(k+2)(k+3)T_{k+1}(x)-(k+1)(k+2)T_k(x)+\frac{1}{2}(k+1)(k+6)T_{k-1}(x)\\
&+6k\sum_{i=1}^{k-2}(-1)^{i+k+1}T_i(x)+(-1)^{k+1}3kT_0(x). \qquad (2.24)
\end{aligned}
$$

This ends the proof. □

We denote by $\mathbb{D}=(d_{k,l})_{0\le k,l\le N}$ the matrix with the element $d_{kl}=(\partial_x p_k,\partial_x^2(q_l\omega))$. By using (2.22), (2.17), (2.2) and integration by parts, we can obtain the nonzero elements of $\mathbb{D}$ as following,

$$
\begin{aligned}
d_{kk}=&\frac{\pi}{8}(k+1)(k+2)((k+3)(1+3\delta_{k,0})-\delta_{k,1})\\
&-\frac{\pi}{8}(k^2-1)(k+6)c_{k-1}+\frac{\pi}{4}3k(k-2)c_{k-2},\\
d_{k,k+1}=&\frac{\pi}{8}\Big((k+2)(k+3)\delta_{k,0}+2k(k+1)(k+2)\Big)-\frac{\pi}{16}(k^2-1)(k+6)c_{k-1},\\
d_{k+1,k}=&-\frac{\pi}{16}\Big(3k(k+2)(k+3)(1+\delta_{k,0})-(k+2)(k+7)\delta_{k,1}\Big)\\
&+\frac{3\pi}{2}(k^2-1)c_{k-1}-\frac{3\pi}{4}(k+1)(k-2)c_{k-2},\\
d_{k,k+2}=&-\frac{\pi}{8}(k+1)(k+2)(3+(k+3)\delta_{k,0}),\\
d_{k+2,k}=&\frac{\pi}{8}\Big(3k(k+3)(1+\delta_{k,0})-6(k+2)\delta_{k,1}\Big)-\frac{3\pi}{2}(k-1)(k+2)c_{k-1}\\
&+\frac{3\pi}{4}(k^2-4)c_{k-2},\\
d_{k,k+3}=&-\frac{\pi}{16}(k+1)(k+2)(k+3)(1+\delta_{k,0}),\\
d_{k+3,k}=&\frac{\pi}{16}\Big(k(k^2-9k-34)(1+\delta_{k,0})+12(k+3)\delta_{k,1}\Big)\\
&+\frac{3\pi}{2}(k-1)(k+3)c_{k-1}-\frac{3\pi}{4}(k-2)(k+3)c_{k-2}. \qquad (2.25)
\end{aligned}
$$

## 3. Chebyshev Dual-Petrov-Galerkin Methods

In this section, we propose new Chebyshev dual-Petrov-Galerkin methods for solving the odd-order equations. The main idea is to find bi-orthogonal polynomials with respect to the bilinear forms, such that both the exact solution and the approximate solution can be expressed explicitly.

### 3.1. First-order equation

To illustrate the attractive properties of new Chebyshev dual-Petrov-Galerkin method, we first consider the following first-order linear hyperbolic equation (cf. [26]):

$$\begin{cases} u_t + au_x = f, & (x,t) \in I \times (0,T], \\ u(-1,t) = c(t), & t \in [0,T], \\ u(x,0) = u_0(x), & x \in \bar{I}, \end{cases} \tag{3.1}$$

where $a$ is a positive constant. Since the non-homogeneous boundary condition $u(-1,t) = c(t)$ can be easily homogenized by subtracting a simple linear function from the exact solution, we shall only consider, without loss of generality, the case $c(t) = 0$. The existence and uniqueness of the solution for (3.1) can be found in [9]. Moreover, the dual-Petrov-Galerkin method is always stable without any sign restriction on the coefficient $a$ as shown in [26].

Let $P_N(I)$ be the space of all polynomials of degree $\le N$. We define

$$X(I) = \{u \in H^1_\omega(I) : u(-1) = 0\}, \qquad X_N(I) = X(I) \cap P_N(I), \tag{3.2a}$$

$$X^*(I) = \{u \in H^1_\omega(I) : u(1) = 0\}, \qquad X^*_N(I) = X^*(I) \cap P_N(I), \tag{3.2b}$$

and denote by $\tau$ the time step size, $M = [\frac{T}{\tau}]$ and $u^{(k)}(x) = u(x, k\tau),\ k = 0, 1, \cdots, M$. Then a standard centered difference scheme in time is given by

$$\begin{cases} \dfrac{u^{(k+1)}(x) - u^{(k)}(x)}{\tau} + a\dfrac{\partial_x u^{(k+1)}(x) + \partial_x u^{(k)}(x)}{2} = \dfrac{f^{(k+1)}(x) + f^{(k)}(x)}{2}, \\ u^{(k)}(-1) = 0, \qquad k = 0, 1, \cdots, M, \\ u^{(0)}(x) = u_0(x), \qquad x \in I. \end{cases} \tag{3.3}$$

A weak formulation of (3.3) is to find $u^{(k+1)} \in X(I)$ such that

$$\mathcal{A}_a(u^{(k+1)}, v) := 2(u^{(k+1)}, v)_\omega + a\tau(\partial_x u^{(k+1)}, v)_\omega = (g^{(k)}, v)_\omega, \quad \forall v \in X^*(I), \tag{3.4a}$$

where

$$g^{(k)}(x) = \tau f^{(k+1)}(x) + \tau f^{(k)}(x) + 2u^{(k)}(x) - a\tau\partial_x u^{(k)}(x). \tag{3.4b}$$

The Chebyshev dual-Petrov-Galerkin scheme for (3.4) is to find $u_N^{(k+1)} \in X_N(I)$ such that

$$\mathcal{A}_a(u_N^{(k+1)}, \phi) = (g_N^{(k)}, \phi), \qquad \phi \in X^*_N(I), \tag{3.5a}$$

where

$$g_N^{(k)}(x) = \tau f^{(k+1)}(x) + \tau f^{(k)}(x) + 2u_N^{(k)}(x) - a\tau\partial_x u_N^{(k)}(x). \tag{3.5b}$$

To propose efficient Chebyshev dual-Petrov-Galerkin approximation scheme for (3.5), we need to construct two kinds of basis functions $\{\varphi_k\}$ and $\{\psi_k\}$, which are bi-orthogonal with respect to the bilinear operator $\mathcal{A}_a(\cdot,\cdot)$.

**Lemma 3.1.** *Let $\varphi_k \in X_{k+1}(I)$ and $\psi_k \in X^*_{k+1}(I)$ be the bi-orthogonal Chebyshev polynomials such that $\varphi_k - s_k \in X_k(I),\ \psi_k - t_k \in X^*_k(I)$ and*

$$\mathcal{A}_a(\varphi_k, \psi_l) = \sigma_k \delta_{k,l}, \qquad \forall k,\ l \ge 0. \tag{3.6}$$

*Then we have*

$$\varphi_k(x) = s_k(x) + e_{k,1}\varphi_{k-1}(x) + e_{k,2}\varphi_{k-2}(x), \tag{3.7a}$$
$$\psi_k(x) = t_k(x) + f_{k,1}\psi_{k-1}(x) + f_{k,2}\psi_{k-2}(x), \tag{3.7b}$$

*where* $\varphi_k(x) = \psi_k(x) \equiv 0,\ \sigma_k = 0$ *for* $k < 0,\ e_{k,i} = f_{k,i} = 0$ *for* $k < i$, *and*

$$\begin{aligned}\sigma_k =& -\frac{1+3\delta_{k,0}}{4}\pi + \frac{a\tau k + a\tau + 2}{2}c_k\pi - \frac{2a\tau k + \delta_{k,1} + 1}{4(1+\delta_{k,1})}c_{k-1}\pi \\ &- e_{k,1}f_{k,1}\sigma_{k-1} - e_{k,2}f_{k,2}\sigma_{k-2},\end{aligned} \tag{3.8a}$$
$$e_{k,1} = \frac{1}{\sigma_{k-1}}\left(-\frac{a\tau(k-1)}{4}\pi + e_{k,2}f_{k-1,1}\sigma_{k-2}\right), \qquad e_{k,2} = \frac{1+\delta_{k,2}}{4\sigma_{k-2}}\pi, \tag{3.8b}$$
$$f_{k,1} = \frac{1}{\sigma_{k-1}}\left(\frac{a\tau k}{4}c_{k-1}\pi + e_{k-1,1}f_{k,2}\sigma_{k-2}\right), \qquad f_{k,2} = \frac{1+\delta_{k,2}}{4\sigma_{k-2}}\pi. \tag{3.8c}$$

*Proof.* Let

$$\varphi_k(x) = s_k(x) + \sum_{i=1}^{k} e_{k,i}\varphi_{k-i}(x), \qquad \psi_k(x) = t_k(x) + \sum_{i=1}^{k} f_{k,i}\psi_{k-i}(x). \tag{3.9}$$

Then, by (2.7), (2.8) and (2.2), we deduce that for any $0 \le l \le k-3$,

$$(\partial_x s_k, t_l)_\omega = 0.$$

Hence, by (3.4), (2.6) and (2.2), we further derive that for any $0 \le l \le k-3$,

$$\mathcal{A}_a(s_k, \psi_l) = 2(s_k, \psi_l)_\omega + a\tau(\partial_x s_k, \psi_l)_\omega = 0. \tag{3.10}$$

On the other hand, by (3.9) and (3.6) we get that for $0 \le l \le k-3$,

$$\mathcal{A}_a(s_k, \psi_l) = \mathcal{A}_a(\varphi_k - \sum_{i=1}^{k} e_{k,i}\varphi_{k-i}, \psi_l) = -e_{k,k-l}\sigma_l. \tag{3.11}$$

Thus, $e_{k,k-l} = 0$ for any $0 \le l \le k-3$. This means

$$\varphi_k(x) = s_k(x) + e_{k,1}\varphi_{k-1}(x) + e_{k,2}\varphi_{k-2}(x). \tag{3.12}$$

Similarly, we deduce that

$$\psi_k(x) = t_k(x) + f_{k,1}\psi_{k-1}(x) + f_{k,2}\psi_{k-2}(x). \tag{3.13}$$

It remains to confirm the coefficients $e_{k,i},\ f_{k,i}$ and $\sigma_k$. By (3.4), (2.6)–(2.8) and (2.2) we know that

$$\mathcal{A}_a(s_k, t_{k-2}) = 2(s_k, t_{k-2})_\omega + a\tau(\partial_x s_k, t_{k-2})_\omega = -\frac{1+\delta_{k,2}}{4}\pi. \tag{3.14}$$

On the other hand, by (3.4), (3.12), (3.13) and (3.6) we get

$$\begin{aligned}&\mathcal{A}_a(s_k, t_{k-2}) \\ =& A(\varphi_k - e_{k,1}\varphi_{k-1} - e_{k,2}\varphi_{k-2}, \psi_{k-2} - f_{k-2,1}\psi_{k-3} - e_{k-2,2}\psi_{k-4}) \\ =& - e_{k,2}\sigma_{k-2}.\end{aligned} \tag{3.15}$$

Thus $e_{k,2} = \dfrac{1+\delta_{k,2}}{4\sigma_{k-2}}\pi$. Similarly, we have

$$\begin{aligned}
\mathcal{A}_a(s_k, t_{k-1}) &= -e_{k,1}\sigma_{k-1} + e_{k,2}f_{k-1,1}\sigma_{k-2} = \frac{\pi}{4}a\tau(k-1),\\
\mathcal{A}_a(s_k, t_k) &= \sigma_k + e_{k,1}f_{k,1}\sigma_{k-1} + e_{k,2}f_{k,2}\sigma_{k-2}\\
&= -\frac{\pi}{4}\Big(1+3\delta_{k,0}\Big) + \frac{\pi}{2}\Big(a\tau k + a\tau + 2\Big)c_k - \frac{\pi}{4}\frac{2a\tau k + \delta_{k,1} + 1}{(1+\delta_{k,1})}c_{k-1},\\
\mathcal{A}_a(s_k, t_{k+1}) &= -f_{k+1,1}\sigma_k + e_{k,1}f_{k+1,2}\sigma_{k-1} = -\frac{\pi}{4}a\tau(k+1)c_k,\\
\mathcal{A}_a(s_k, t_{k+2}) &= -f_{k+2,2}\sigma_k = -\frac{\pi}{4}(1+\delta_{k,0}).
\end{aligned}$$

These give the results of (3.8).

Obviously, $X_N(I) = \{\varphi_k(x) : 0 \le k \le N-1\}$ and $X_N^*(I) = \{\psi_k(x) : 0 \le k \le N-1\}$. Thus the variational formulation (3.5) together with the biorthogonality of $\{\varphi_k(x)\}$ and $\{\psi_k(x)\}$ leads to the following main theorem in this subsection.

**Theorem 3.1.** *Let $u_N^{(k+1)}(x)$ be the solution of* (3.5). *Then we have*

$$u_N^{(k+1)}(x) = \sum_{l=0}^{N-1} \hat{u}_l^{(k+1)}\varphi_l(x), \qquad \hat{u}_l^{(k+1)} = \frac{1}{\sigma_l}\mathcal{A}_a(u_N^{(k+1)}, \psi_l) = \frac{1}{\sigma_l}(g_N^{(k)}, \psi_l), \qquad l \ge 0. \tag{3.16}$$

**Remark 3.1.** The convergence of a semi-discrete Chebyshev dual-Petrov-Galerkin scheme for problem (3.1) can be found in [26], which states

$$\begin{aligned}
&\|u-u_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \|u-u_N\|_{L^2(0,T;L^2_{\omega_1}(I))}\\
\le& cN^{1-r}\Big(\|\partial_t\partial_x^{r-1}u\|_{L^2(0,T;L^2_{\omega^{r-5/2,r-5/2}})} + \|\partial_x^r u\|_{L^\infty(0,T;L^2_{\omega^{r-3/2,r-3/2}}(I))}\Big),
\end{aligned}$$

where $u_N$ is the numerical solution of the semi-discrete Chebyshev dual-Petrov-Galerkin scheme, $\omega_0(x) = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{3}{2}}$, $\omega_1(x) = (1-x)^{-\frac{1}{2}}(1+x)^{-\frac{5}{2}}$ and $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$.

### 3.2. Third-order equation

We next consider the following third-order equation (cf. [23]):

$$\begin{cases} \alpha u - \beta u_x - \gamma u_{xx} + u_{xxx} = f, \quad x \in I = (-1,1),\\ u(\pm 1) = u_x(1) = 0, \end{cases} \tag{3.17}$$

where $\alpha, \beta, \gamma$ are given constants. Without loss of generality, we consider only homogeneous boundary conditions, since the nonhomogeneous boundary conditions $u(-1) = c_1, u(1) = c_2$ and $u_x(1) = c_3$ can be handled easily by considering $v = u - \hat{u}$, where $\hat{u}$ is the unique quadratic polynomial satisfying the nonhomogeneous boundary conditions.

Define

$$\begin{aligned}
&V(I) = \{u \in H^1_\omega(I) : u(\pm 1) = u_x(1) = 0\}, && V_N(I) = V(I) \cap P_N(I),\\
&V^*(I) = \{u \in H^1_\omega(I) : u(\pm 1) = u_x(-1) = 0\}, && V_N^*(I) = V^*(I) \cap P_N(I).
\end{aligned} \tag{3.18}$$

A weak formulation of (3.17) is to find $u \in V(I)$ such that

$$(\alpha u - \beta u_x - \gamma u_{xx} + u_{xxx}, v)_\omega = (f, v)_\omega, \qquad \forall v \in V^*(I). \tag{3.19}$$

The Chebyshev dual-Petrov-Galerkin scheme for (3.19) is to find $u_N \in V_N(I)$ such that

$$\begin{aligned}\mathcal{A}_{\alpha,\beta,\gamma}(u_N,\phi) :=& \alpha(u_N,\phi)_\omega + \beta(u_N,\partial_x(\phi\omega)) + \gamma(\partial_x u_N, \partial_x(\phi\omega)) \\ &+ (\partial_x u_N, \partial_x^2(\phi\omega)) = (f,\phi)_\omega, \qquad \forall \phi \in V_N^*(I). \end{aligned} \tag{3.20}$$

To propose an efficient approximation scheme for (3.20), we need to construct two kinds of basis functions $\{\Phi_k\}$ and $\{\Psi_k\}$, which are bi-orthogonal with respect to the bilinear operator $\mathcal{A}_{\alpha,\beta,\gamma}(\cdot,\cdot)$.

**Lemma 3.2.** *Let $\Phi_k \in V_{k+3}(I)$ and $\Psi_k \in V_{k+3}^*(I)$ be the bi-orthogonal Chebyshev polynomials such that $\Phi_k - p_k \in V_{k+2}(I)$, $\Psi_k - q_k \in V_{k+2}^*(I)$ and*

$$\mathcal{A}_{\alpha,\beta,\gamma}(\Phi_k,\Psi_l) = \eta_k \delta_{k,l}, \qquad \forall k,l \geq 0. \tag{3.21}$$

*Then we have*

$$\Phi_k(x) = p_k(x) + \sum_{i=1}^{6} g_{k,i}\Phi_{k-i}(x), \qquad \Psi_k(x) = q_k(x) + \sum_{i=1}^{6} h_{k,i}\Psi_{k-i}(x), \tag{3.22}$$

*where $\Phi_k(x) \equiv \Psi_k(x) \equiv 0$, $\eta_k = 0$ for $k < 0$, $g_{k,i} = h_{k,i} = 0$ for $k < i$, and*

$$\eta_k = \alpha a_{kk} + \beta b_{kk} + \gamma c_{kk} + d_{kk} - \sum_{i=1}^{6} g_{k,i}h_{k,i}\eta_{k-i}, \tag{3.23a}$$

$$g_{k,1} = \frac{1}{\eta_{k-1}}\Big( - \beta b_{k,k-1} - \gamma c_{k,k-1} - d_{k,k-1} + \sum_{i=2}^{6} g_{k,i}h_{k-1,i-1}\eta_{k-i}\Big), \tag{3.23b}$$

$$g_{k,2} = \frac{1}{\eta_{k-2}}\Big( - \alpha a_{k,k-2} - \beta b_{k,k-2} - \gamma c_{k,k-2} - d_{k,k-2} + \sum_{i=3}^{6} g_{k,i}h_{k-2,i-2}\eta_{k-i}\Big), \tag{3.23c}$$

$$g_{k,3} = \frac{1}{\eta_{k-3}}\Big( - \beta b_{k,k-3} - \gamma c_{k,k-3} - d_{k,k-3} + \sum_{i=4}^{6} g_{k,i}h_{k-3,i-3}\eta_{k-i}\Big), \tag{3.23d}$$

$$g_{k,4} = \frac{1}{\eta_{k-4}}\Big( - \alpha a_{k,k-4} - \beta b_{k,k-4} - \gamma c_{k,k-4} + \sum_{i=5}^{6} g_{k,i}h_{k-4,i-4}\eta_{k-i}\Big), \tag{3.23e}$$

$$g_{k,5} = \frac{1}{\eta_{k-5}}(-\beta b_{k,k-5} + g_{k,6}h_{k-5,1}\eta_{k-6}), \qquad g_{k,6} = \frac{-\alpha a_{k,k-6}}{\eta_{k-6}}, \tag{3.23f}$$

$$h_{k,1} = \frac{1}{\eta_{k-1}}\Big( - \beta b_{k-1,k} - \gamma c_{k-1,k} - d_{k-1,k} + \sum_{i=2}^{6} h_{k,i}g_{k-1,i-1}\eta_{k-i}\Big), \tag{3.23g}$$

$$h_{k,2} = \frac{1}{\eta_{k-2}}\Big( - \alpha a_{k-2,k} - \beta b_{k-2,k} - \gamma c_{k-2,k} - d_{k-2,k} + \sum_{i=3}^{6} h_{k,i}g_{k-2,i-2}\eta_{k-i}\Big), \tag{3.23h}$$

$$h_{k,3} = \frac{1}{\eta_{k-3}}\Big( - \beta b_{k-3,k} - \gamma c_{k-3,k} - d_{k-3,k} + \sum_{i=4}^{6} h_{k,i}g_{k-3,i-3}\eta_{k-i}\Big), \tag{3.23i}$$

$$h_{k,4} = \frac{1}{\eta_{k-4}}\Big( - \alpha a_{k-4,k} - \beta b_{k-4,k} - \gamma c_{k-4,k} + \sum_{i=5}^{6} h_{k,i}g_{k-4,i-4}\eta_{k-i}\Big), \tag{3.23j}$$

$$h_{k,5} = \frac{1}{\eta_{k-5}}(-\beta b_{k-5,k} + h_{k,6}g_{k-5,1}\eta_{k-6}), \qquad h_{k,6} = \frac{-\alpha a_{k-6,k}}{\eta_{k-6}}. \tag{3.23k}$$

*Proof.* Let

$$\Phi_k(x) = p_k(x) + \sum_{i=1}^{k} g_{k,i}\Phi_{k-i}(x), \qquad \Psi_k(x) = q_k(x) + \sum_{i=1}^{k} h_{k,i}\Psi_{k-i}(x). \tag{3.24}$$

Then, by (3.20), (2.15), (2.20), (2.21), (2.25) and integration by parts, we deduce that for any $0 \le l \le k-7$,

$$\begin{aligned}&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, \Psi_l)\\ =&\alpha(p_k, \Psi_l)_\omega + \beta(p_k, \partial_x(\Psi_l\omega)) + \gamma(\partial_x p_k, \partial_x(\Psi_l\omega)) + (\partial_x p_k, \partial_x^2(\Psi_l\omega)) = 0.\end{aligned} \tag{3.25}$$

On the other hand, by (3.24) and (3.21) we get that for $0 \le l \le k-7$,

$$\mathcal{A}_{\alpha,\beta,\gamma}(p_k, \Psi_l) = \mathcal{A}_{\alpha,\beta,\gamma}(\Phi_k - \sum_{i=1}^{k} g_{k,i}\Phi_{k-i}, \Psi_l) = -g_{k,k-l}\eta_l. \tag{3.26}$$

Hence, $g_{k,k-l} = 0$ for any $0 \le l \le k-7$. This means

$$\Phi_k(x) = p_k(x) + \sum_{i=1}^{6} g_{k,i}\Phi_{k-i}(x).$$

Similarly, we deduce that

$$\Psi_k(x) = q_k(x) + \sum_{i=1}^{6} h_{k,i}\Psi_{k-i}(x).$$

It remains to confirm the coefficients $g_{k,i}, h_{k,i}$ and $\eta_k$. By (3.20), (2.15), (2.20), (2.21) and (2.25) we know that

$$\begin{aligned}&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-6})\\ =&\alpha(p_k, q_{k-6})_\omega + \beta(p_k, \partial_x(q_{k-6}\omega)) + \gamma(\partial_x p_k, \partial_x(q_{k-6}\omega)) + (\partial_x p_k, \partial_x^2(q_{k-6}\omega))\\ =&\alpha a_{k,k-6}.\end{aligned} \tag{3.27}$$

On the other hand, by (3.21) and (3.22) we get

$$\begin{aligned}&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-6})\\ =&\mathcal{A}_{\alpha,\beta,\gamma}\left(\Phi_k - \sum_{i=1}^{6} g_{k,i}\Phi_{k-i}, \Psi_{k-6} - \sum_{i=1}^{6} h_{k-6,i}\Psi_{k-6-i}\right) = -g_{k,6}\eta_{k-6}.\end{aligned} \tag{3.28}$$

Thus $g_{k,6} = \dfrac{-\alpha a_{k,k-6}}{\eta_{k-6}}$. Similarly, we have

$$\begin{aligned}
&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-5}) = -g_{k,5}\eta_{k-5} + g_{k,6}h_{k-5,1}\eta_{k-6} = \beta b_{k,k-5},\\
&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-4}) = -g_{k,4}\eta_{k-4} + \sum_{i=5}^{6} g_{k,i}h_{k-4,i-4}\eta_{k-i} = \alpha a_{k,k-4} + \beta b_{k,k-4} + \gamma c_{k,k-4},\\
&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-3}) = -g_{k,3}\eta_{k-3} + \sum_{i=4}^{6} g_{k,i}h_{k-3,i-3}\eta_{k-i} = \beta b_{k,k-3} + \gamma c_{k,k-3} + d_{k,k-3},\\
&\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-2}) = -g_{k,2}\eta_{k-2} + \sum_{i=3}^{6} g_{k,i}h_{k-2,i-2}\eta_{k-i}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha a_{k,k-2} + \beta b_{k,k-2} + \gamma c_{k,k-2} + d_{k,k-2},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k-1}) &= -g_{k,1}\eta_{k-1} + \sum_{i=2}^{6} g_{k,i}h_{k-1,i-1}\eta_{k-i} = \beta b_{k,k-1} + \gamma c_{k,k-1} + d_{k,k-1},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_k) &= \eta_k + \sum_{i=1}^{6} g_{k,i}h_{k,i}\eta_{k-i} = \alpha a_{kk} + \beta b_{kk} + \gamma c_{kk} + d_{kk},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k+1}) &= -h_{k+1,1}\eta_k + \sum_{i=1}^{5} g_{k,i}h_{k+1,1+i}\eta_{k-i} = \beta b_{k,k+1} + \gamma c_{k,k+1} + d_{k,k+1},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k+2}) &= -h_{k+2,2}\eta_k + \sum_{i=1}^{4} g_{k,i}h_{k+2,2+i}\eta_{k-i}\\
&= \alpha a_{k,k+2} + \beta b_{k,k+2} + \gamma c_{k,k+2} + d_{k,k+2},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k+3}) &= -h_{k+3,3}\eta_k + \sum_{i=1}^{3} g_{k,i}h_{k+3,3+i}\eta_{k-i} = \beta b_{k,k+3} + \gamma c_{k,k+3} + d_{k,k+3},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k+4}) &= -h_{k+4,4}\eta_k + \sum_{i=1}^{2} g_{k,i}h_{k+4,4+i}\eta_{k-i} = \alpha a_{k,k+4} + \beta b_{k,k+4} + \gamma c_{k,k+4},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k+5}) &= -h_{k+5,5}\eta_k + g_{k,1}h_{k+5,6}\eta_{k-1} = \beta b_{k,k+5},\\
\mathcal{A}_{\alpha,\beta,\gamma}(p_k, q_{k+6}) &= -h_{k+6,6}\eta_k = \alpha a_{k,k+6}.
\end{aligned}
$$

These lead to the desired results in (3.23). □

Obviously, $V_N(I) = \{\Phi_k(x) : 0 \le k \le N-3\}$ and $V_N^*(I) = \{\Psi_k(x) : 0 \le k \le N-3\}$. Thus by (3.19) and (3.20) and the biorthogonality of $\{\Phi_k(x)\}$ and $\{\Psi_k(x)\}$, we obtain the following main theorem in this subsection.

**Theorem 3.2.** *Let $u(x)$ and $u_N(x)$ be the solutions of* (3.19) *and* (3.20), *respectively. Then both $u(x)$ and $u_N(x)$ have the explicit representations in* $\{\Phi_k(x)\}$,

$$
u(x) = \sum_{k=0}^{\infty} \hat{u}_k \Phi_k(x), \qquad u_N(x) = \sum_{k=0}^{N-3} \hat{u}_k \Phi_k(x), \tag{3.29a}
$$

$$
\hat{u}_k = \frac{1}{\eta_k}\mathcal{A}_{\alpha,\beta,\gamma}(u, \Psi_k) = \frac{1}{\eta_k}(f, \Psi_k)_\omega, \qquad k \ge 0. \tag{3.29b}
$$

**Remark 3.2.** According to Theorem 2.2 of [23], for any $\alpha, \beta \ge 0$ and $-\frac{1}{3} < \gamma < \frac{1}{6}$, there exists a unique solution for the system (3.20), satisfying

$$
\begin{aligned}
&\alpha\|u - u_N\|_{\omega^{-1,1}} + N^{-1}\|\partial_x(u - u_N)\|_{\omega^{-1,0}}\\
&\le c(1 + |\gamma|N)N^{-m}\|\partial_x^m u\|_{\omega^{m-2,m-1}}, \quad m \ge 1,
\end{aligned}
$$

where $\omega^{a,b}(x) = (1-x)^a(1+x)^b$.

### 3.3. Application to the KdV equation

There exist a large body of literature on the theoretical and numerical results of the KdV type equations (see, e.g., [3,6,19] and the references therein). As an example of Chebyshev dual-Petrov-Galerkin method for nonlinear problems, we consider the third-order KdV equation on a finite interval (cf. [23]):

$$\begin{cases} \epsilon u_t + \nu u_x + \mu u u_x + u_{xxx} = f, & (x,t) \in I \times (0,T], \\ u(\pm 1, t) = u_x(1,t) = 0, & t \in [0,T], \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{3.30}$$

Denote by $\tau$ the time step size, $M = \left[\frac{T}{\tau}\right]$ and $u^{(k)}(x) = u(x, k\tau)$, $k = 0, 1, \cdots, M$. Then a standard centered difference scheme in time is given by

$$\begin{cases} \epsilon \dfrac{u^{(k+1)} - u^{(k)}}{\tau} + \nu \dfrac{u_x^{(k+1)} + u_x^{(k)}}{2} + \mu \dfrac{u^{(k+1)} u_x^{(k+1)} + u^{(k)} u_x^{(k)}}{2} \\ \qquad + \dfrac{u_{xxx}^{(k+1)} + u_{xxx}^{(k)}}{2} = \dfrac{f^{(k+1)} + f^{(k)}}{2}, \quad x \in I, \quad k = 0, 1, \cdots, M, \\ u^{(k)}(\pm 1) = u_x^{(k)}(1) = 0, \qquad k = 0, 1, \cdots, M, \\ u^{(0)}(x) = u_0(x), \qquad x \in I. \end{cases} \tag{3.31}$$

A weak formulation of (3.31) is to find $u^{(k+1)} \in V(I)$ such that

$$(2\epsilon u^{(k+1)} + \nu\tau \partial_x u^{(k+1)} + \tau \partial_x^3 u^{(k+1)}, v)_\omega = (g^{(k)}, v)_\omega, \quad \forall v \in V^*(I), \tag{3.32}$$

where

$$\begin{aligned} g^{(k)} =& \tau f^{(k+1)} + \tau f^{(k)} + 2\epsilon u^{(k)} - \nu\tau \partial_x u^{(k)} - \mu\tau (u^{(k+1)} \partial_x u^{(k+1)} \\ &+ u^{(k)} \partial_x u^{(k)}) - \tau \partial_x^3 u^{(k)}. \end{aligned}$$

The Chebyshev dual-Petrov-Galerkin scheme for (3.32) is to find $u_N^{(k+1)} \in V_N(I)$ such that

$$\begin{aligned} &\mathcal{B}_{\epsilon,\nu,\mu}(u_N^{(k+1)}, v) \\ :=& 2\epsilon (u_N^{(k+1)}, v)_\omega - \nu\tau (u_N^{(k+1)}, \partial_x(v\omega)) + \tau (\partial_x u_N^{(k+1)}, \partial_x^2(v\omega)) \\ =& (g_N^{(k)}, v)_\omega, \quad \forall v \in V_N^*(I), \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} g_N^{(k)} =& \tau f^{(k+1)} + \tau f^{(k)} + 2\epsilon u_N^{(k)} - \nu\tau \partial_x u_N^{(k)} - \mu\tau (u_N^{(k+1)} \partial_x u_N^{(k+1)} \\ &+ u_N^{(k)} \partial_x u_N^{(k)}) - \tau \partial_x^3 u_N^{(k)}. \end{aligned}$$

Let $\Phi_k$ and $\Psi_k$ be the bi-orthogonal Chebyshev polynomials as defined in Lemma 3.2 with $\alpha = \frac{2\epsilon}{\tau}$, $\beta = -\nu$ and $\gamma = 0$. It is clear that

$$\mathcal{B}_{\epsilon,\nu,\mu}(\Phi_k, \Psi_l) = \tau \mathcal{A}_{\alpha,\beta,0}(\Phi_k, \Psi_l) = \tau \eta_k \delta_{k,l}, \qquad \forall k, l \geq 0, \tag{3.34}$$

**Theorem 3.3.** *Let $u_N^{(k+1)}(x)$ be the solution of* (3.33)*. Then we have*

$$u_N^{(k+1)}(x) = \sum_{l=0}^{N-3} \hat{u}_l^{(k+1)} \Phi_l(x), \tag{3.35a}$$

where

$$\hat{u}_l^{(k+1)} = \frac{1}{\tau \eta_l} \mathcal{B}_{\epsilon,\nu,\mu}(u_N^{(k+1)}, \Psi_l) = \frac{1}{\tau \eta_l} (g_N^{(k)}, \Psi_l), \qquad l \geq 0. \tag{3.35b}$$

## 4. Numerical Experiments

In this section, we examine the effectiveness and accuracy of the Chebyshev dual-Petrov-Galerkin spectral method for solving the odd-order equations.

Let $x_{N,k} = -\cos\frac{(2k+1)\pi}{2N+2}$, $0 \le k \le N$ be the zeros of the Chebyshev polynomials $T_{N+1}(x)$, and $\omega_{N,k} = \frac{\pi}{N+1}$, $0 \le k \le N$ stand for the corresponding weights of the Chebyshev-Gauss quadrature. The discrete $L^2_\omega(I)$ and $L^\infty(I)$ errors are measured by

$$E_{N,1} = \left(\sum_{k=0}^{N}(u(x_{N,k}) - u_N(x_{N,k}))^2\omega_{N,k}\right)^{\frac{1}{2}}, \qquad E_{N,2} = \max_{0\le k\le N}|u(x_{N,k}) - u_N(x_{N,k})|.$$

### 4.1. First-order equation

We first use (3.16) to solve problem (3.1) with $a = 1$ and take the exact solution

$$u(x,t) = (1+x)\sin(kx+t), \tag{4.1}$$

which oscillates seriously for large $k$. In Fig. 4.1, we plot the $\log_{10}$ of the discrete $L^2_\omega(I)$ errors vs. $N$ with $k = 4$ and $\tau = 0.1, 0.01, 0.001, 0.0001$. Clearly, the numerical errors decay rapidly as $N$ increases and $\tau$ decreases. In Fig. 4.2, we plot the $\log_{10}$ of the discrete $L^2_\omega(I)$ errors vs $N$ with $k = 20$ and $\tau = 0.1, 0.01, 0.001, 0.0001$, which shows that our new spectral approach still works well even for the solutions oscillating seriously. In Fig. 4.3, we plot the values of discrete $L^2_\omega(I)$ and $L^\infty(I)$ errors for $0 \le t \le 100$ with $k = 20$ and $\tau = 0.0001$. It demonstrates the stability of long-time calculations of the suggested Chebyshev spectral approach (3.16).

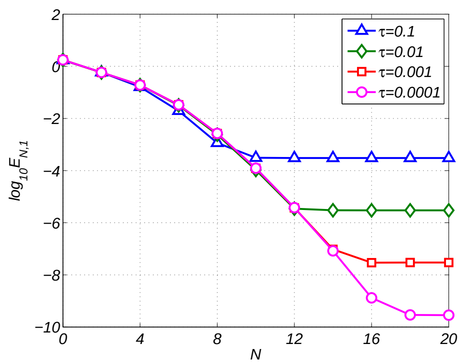


Fig. 4.1. The discrete $L^2_\omega(I)$ errors with the exact solution (4.1) and $k = 4$.

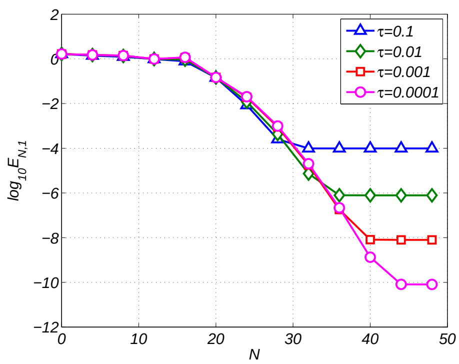


Fig. 4.2. The discrete $L^2_\omega(I)$ errors with the exact solution (4.1) and $k = 20$.

### 4.2. Third-order equation

We next use (3.27) to solve problem (3.17) with $\alpha = \beta = \gamma = 1$ and take the exact solution

$$u(x) = (1+x)(1-x)^2\sin(kx), \tag{4.2}$$

which also oscillates seriously for large $k$. In Fig. 4.4, we plot the $\log_{10}$ of the discrete $L^2_\omega(I)$ and $L^\infty(I)$ errors vs. $N$ with $k = 10$. The near straight lines indicate a geometric convergence rate. In Fig. 4.5, we also compare the discrete $L^2_\omega(I)$ errors of our new method with the classical

Chebyshev dual-Petrov-Galerkin method for $k = 20$. Clearly, the numerical errors are almost the same. The main reason is that the approximation spaces of these two methods are exactly the same.

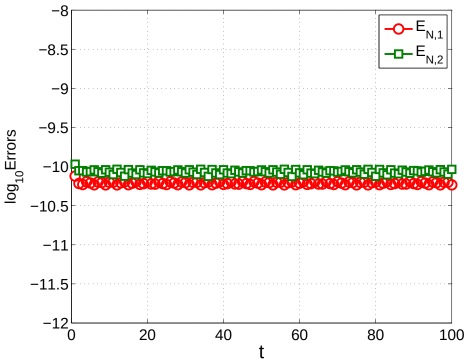


Fig. 4.3. Stability of (3.16) with the exact solution (4.1) and $k = 20$, $\tau = 0.0001$.

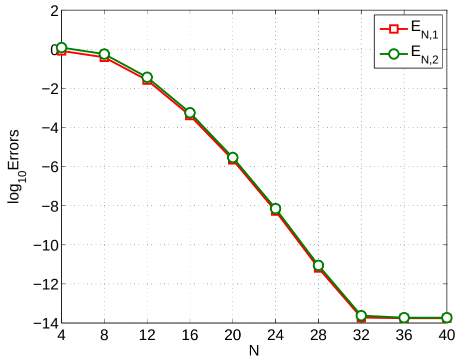


Fig. 4.4. The discrete errors with the exact solution (4.2) and $k = 10$.

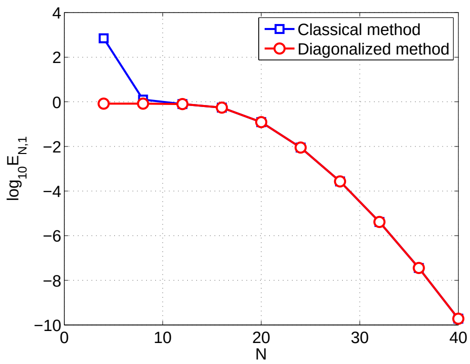


Fig. 4.5. Numerical comparison of our method with the classical method.

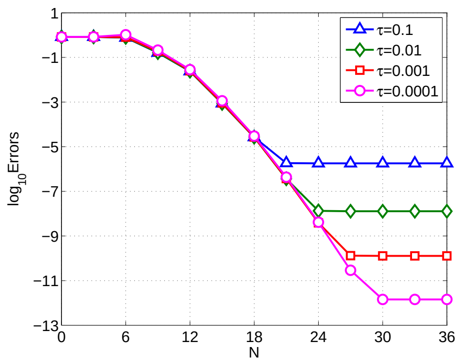


Fig. 4.6. The discrete $L^2_\omega(I)$ errors with the exact solution (4.3) and $k = 10$.

We now use (3.35) to solve the third-order KdV equation (3.30) with $\epsilon = \nu = \mu = 1$ and take the exact solution

$$u(x,t) = (1+x)(1-x)^2 \sin(kx+t), \tag{4.3}$$

which oscillates seriously for large $k$. In Fig. 4.6, we plot the $\log_{10}$ of the discrete $L^2_\omega(I)$ errors vs. $N$ with $\tau = 0.1, 0.01, 0.001, 0.0001$ and $k = 10$. Clearly, a geometric convergence rate is observed. It also indicates that the smaller the time step size $\tau$, the smaller the numerical errors would be. In Fig. 4.7, we plot the values of discrete $L^2_\omega(I)$ and $L^\infty(I)$ errors for $0 \le t \le 100$ with $k = 10$ and $\tau = 0.0001$. It demonstrates the stability of long-time calculation of our new Chebyshev spectral approach (3.35).

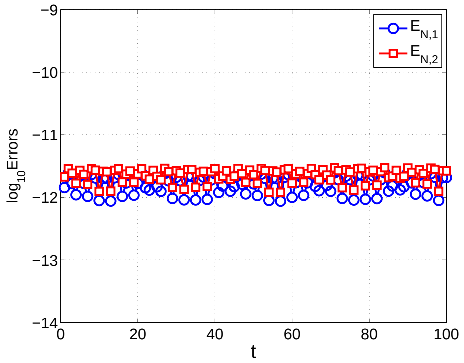


Fig. 4.7. Stability of (3.35) with the exact solution (4.3) and $k = 10,\ \tau = 0.0001$.

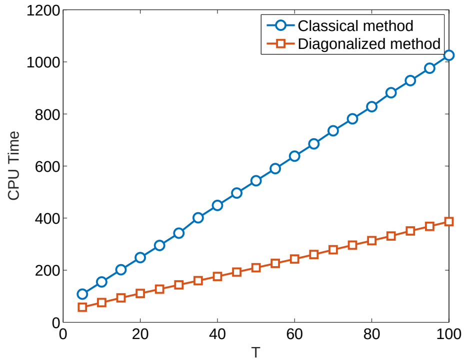


Fig. 4.8. Comparison of the CPU time (the unit is seconds).

### 4.3. An extension to the fifth-order equation

The suggested methods can also be extended to the fifth-order equation. For instance, let us consider the solitary wave solutions of the following Kawahara equation (cf. [15,28]),

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \quad u(x,0) = u_{ex}(x,0), \quad x \in (-\infty, +\infty), \tag{4.4}$$

where

$$u_{ex}(x,t) = \frac{105}{169}\text{sech}^4\Big(\frac{1}{2\sqrt{13}}(x - \frac{36t}{169} - x_0)\Big)$$

is an exact soliton solution of (4.4).

In order to apply the dual-Petrov-Galerkin method, we fix $x_0 = 0$ and restrict problem (4.4) to the finite interval $[-L, L]$ with $L$ sufficiently large such that the solution $u_{ex}(\pm L, t)$, $\partial_x u_{ex}(\pm L, t)$ and $\partial_x^2 u_{ex}(L, t)$ are essentially zero for $t \in [0, T]$ (where $T$ is given). As in [28], we apply the scaling $(\tilde{x}, \tilde{t}) = (L^{-1}x, L^{-1}t)$, and for the sake of simplicity, we still use $(x,t)$ to denote $(\tilde{x}, \tilde{t})$. Then, we are led to consider the following scaled Kawahara equation:

$$\begin{cases} u_t + uu_x + \frac{1}{L^2}u_{xxx} - \frac{1}{L^4}u_{xxxxx} = 0, \quad x \in I = (-1,1), \\ u(\pm 1, t) = u_x(\pm 1, t) = u_{xx}(1,t) = 0, \\ u(x,0) = \frac{105}{169}\text{sech}^4(\frac{L}{2\sqrt{13}}x). \end{cases} \tag{4.5}$$

Denote by $\tau$ the time step size, $M = \big[\frac{T}{\tau}\big]$ and $u^{(k)}(x) = u(x, k\tau)$. The second-order Crank-Nicolson leap-frog scheme in time of (4.5) is given by

$$\begin{cases} \frac{1}{2\tau}(u^{(k+1)} - u^{(k-1)}) + u^{(k)}\partial_x u^{(k)} + \frac{1}{2L^2}(\partial_x^3 u^{(k+1)} + \partial_x^3 u^{(k-1)}) \\ \qquad -\frac{1}{2L^4}(\partial_x^5 u^{(k+1)} + \partial_x^5 u^{(k-1)}) = 0, \quad x \in I, \quad k = 1, \cdots, M, \\ u^{(k)}(\pm 1) = u_x^{(k)}(\pm 1) = u_{xx}^{(k)}(1) = 0, \qquad k = 1, \cdots, M, \\ u^{(0)}(x) = \frac{105}{169}\text{sech}^4(\frac{L}{2\sqrt{13}}x), \qquad x \in I. \end{cases} \tag{4.6}$$

In order to apply the dual-Petrov-Galerkin method for the spatial discretization of (4.6), we choose the basis functions $\varphi_i \in P_N(I)$ and $\psi_j \in P_N(I)$, such that

$$\begin{aligned}
&\varphi_i(\pm 1) = \partial_x \varphi_i(\pm 1) = \partial_x^2 \varphi_i(1) = 0,\\
&\psi_j(\pm 1) = \partial_x \psi_j(\pm 1) = \partial_x^2 \psi_j(-1) = 0,\\
&(\varphi_i + \tfrac{\tau}{L^2}\partial_x^3 \varphi_i - \tfrac{\tau}{L^4}\partial_x^5 \varphi_i, \psi_j)_\omega = \eta_i \delta_{i,j}, \qquad \forall i, j \geq 0.
\end{aligned}$$

To shorten the length of the paper, we omit specific details. In Table 4.1, we take $L = 200$ and $N = 500$ in our dual-Petrov-Galerkin scheme, and list the $L^2$-errors at different times with two different time steps. In Table 4.2, we also list the corresponding numerical results given in [28] with $L = 200$ and $N = 1000$. It can be observed that we get almost the same numerical results as in [28] with only half of the basis functions.

Table 4.1: $L^2$-errors with $L = 200$ and $N = 500$.

| Time | $L^2$-errors with $\tau = 10^{-4}$ | $L^2$-errors with $\tau = 2 \times 10^{-4}$ |
|---|---|---|
| 0.5 | 3.445e-07 | 1.378e-06 |
| 1.0 | 5.956e-07 | 2.382e-06 |
| 2.0 | 1.150e-06 | 4.601e-06 |
| 3.0 | 1.829e-06 | 7.317e-06 |
| 4.0 | 2.981e-06 | 1.173e-05 |

Table 4.2: $L^2$-errors given in [28] with $L = 200$ and $N = 1000$.

| Time | $L^2$-errors with $\tau = 10^{-4}$ | $L^2$-errors with $\tau = 2 \times 10^{-4}$ |
|---|---|---|
| 0.5 | 3.440e-07 | 1.374e-06 |
| 1.0 | 5.926e-07 | 2.358e-06 |
| 2.0 | 1.104e-06 | 4.389e-06 |
| 4.0 | 2.147e-06 | 8.494e-06 |

Table 4.3: Condition numbers of the classical Chebyshev dual-Petrov-Galerkin method.

| Matrices | $N = 20$ | $N = 40$ | $N = 60$ | $N = 80$ | $N = 100$ |
|---|---|---|---|---|---|
| $\alpha(p_k, q_l)_\omega$ | 2.1425e+05 | 8.4529e+06 | 8.1079e+07 | 4.1712e+08 | 1.5081e+09 |
| $\beta(p_k, \partial_x(q_l\omega))$ | 7.2978e+04 | 2.0854e+06 | 1.5369e+07 | 6.4014e+07 | 1.9432e+08 |
| $\gamma(\partial_x p_k, \partial_x(q_l\omega))$ | 3.0807e+04 | 5.1311e+05 | 2.6905e+06 | 8.7219e+06 | 2.1702e+07 |
| $(\partial_x p_k, \partial_x^2(q_l\omega))$ | 1.5696e+03 | 1.5886e+04 | 6.0025e+04 | 1.5242e+05 | 3.1217e+05 |
| $\mathcal{A}(p_k, q_l)$ | 1.5597e+03 | 1.5757e+04 | 5.9601e+04 | 1.5148e+05 | 3.1044e+05 |

Table 4.4: Condition numbers of our Chebyshev dual-Petrov-Galerkin method.

| Matrices | $N = 20$ | $N = 40$ | $N = 60$ | $N = 80$ | $N = 100$ |
|---|---|---|---|---|---|
| $\alpha(\Phi_k, \Psi_l)_\omega$ | 1.4185e+05 | 6.0063e+06 | 5.9768e+07 | 3.1486e+08 | 1.1580e+09 |
| $\beta(\Phi_k, \partial_x(\Psi_l\omega))$ | 2.5813e+03 | 3.3275e+04 | 1.5946e+05 | 4.9417e+05 | 1.1977e+06 |
| $\gamma(\partial_x\Phi_k, \partial_x(\Psi_l\omega))$ | 1.0592e+02 | 3.6569e+02 | 7.9066e+02 | 1.3843e+03 | 2.1490e+03 |
| $(\partial_x\Phi_k, \partial_x^2(\Psi_l\omega))$ | 1.3826e+00 | 1.3842e+00 | 1.3846e+00 | 1.3848e+00 | 1.3848e+00 |
| $\mathcal{A}(\Phi_k, \Psi_l)$ | 1.0000e+00 | 1.0000e+00 | 1.0000e+00 | 1.0000e+00 | 1.0000e+00 |

### 4.4. Comparisons of condition numbers and computational costs

To demonstrate the essential superiority of our new Chebyshev dual-Petrov-Galerkin method to the classical Chebyshev dual-Petrov-Galerkin method, we examine the issues on the 2-norm condition numbers for the resulting algebraic systems and the computational costs.

For the classical Chebyshev dual-Petrov-Galerkin method, the basis functions are chosen as $\{p_k(x)\}_{k=0}^{N-3}$ and $\{q_k(x)\}_{k=0}^{N-3}$ for problem (3.17). The corresponding matrices have off-diagonal entries. In Tables 4.3 and 4.4, we list the condition numbers of two kinds of numerical methods for problem (3.17) with $\alpha = \beta = \gamma = 1$. Notice that the condition numbers of the mass matrices are almost the same, but the condition numbers of the total matrices are very different. Particularly, the condition numbers of the total matrices for the classical Chebyshev dual-Petrov-Galerkin method increase asymptotically as $O(N^3)$, while the condition numbers of the total matrices for our Chebyshev dual-Petrov-Galerkin method with respect to the basis functions $\{\frac{1}{\sqrt{\eta_k}}\Phi_k(x)\}_{k=0}^{N-3}$ and $\{\frac{1}{\sqrt{\eta_k}}\Psi_k(x)\}_{k=0}^{N-3}$ are always equal to 1.

In Fig. 4.8, we consider the problem (3.1) with $N = 60$, $T = 100$ and $\tau = 0.0001$ and compare the computation costs of our method with that of the classical Chebyshev dual-Petrov-Galerkin spectral method. Clearly, our method costs much less CPU time.

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