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SCHWARZ METHOD FOR FINANCIAL ENGINEERING*

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Abstract

Schwarz method is put forward to solve second order backward stochastic differential equations (2BSDEs) in this work. We will analyze uniqueness, convergence, stability and optimality of the proposed method. Moreover, several simulation results are presented to demonstrate the effectiveness; several applications of the 2BSDEs are investigated. It is concluded from these results that the proposed the method is powerful to calculate the 2BSDEs listing from the financial engineering.

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1. Introduction

The search for fast and efficient schemes of the backward stochastic differential equations (BSDEs) is a challenging task. The study of parallel and distributed solutions is thus important. These solutions employ two basic forms: domain decomposition and function decomposition (task decomposition). The former is based on original BSDEs. It yields several sub-systems in parallel, on each subdomain. The latter sub-divides the system of BSDEs into many components (or task) to be parallelized.

Motivated by applications and probabilistic numerical methods for second order BSDEs (2BSDEs), Cheridito et al. (2007) considered the connection between the 2BSDEs and fully nonlinear parabolic PDEs. This connection is found through the dependence of a drift part. In addition, Soner et al. (2012) proposed a form of 2BSDEs in connection with G-expectations and G-martingales. We now present our discussed 2BSDEs, written by

$$\begin{cases} dY_t = -f(t, X_t, Y_t, Z_t, \Gamma_t)dt + Z_t dB_t, \\ dZ_t = -A_t dt - \Gamma_t dB_t, \quad t \in [0, T], \end{cases} \tag{1.1}$$

where $Y_T = \phi(X_T)$ and $Z_T = z$. A_t and Γ are all measurable processes. ϕ and f are all deterministic functions. X_t is a diffusion process. $B_t = (B_t^1, \cdots, B_t^r)^T$ is a r-dimensional Brownian motion. Here $f(t, X_t, Y_t, Z_t, \Gamma_t) = f(t, X_t, Y_t, Z_t) + Tr(\Gamma_t)/2$. It is worthy noting

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that, the system of 2BSDEs (1.1) is a special case of G-BSDEs mentioned by Hu et al. (2014). Based on stochastic integral theory, the 2BSDEs can be listed as

$$Y_t = \phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, \Gamma_s)ds - \int_t^T Z_s dB_s, \tag{1.2a}$$

$$Z_t = z + \int_t^T A_s ds + \int_t^T \Gamma_s dB_s. \tag{1.2b}$$

With $|\pi| = \max_i |t_{i+1} - t_i|$, and the partition $\pi = \{0 = t_0 \leq \cdots \leq t_i \leq \cdots \leq t_N = T\}$ on $[0, T]$. $Y_T^\pi = \phi(X_T^\pi)$. X^π is a corresponding discretisation of X. We then have the following time Euler discretisation of the 2BSDEs:

$$\begin{cases} Y_{t_i}^\pi = E_i^\pi[Y_{t_{i+1}}^\pi] + f(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi, \Gamma_{t_i}^\pi)(t_{i+1} - t_i), \\ Z_{t_i}^\pi = \dfrac{1}{(t_{i+1} - t_i)} E_i^\pi[Y_{t_{i+1}}^\pi (B_{t_{i+1}} - B_{t_i})], \\ \Gamma_{t_i}^\pi = \dfrac{1}{(t_{i+1} - t_i)} E_i^\pi[Z_{t_{i+1}}^\pi (B_{t_{i+1}} - B_{t_i})], \\ A_{t_i}^\pi = \dfrac{1}{(t_{i+1} - t_i)} E_i^\pi[Z_{t_{i+1}}^\pi]; i \in [0, N-1], \end{cases} \tag{1.3}$$

which is similar to the works of Cheridito et al. (2007) and Soner et al. (2012). Under several regularity conditions, a solution exists on $\hat{Y}_t = u(t, X_t)$. We consider the following backward second order parabolic PDEs on $[0, T) \times \mathbb{R}^r$, given by

$$\partial_t u(t, x) + f(t, x, u(t, x), Du(t, x), D^2 u(t, x)) = 0, \tag{1.4a}$$

$$u(T, x) = \phi(x), \tag{1.4b}$$

where $x = (x^1, \cdots, x^r) \in \mathbb{R}^r$. Here

$$\partial_t = \frac{\partial}{\partial t}, \quad Du = (D_i u), \quad D^2 u = (D_{ij} u), \quad D_{ij} = D_i D_j, \quad D_i = \frac{\partial}{\partial x^i}.$$

Then

$$\hat{Y}_t = u(t, X_t), \quad \hat{Z}_t = Du(t, X_t), \quad \hat{\Gamma}_t = D^2 u(t, X_t), \quad \hat{A}_t = \mathcal{L} Du(t, X_t)$$

is a solution of the 2BSDEs. The Dynkin operator $\mathcal{L}$ of X is without the drift term, see also Cheridito et al. (2007).

As to the aforementioned 2BSDEs, we give several conditions useful for the uniqueness of solution.

(A0). For $(t, x, y, z) \in [0, T] \times \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^d$, $f(t, x, y, z, \gamma) \geq f(t, x, y, z, \tilde{\gamma})$ whenever $\gamma \leq \tilde{\gamma}$ with $\gamma, \tilde{\gamma} \in \mathbb{R}^d$. For all $t \in [0, T]$, the PDEs (1.4) satisfy the comparison principle.

(A1). f satisfies Lipschitz condition when $\|v\|_2 + \|N_0\|_2 + \|Q\|_2$ is bounded. That is, for fixed t, there exists a constant K_0 (> 0) satisfying

$$\Big\| f(t, x, u + v, P + Q, M + N_0) - f(t, x, u, P, M) \Big\|_2 \leq \frac{K_0}{T - t} \Big(\|v\|_2 + \|N_0\|_2 + \|Q\|_2 \Big).$$

(A0) constraints the viscosity of Equations (1.4). (A1) constraints the elements of $f(t, x, \cdot, \cdot, \cdot)$, and is a stronger consistency condition. In this case, Equations (1.4) have a solution. A uniqueness theorem and a simplified proof are respectively as follows.

Theorem 1.1 (Uniqueness of the associated PDE). *Assume that* $(A0)$ *and* $(A1)$ *hold on* $[0,T]\times\mathbb{R}^r$ *and that the* $2BSDEs$ (1.1) *have a solution* $(\hat{Y}_t,\hat{Z}_t,\hat{\Gamma}_t;\hat{A}_t)$. *Then the associated PDEs have a unique solution* u *on* $[0,T]\times\mathbb{R}^r$, *with*

$$\hat{Y}_t = u(t,X_t).$$

Moreover, $(\hat{Y}_t,\hat{Z}_t,\hat{\Gamma}_t,\hat{A}_t)$ *is a unique solution of the equations.*

Proof. Note that $(\hat{Y}_t,\hat{Z}_t,\hat{\Gamma}_t,\hat{A}_t)$ is a solution of the 2BSDEs. Using dynamic programming principle, it follows that, for all $t\in[0,T]$,

$$\hat{Y}_t \geq \overline{U}(t,X_t) \quad \text{and} \quad \hat{Y}_t \leq \underline{U}(t,X_t),$$

where

$$\underline{U}(t,X_t) = \inf\{y : y \geq \varphi(X_T)\} \quad \text{and} \quad \overline{U}(t,X_t) = \sup\{y : y \leq \varphi(X_T)\}.$$

Observe that $X_t = x+B_t$ on $[0,T]$, so X_t has full support. We also have $\underline{U}(t,X_t)\leq\overline{U}(t,X_t)$. We set

$$\underline{U_*}(t,x) = \liminf_{(\tilde{t},\tilde{x})\to(t,x)} \underline{U}(\tilde{t},\tilde{x})m, \quad \overline{U_*}(t,x) = \limsup_{(\tilde{t},\tilde{x})\to(t,x)} \overline{U}(\tilde{t},\tilde{x}), \quad (t,x)\in[0,T]\times\mathbb{R}^r.$$

Thus $\underline{U_*}$ and $\overline{U_*}$ are viscosity supersolution and subsolution of Eq. (1.4), respectively. Hence, we have $\overline{U_*}\leq\underline{U_*}$ through (A0). Then $u=\overline{U_*}=\overline{U}=\underline{U}=\underline{U_*}$ is a unique viscosity solution of Equations (1.4).

By the above statement, $\hat{Y}_t = u(t,X_t)$ on $[0,T]$. $(\hat{Y}_t,\hat{Z}_t,\hat{\Gamma}_t,\hat{A}_t)$ is then a unique solution of the equations. $\square$

With f satisfying the above equations, Cheridito et al. (2007) summarized the relation of the 2BSDEs and the associated PDEs in the linear case, the semi-linear case, and the quasi-linear case and the fully nonlinear case. In the work, we propose a novel Schwarz method to solve the mentioned 2BSDEs. We have organized the remainder as follows. In Section 2, with a given time window, we present a Schwarz method to solve the associated PDEs. We also discuss its properties. In Section 3, we introduce Schwarz waveform relaxation, which is applied to the 2BSDEs. In Section 4, we consider convergence of the proposed method. To examine the method, Section 5 presents several simulations studies. Section 6 presents an application of the 2BSDEs in finance. Conclusion is presented in Section 7.

2. Schwarz Method for the Associated PDEs on Time Windows

We now present a Schwarz method as a domain decomposition method to solve the 2BSDEs. Bounded domain Ω in $\mathbb{R}^r$ is split into m (≥ 2) subdomains. For example, with $\Omega=\bigcup_{j=1}^m\Omega_j$, Ω_j indicates a one-dimensional decomposition. Let $\delta_j^+ = \{x\in\Omega_j\bigcap\Omega_{j+1} : x\geq y, y\in\Omega_j\bigcap\Omega_{j+1}\}$ and $\delta_j^- = \{x\in\Omega_j\bigcap\Omega_{j+1} : x\leq y, y\in\Omega_j\bigcap\Omega_{j+1}\}$. We set $U_j=[t_i,t_{i+1})\times\Omega_j$ for $j\in[1,m]$, and have $U=[t_i,t_{i+1})\times\Omega=\bigcup_{j=1}^m U_j$. We are interested in the solution of the associated PDEs (1.4) on $[t_i,t_{i+1})$. The proposed Schwarz method is defined on $[t_i,t_{i+1})$ by, for $n\geq 1$,

$$\begin{cases} \partial_t u_j^{(n)}(t,x) = -f(t,x,u_j^{(n)},Du_j^{(n)},D^2u_j^{(n)}), \quad (t,x)\in U_j; \\ u_j^{(n)}(t,x) = u_{j+1}^{(n-1)}(t,x), \quad x=\delta_j^+, \\ u_j^{(n-1)}(t,x) = u_{j+1}^{(n)}(t,x), \quad x=\delta_j^-; \\ u_j^{(n)}(T,x) = \phi(x), \quad x\in\Omega_j, j=1,\cdots,m, \end{cases} \tag{2.1}$$

where an initial guess $u_j^{(0)}(t,0)$ (often abbreviated $u_j^{(0)}$) has to be provided. For a guess u_0, we connect each subdomain Ω_j with $u_j^{(0)} = u_0$ on $\Omega_j \bigcap \Omega_{j+1}$.

For classical Schwarz waveform with $m = 2, r = 1$, we deduce that

$$\begin{cases} \partial_t u_j^{(n)}(t,x) = -f(t,x,u_j^{(n)}, Du_j^{(n)}, D^2 u_j^{(n)}), \quad (t,x) \in U_j; \\ u_1^{(n)}(t,x) = u_2^{(n-1)}(t,x), \quad x = \delta_1^+, \\ u_1^{(n-1)}(t,x) = u_2^{(n)}(t,x), \quad x = \delta_1^-; \\ u_j^{(n)}(T,x) = \phi(x), \quad x \in \Omega_j, j = 1,2. \end{cases} \tag{2.2}$$

Let $d_j^{(n)} = u_j^{(n)} - u_j$ and $d_j^{(0)} = u_j^{(0)} - u_j$. For $x_M = \delta_j^+$ $(j \geq 1)$, $x_m = \delta_j^-$, we have

$$d_j^{(n)}(t,x_M) = d_{j+1}^{(n-1)}(t,x_M) \quad \text{and} \quad d_j^{(n-1)}(t,x_m) = d_{j+1}^{(n)}(t,x_m).$$

The aforementioned method is characterized by two properties under the following conditions.

(A2). For fixed $u_j^{(0)}$, there is a monotone iterative sequence $\{u_j^{(n)}\}_{n=0}^{\infty}$ which is uniformly convergent on each t and x. Moreover,

$$\lim_{n\to\infty} u_j^{(n)} = u_j, \quad \|u_j^{(n)} - u_j\|_2 \leq p_0 \|u_j^{(n-1)} - u_j\|_2,$$
$$\|D(u_j^{(n)} - u_j)\|_2 \leq p_1 \|u_j^{(n-1)} - u_j\|_2 \text{ for } p_0, p_1 \in (0,1).$$

(A3). For any j,x_m and x_M in Equations (2.2), we have

$$\|d_j^{(n-1)}(t,x_m)\|_2 = \rho_1 \min\left\{\|d_j^{(n-1)}(t,x_M)\|_2, \|d_{j+1}^{(n-1)}(t,x_m)\|_2\right\}, \tag{2.3a}$$

$$\|d_{j+1}^{(n-1)}(t,x_M)\|_2 = \rho_1 \max\left\{\|d_j^{(n-1)}(t,x_M)\|_2, \|d_{j+1}^{(n-1)}(t,x_m)\|_2\right\} \quad \text{for } \rho_1 \in (0,1). \tag{2.3b}$$

If the above conditions are satisfied, we get convergence and bound stability of the proposed method. (A2) is a monotonicity condition, and gives the convergence range of $u_j^{(n)}$ and $u_j^{(n-1)}$. (A3) is a bound monotonicity condition, and provides the confining region of u_j and u_{j+1}. We have the following convergence under the above conditions.

Theorem 2.1 (Convergence). *Assume that* (A1) *and* (A2) *hold. Then the Schwarz method given by Eq.* (2.1), *converges, namely,*

$$\lim_{n\to\infty} \|d_j^{(n)}\|_2 = 0. \tag{2.4}$$

Proof. With fixed j,

$$d_j^{(n)} = u_j^{(n)} - u_j = \int_t^T f(s,x,u_j^{(n)}, Du_j^{(n)}, D^2u_j^{(n)}) - f(s,x,u_j,Du_j,D^2u_j)ds. \tag{2.5}$$

We observe that

$$\begin{aligned} &f(s,x,u_j^{(n)}, Du_j^{(n)}, D^2u_j^{(n)}) - f(s,x,u_j,Du_j,D^2u_j) \\ &= f(s,x,u_j^{(n)}, Du_j^{(n)}, D^2u_j^{(n)}) - f(s,x,u_j,Du_j^{(n)}, D^2u_j^{(n)}) \\ &\quad + f(s,x,u_j,Du_j^{(n)}, D^2u_j^{(n)}) - f(s,x,u_j,Du_j,D^2u_j^{(n)}) \\ &\quad + f(s,x,u_j,Du_j,D^2u_j^{(n)}) - f(s,x,u_j,Du_j,D^2u_j). \end{aligned} \tag{2.6}$$

Through (A1), we have

$$\begin{aligned}&\left\|f(s,x,u_j^{(n)},Du_j^{(n)},D^2u_j^{(n)})-f(s,x,u_j,Du_j,D^2u_j)\right\|_2\\&\le\frac{K_0}{T-t}\Big(\|u_j^{(n)}-u_j\|_2+\|D(u_j^{(n)}-u_j)\|_2+\|D^2(u_j^{(n)}-u_j)\|_2\Big).\end{aligned}\tag{2.7}$$

Consequently,

$$d_j^{(n)}\le K_0\Big(\|u_j^{(n)}-u_j\|_2+\|D(u_j^{(n)}-u_j)\|_2+\|D^2(u_j^{(n)}-u_j)\|_2\Big).\tag{2.8}$$

Through (A2), we obtain

$$\|D^2(u_j^{(n)}-u_j)\|_2\le p_1\|D(u_j^{(n-1)}-u_j)\|_2\le p_1^2\|u_j^{(n-2)}-u_j\|_2,\tag{2.9a}$$

$$\begin{aligned}&\|u_j^{(n)}-u_j\|_2+\|D(u_j^{(n)}-u_j)\|_2+\|D^2(u_j^{(n)}-u_j)\|_2\\&\qquad\le\Big(p_0^n+p_1p_0^{n-1}+p_1^2p_0^{n-2}\Big)\cdot\|u_j^{(0)}-u_j\|_2.\end{aligned}\tag{2.9b}$$

Due to

$$\lim_{n\to\infty}\big(p_0^n+p_1p_0^{n-1}+p_1^2p_0^{n-2}\big)=0,\tag{2.10}$$

we conclude that (2.4) holds. □

The stability of the method is presented under the above given conditions.

Theorem 2.2 (Bound stability). *Assume that* (A3) *holds, then the Schwarz method* (2.1) *converges. That is, for* ρ_1 (<1),

$$\left\|d_j^{(n)}(t,x_M)\|_2+\|d_{j+1}^{(n)}(t,x_m)\right\|_2\le\rho_1^n\Big(\|d_j^{(0)}(t,x_M)\|_2+\|d_{j+1}^{(0)}(t,x_m)\|_2\Big).\tag{2.11}$$

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Proof. For fixed j and any n, we set $r_j^{(n)}=\|d_j^{(n)}(t,x_M)\|_2+\|d_{j+1}^{(n)}(t,x_m)\|_2$. Observe that

$$\|d_j^{(n)}(t,x_M)\|_2=\|u_j^{(n)}(t,x_M)-u\|_2=\|u_j^{(n-1)}(t,x_M)-u\|_2=\|d_{j+1}^{(n-1)}(t,x_M)\|_2,\tag{2.12a}$$
$$\|d_{j+1}^{(n)}(t,x_m)\|_2=\|u_{j+1}^{(n)}(t,x_m)-u\|_2=\|u_j^{(n-1)}(t,x_m)\|_2=\|d_j^{(n-1)}(t,x_m)\|_2.\tag{2.12b}$$

Then,

$$r_j^{(n)}=\|d_j^{(n)}(t,x_M)\|_2+\|d_{j+1}^{(n)}(t,x_m)\|_2=\|d_j^{(n-1)}(t,x_m)\|_2+\|d_{j+1}^{(n-1)}(t,x_M)\|_2.\tag{2.13}$$

Through (A3), we deduce

$$r_j^{(n)}=\rho_1\bigg(\|d_j^{(n-1)}(t,x_M)\|_2+\|d_{j+1}^{(n-1)}(t,x_m)\|_2\bigg)=\rho_1r_j^{(n-1)}=\rho_1^nr_j^{(0)}.\tag{2.14}$$

Therefore, the theorem is proven. □

Although we present two properties of our Schwarz method, we only indirectly obtain the solution of the 2BSDEs. To solve the 2BSDEs in directly, we will give a novel Schwarz method to solve these equations in the next section.

3. Schwarz Method for 2BSDEs with Time Windows

Schwarz-like methods are very popular in scientific computing, see also, Tran (2011), Guo (2012, 2018), Guo and Zhao (2012), Kim and Zhang (2015), Dryja, et al. (2016), Yin, et al. (2016), Ciaramella, et al. (2017), Magoules, et al. (2017), Axelsson and Gustafsson (2019), among others. They decomposed an original computational problem into several sub-problems easier to solve it. We are now interested in the solution problem of the 2BSDEs (1.2). If we denote the columns of Z_t and Γ_t by, Z_t^l and Γ_t^l, respectively, then the 2BSDEs (1.2) can be written as

$$Y_t = \phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, \Gamma_s)ds - \int_t^T \sum_{l=1}^d Z_s^l dB_s^l, \tag{3.1a}$$

$$Z_t^l = z^l + \int_t^T A_s^l ds + \int_t^T \Gamma_s^l dB_s^l \text{ for } l \in [1, d]. \tag{3.1b}$$

We consider the case of m-subdomain, and choose initial guesses Y^0, Z^0 and Γ^0. We obtain $Y^0_{t(j)}$, $Z^0_{t(j)}$ and $\Gamma^0_{t(j)}$ on subdomain Ω_j, and set $Y^0_{t(j)} = Y^0$, $Z^0_{t(j)} = Z^0$ and $\Gamma^0_{t(j)} = \Gamma^0$ on $\Omega_j \bigcap \Omega_{j+1}$. For $n \geq 1$, the proposed Schwarz method is defined on $[t_i, t_{i+1})$,

$$\begin{cases} Y^{(n)}_{t(j)} = \phi(X_T) + \int_t^T f(s, X_s, Y^{(n)}_{s(j)}, Z^{(n)}_{s(j)}, \Gamma^{(n)}_{s(j)})ds + \int_t^T \sum_{l=1}^d Z^{l(n)}_{s(j)} dB_s^l, \\ Z^{l(n)}_{t(j)} = z^l + \int_t^T \Gamma^{l(n-1)}_{s(j)} dB_s^l, A^{l(n)}_{t(j)} = \frac{1}{t_{i+1} - t_i} E\big(Z^{l(n-1)}_{t(j)}\big), \\ \Gamma^{l(n)}_{t(j)} = \frac{1}{t_{i+1} - t_i} E\Big[Z^{l(n-1)}_{t(j)}(B^l_{t_{i+1}} - B^l_{t_i})\Big], \quad l \in [1, d], (t, x) \in U_j; \\ Y^{(n)}_{t(j)} = Y^{(n-1)}_{t(j+1)}, \quad Z^{(n)}_{t(j)} = Z^{(n-1)}_{t(j+1)}, \quad \Gamma^{(n)}_{t(j)} = \Gamma^{(n-1)}_{t(j+1)}, \quad A^{(n)}_{t(j)} = A^{(n-1)}_{t(j+1)}, \quad x \in \delta_j^+; \\ Y^{(n-1)}_{t(j)} = Y^{(n)}_{t(j+1)}, \quad Z^{(n-1)}_{t(j)} = Z^{(n)}_{t(j+1)}, \quad \Gamma^{(n-1)}_{t(j)} = \Gamma^{(n)}_{t(j+1)}, \quad A^{(n-1)}_{t(j)} = A^{(n)}_{t(j+1)}, \quad x \in \delta_j^-; \\ Y^{(n)}_{T(j)} = \phi(x), \quad Z^{(n)}_{T(j)} = z, \quad x \in \Omega_j, j = 1, \cdots, m, \end{cases} \tag{3.2}$$

where $Y^0_{t(j)}$, $Z^0_{t(j)}$ and $\Gamma^0_{t(j)}$ are given. If $j = 2$ and $d = 1$ in Eq. (3.2), we obtain classical Schwarz waveform to solve the 2BSDEs.

We define, for $n \geq 1$,

$$e^{(n)}_{t(j)} = Y^{(n)}_{t(j)} - Y_t, \quad \varepsilon^{l(n)}_{t(j)} = Z^{l(n)}_{t(j)} - Z_t^l, \quad p\epsilon^{l(n)}_{t(j)} = \Gamma^{l(n)}_{t(j)} - \Gamma_t^l, \quad \omega^{l(n)}_{t(j)} = A^{l(n)}_{t(j)} - A_t^l.$$

We also set

$$f_j(s) = f(s, X_s, Y_{s(j)}, Z_{s(j)}, \Gamma_{s(j)}), \quad f_j(s)^{(n)} = f(s, X_s, Y^{(n)}_{s(j)}, Z^{(n)}_{s(j)}, \Gamma^{(n)}_{s(j)}).$$

The main simplified assumptions are as follows:

(A4). For fixed s, $f_j(s)$ satisfies the following Lipschitz condition,

$$\sup_{(s,x)\in U_j} \big|e^{(n)}_{s(j)}\big|^2 + \sum_{l=1}^d \Big(\big|\varepsilon^{l(n)}_{s(j)}\big|^2 + \big|\epsilon^{l(n)}_{s(j)}\big|^2\Big) < +\infty. \tag{3.3}$$

There is a given constant K_1 such that

$$\big|f_j(s) - f_j(s)^{(n)}\big|^2 \leq K_1 \Big[\sup_{(s,x)\in U_j} \big|e^{(n-1)}_{s(j)}\big|^2 + \sum_{l=1}^d \Big(\big|\varepsilon^{l(n-1)}_{s(j)}\big|^2 + \big|\epsilon^{l(n-1)}_{s(j)}\big|^2\Big)\Big], \tag{3.4}$$

where $K_1 \in (0,1)$ and $K_1^2 + (T-t_i)^2 < 1$.

(A5). For every j, $\{\Gamma^{l(n)}_{t(j)}\}$ is a monotone iteration sequence on $[t_i, t_{i+1})$, which converge uniformly such that

$$\lim_{n\to\infty} \Gamma^{l(n)}_{t(j)} = \Gamma^l_t. \tag{3.5}$$

That is, there exists a given K_2 such that

$$\big|\epsilon^{l(n)}_{t(j)}\big|^2 \le K_2 \big|\epsilon^{l(n-1)}_{t(j)}\big|^2 \quad \text{for } K_2 \in (0,1). \tag{3.6}$$

(A6). There are given ρ_2 and ρ_3 between 0 and 1, for any j,

$$\|e^{(n)}_j(t,x_m)\|_2 = \rho_2 \min\Big\{\|e^{(n-1)}_j(t,x_M)\|_2, \|e^{(n-1)}_{j+1}(t,x_m)\|_2\Big\}, \tag{3.7a}$$

$$\|e^{(n)}_{j+1}(t,x_M)\|_2 = \rho_2 \max\Big\{\|e^{(n-1)}_j(t,x_M)\|_2, \|e^{(n-1)}_{j+1}(t,x_m)\|_2\Big\}. \tag{3.7b}$$

and

$$\|\varepsilon^{(n)}_j(t,x_m)\|_2 = \rho_3 \min\Big\{\|\varepsilon^{(n-1)}_j(t,x_M)\|_2, \|\varepsilon^{(n-1)}_{j+1}(t,x_m)\|_2\Big\}, \tag{3.8a}$$

$$\|\varepsilon^{(n)}_{j+1}(t,x_M)\|_2 = \rho_3 \max\Big\{\|\varepsilon^{(n-1)}_j(t,x_M)\|_2, \|\varepsilon^{(n-1)}_{j+1}(t,x_m)\|_2\Big\}. \tag{3.8b}$$

If three above conditions are satisfied, we present several related convergences of the proposed method. (A4) is a stronger consistency condition, and provides a three-element constraint range of $f(t, X_t, \cdot, \cdot, \cdot)$ on (Y_t, Z_t, Γ_t). (A5) is a monotonicity and constraint condition on Γ_t. (A6) is a bound monotonicity condition, and provides the range of Y_t and Z_t in the overlapping case. Before obtaining these related convergences, we present two following lemmas.

Lemma 3.1 (Doob Martingale Inequality [Revuz and Yor, 1999]). *For $p \ge 2$ and $[t_i, t_{i+1}) \in [T, 0]$,*

$$E\left[\sup_{t_i \le t < t_{i+1}} \Big|\int_{t_i}^{t_{i+1}} f(s,X_s)dB_s\Big|^p\right] \le C_p \left[\int_{t_i}^{t_{i+1}} E|f(s,X_s)|^2 ds\right]^{p/2}, \tag{3.9a}$$

where

$$C_p \le \left[\Big(\frac{p}{p-1}\Big)^p \Big(\frac{p(p-1)}{2}\Big)\right]^{p/2}. \tag{3.9b}$$

In the setting of $p = 2$, we have, for $C_1 \in (0,4]$,

$$E\left[\sup_{t_i \le t < t_{i+1}} \left(\int_{t_i}^{t_{i+1}} f(s,X_s)dB_s\right)^2\right] \le C_1 \left[\int_{t_i}^{t_{i+1}} E|f(s,X_s)|^2 ds\right]. \tag{3.10}$$

Lemma 3.2. *For $a_l > 0$ and $p \ge 2$, we have*

$$\left(\sum_{l=1}^d a_l\right)^p \le d^{p-1} \sum_{l=1}^d a_l^p.$$

We now prove convergence of the proposed method on $[t_i, t_{i+1})$.

Theorem 3.1 (Convergence of Z_t). *For the 2BSDEs* (1.1)*, the Schwarz method* (3.2)*, on the sequence* $\{Z_{t(j)}^{l(n)}\}_{n,j}$*, converges in mean square, namely,*

$$\lim_{n\to\infty} E\left[\sup_{t_i\le t<t_{i+1}} |\varepsilon_{t(j)}^{l(n)}|^2\right] = 0. \tag{3.11}$$

Proof. We consider the following errors and the mean squared errors of Z_t at step n:

$$\varepsilon_{t(j)}^{l(n)} = \int_t^T \epsilon_{s(j)}^{l(n-1)} dB_s^l, \quad |\varepsilon_{t(j)}^{l(n)}|^2 = \left|\int_t^T \epsilon_{s(j)}^{l(n-1)} dB_s^l\right|^2. \tag{3.12}$$

By (A5), Lemmas 3.1 and 3.2, we have, for $C_2 \in (0,4]$,

$$\begin{aligned} E\left[\sup_{t_i\le t<t_{i+1}} |\varepsilon_{t(j)}^{l(n)}|^2\right] &= E\left[\sup_{t_i\le t<t_{i+1}} \left|\int_t^T \epsilon_{s(j)}^{l(n-1)} dB_s^l\right|^2\right] \le C_2\left(\int_t^T E|\epsilon_{s(j)}^{l(n-1)}|^2 ds\right) \\ &\le C_2 \int_{t_i}^T ds \int_s^T |\epsilon_{s(j)}^{l(n-1)}|^2 ds \le C_2 K_2 \int_{t_i}^T ds \int_s^T |\epsilon_{s(j)}^{l(n-2)}|^2 ds \\ &\le \cdots \le C_2 K_2^{(n-1)} (T-t_i)^2 \sup_{t_i\le t<t_{i+1}} |\epsilon_{t(j)}^{l(0)}|^2. \end{aligned} \tag{3.13}$$

Using the above inequality, we have the theorem. □

Theorem 3.2 (Convergence of Y_t). *For the 2BSDEs by* (1.1)*, the Schwarz method* (3.2)*, on the sequence* $\{Y_{t(j)}^{(n)}\}_{n,j}$*, converges in the mean square, namely,*

$$\lim_{n\to\infty} E\Big[\sup_{t_i\le t<t_{i+1}} |Y_{t(j)}^{(n)} - Y_t|^2\Big] = 0. \tag{3.14}$$

Proof. We consider the following errors and the mean squared errors of Y_t at step n:

$$e_{t(j)}^{(n)} = Y_{t(j)}^{(n)} - Y_t = \int_t^T \Big(f_j(s)^{(n)} - f_j(s)\Big) ds + \int_t^T \sum_{l=1}^d \varepsilon_{s(j)}^{l(n)} dB_s^l, \tag{3.15}$$

$$\big|e_{t(j)}^{(n)}\big|^2 \le 2\left|\int_t^T \Big(f_j(s)^{(n)} - f_j(s)\Big) ds\right|^2 + 2\left|\int_t^T \sum_{l=1}^d \varepsilon_{s(j)}^{l(n)} dB_s^l\right|^2. \tag{3.16}$$

From the above inequality, we conclude that

$$\sup_{t_i\le t<t_{i+1}} \big|e_{t(j)}^{(n)}\big|^2 \le 2(T-t_i)\int_{t_i}^T \big|f_j(s)^{(n)} - f_j(s)\big|^2 ds + 2\sup_{t_i\le t<t_{i+1}} \left|\int_t^T \sum_{l=1}^d \varepsilon_{s(j)}^{l(n)} dB_s^l\right|^2.$$

Using Lemmas 3.1 and 3.2, we deduce that, for $C_3 \in (0,4]$,

$$\begin{aligned} &E\Big[\sup_{t_i\le t<t_{i+1}} |e_{t(j)}^{(n)}|^2\Big] \\ \le& 2(T-t_i) E\left[\int_{t_i}^T \big(f_j(s)^{(n)} - f_j(s)\big)^2 ds\right] + 2C_3(d+2)\sum_{l=1}^d \int_{t_i}^T E\big|\varepsilon_{s(j)}^{l(n)}\big|^2 ds. \end{aligned} \tag{3.17}$$

From (A4) and (A5), we then obtain

$$
\begin{aligned}
&E\Big[\sup_{t_i\le t<t_{i+1}}\big|e^{(n)}_{t(j)}\big|^2\Big]\\
\le&2(T-t_i)\int_{t_i}^{T}E\big|f_j(s)^{(n)}-f_j(s)\big|^2ds\\
&\quad+2C_2C_3(d+2)(T-t_i)^2K_2^{(n-1)}\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\big|\epsilon^{l(0)}_{t(j)}\big|^2\\
\le&2(T-t_i)\int_{t_i}^{T}K_1\cdot E\Big[\sup_{t_i\le t<t_{i+1}}\big|e^{(n-1)}_{s(j)}\big|^2+\sum_{l=1}^{d}\Big(\big|\varepsilon^{l(n-1)}_{s(j)}\big|^2+\big|\epsilon^{l(n-1)}_{s(j)}\big|^2\Big)\Big]ds\\
&\quad+2C_2C_3(d+2)(T-t_i)^2K_2^{(n-1)}\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\big|\epsilon^{l(0)}_{t(j)}\big|^2\\
\le&2K_1(T-t_i)^2E\Big[\sup_{t_i\le t<t_{i+1}}\big|e^{(n-1)}_{t(j)}\big|^2+\sum_{l=1}^{d}\Big(\big|\varepsilon^{l(n-1)}_{t(j)}\big|^2+\big|\epsilon^{l(n-1)}_{t(j)}\big|^2\Big)\Big]\\
&\quad+2C_2C_3(d+2)(T-t_i)^2K_2^{(n-1)}\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\big|\epsilon^{l(0)}_{t(j)}\big|^2\\
\le&2K_1(T-t_i)^2E\Big[\sup_{t_i\le t<t_{i+1}}\big|e^{(n-1)}_{t(j)}\big|^2\Big]+2C_2(T-t_i)^2K_2^{(n-1)}\\
&\quad\cdot\big[C_3(d+2)+K_1(T-t_i)^2+K_1/C_2\big]\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\big|\epsilon^{l(0)}_{t(j)}\big|^2. \qquad (3.18)
\end{aligned}
$$

Using (A5) and Theorem 3.1, we obtain (3.14). □

Similarly, we have the following stability of the proposed method.

Corollary 3.3 (Bound stability). *Assume that* (A6) *holds. Then the Schwarz method* (3.2) *converges sub-linearly. Namely, for* $\rho_2<1$ *and* $\rho_3<1$,

$$\|e^{(n)}_j(t,x_M)\|_2+\|e^{(n)}_{j+1}(t,x_m)\|_2=\rho_2^n\Big(\|e^{(0)}_j(t,x_M)\|_2+\|e^{(0)}_{j+1}(t,x_m)\|_2\Big), \qquad (3.19)$$

$$\|\varepsilon^{(n)}_j(t,x_M)\|_2+\|\varepsilon^{(n)}_{j+1}(t,x_m)\|_2=\rho_3^n\Big(\|\varepsilon^{(0)}_j(t,x_M)\|_2+\|\varepsilon^{(0)}_{j+1}(t,x_m)\|_2\Big). \qquad (3.20)$$

4. Optimization of Schwarz Method in the 2BSDEs

An advantage of the proposed method is that, there exists a solution sequence on subdomains. It is observed that our Schwarz method depends on the mesh size $|\pi|$, the size of subdomain Ω_j, and the coarse space U_0. For Schwarz methd (3.2), the optimization will improve convergence rate.

By (3.18), it is worthy noting that

$$E\Big[\sup_{t_i\le t<t_{i+1}}\big|e^{(n)}_{t(j)}\big|^2\Big]\le\rho E\Big[\sup_{t_i\le t<t_{i+1}}\big|e^{(n-1)}_{t(j)}\big|^2\Big]+I(t_i),\qquad \text{for } i\in[1,N-1]. \qquad (4.1)$$

Here $\rho(K_1,t_i)=2K_1(T-t_i)^2$ is called a guaranteed convergence rate. Moreover,

$$I(t_i)=2C_2(T-t_i)^2K_2^{(n-1)}\cdot\Big[C_3(d+2)+K_1(T-t_i)^2+K_1/C_2\Big]\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\big|\epsilon^{l(0)}_{t(j)}\big|^2.$$

For the min-max problem of $\rho(K_1, t_i)$ and the max-min problem of $I(t_i)$ in the above inequality, we have the following optimization theorem.

Theorem 4.1 (Optimization). *For the Schwarz method* (3.2)*, assume that* K_1 *from* (A4)*,* C_2 *and* C_3 *are constants between* 0 *and* 4*. Then* $\min_{t_i}\max(\rho(K_1,t_i))$ *and* $\max_{d,C_2,C_3}\min(I(t_i))$ *exist.*

Proof. Note that the following inequalities held:

$$2K_1(T-t_i)^2 \le K_1^2 + (T-t_i)^4, \tag{4.2}$$

and

$$\begin{aligned} & C_3(d+2) + K_1(T-t_i)^2 + K_1/C_2 \\ \ge\ & 3\sqrt[3]{C_3(d+2)\cdot K_1(T-t_i)^2\cdot K_1/C_2} \\ \ge\ & 3\sqrt[3]{\frac{C_3}{C_2}K_1^2(d+2)(T-t_i)^2}. \end{aligned} \tag{4.3}$$

We then conclude that $\max(\rho(t_i)) = K_1^2 + (T-t_i)^4$ and

$$\begin{aligned} \min(I(t_i)) &= 2C_2(T-t_i)^2K_2^{(n-1)}\cdot 3\sqrt[3]{\frac{C_3}{C_2}K_1^2(d+2)(T-t_i)^2}\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\left|\epsilon_{t(j)}^{l(0)}\right|^2 \\ &= C_2^{2/3}C_3^{1/3}(d+2)^{1/3}\cdot 6K_1^{2/3}K_2^{(n-1)}(T-t_i)^{8/3}\cdot\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\left|\epsilon_{t(j)}^{l(0)}\right|^2. \end{aligned}$$

The min-max problem of $\rho(K_1,t_i)$ is $\min_{t_i}\max(\rho(K_1,t_i)) = K_1^2 + (T-t_{N-1})^4$, and the max-min problem of $I(t_i)$ is

$$\max_{d,C_2,C_3}\min(I(t_i)) = (C_2^2 + C_3 + d + 2)\cdot 2K_1^{2/3}K_2^{(n-1)}(T-t_i)^{8/3}\cdot\sum_{l=1}^{d}\sup_{t_i\le t<t_{i+1}}\left|\epsilon_{t(j)}^{l(0)}\right|^2.$$

The proof is complete. $\square$

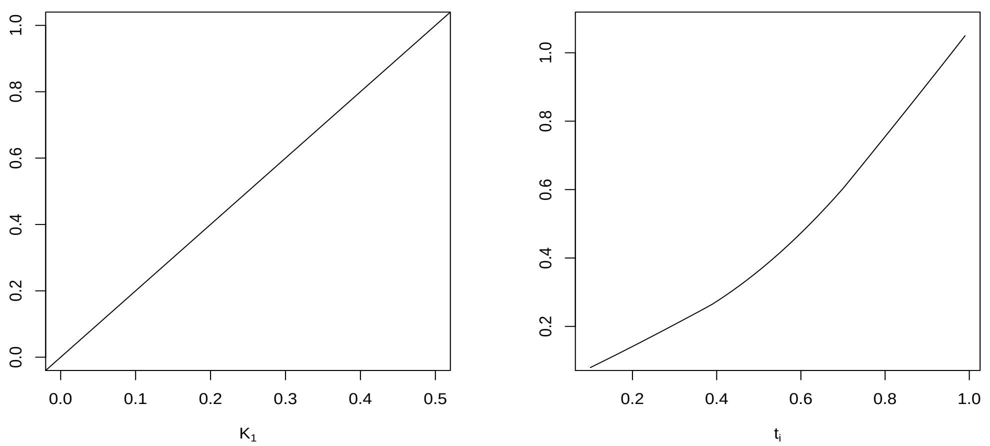

Fig. 4.1. Convergence rates of global Schwarz method for Y_t in 2BSDEs.

From Theorem 4.1, it is shown that the convergence rate $\rho(K_1, t_i)$ satisfies $\rho(K_1, t_i) < 1$ when $K_1^2 + T^4 < 1$, the additional term $I(t_i)$ is bounded. For its theoretical optimal value, we have several difficulties to obtain it. It is also noting that, information may be transferred across many subdomains through coarse grid correction. With the Schwarz method (3.2) on $j = 0, \cdots, m$, we have a global Schwarz method. Using the global method, one can obtain coarse grid correction.

5. Simulations Studies

In this section we present several simulation studies $Tk/10$ to examine our proposed method. Generally, the discretisation grid is equal to $t_k = T\frac{k}{10}$ for $0 \leq k \leq 10$. As a function basis, we fix X_t and the call payoff, namely,

$$X_t = X_T, \quad \phi(X_T) = (X_T - K)^+. \tag{5.1}$$

The put payoff is also $\phi(X_T) = (K - X_T)^+$. The basis situation is that $X_T = 99, Y_T = 1, z^l$ $(l = 1, ..., d), \Gamma_T = 2, N = 10, K = 100, d = 5$ and $T = 1$. We set $f(t, x, y, z, \omega) = y + 2z + 3\omega$. The 2BSDEs can also be written as

$$Y_t = 1 - \int_t^1 (Y_s + 2Z_s + 3\Gamma_s)ds + \int_t^1 \sum_{l=1}^5 Z_s^l dB_s^l, \tag{5.2a}$$

$$Z_t^l = 1/5 + \int_t^1 \Gamma_s^l dB_s^l \text{ for } l \in [1, 5]. \tag{5.2b}$$

In this case, the drift part of Z_t is $1/5$.

In Fig. 4.1, we plot its convergence on K_1 and t_i. On the left, the relation of $\rho(K_1, t_i)$ and K_1 is shown; on the right, the relation of $\rho(K_1, t_i)$ and t_i is given. It is shown that, one can choose $\rho(K_1, t_i)$ to obtain the expected convergence; the chosen values of K_1 and t_i guarantee the stability of the proposed method.

As to $\pi = \{0 = t_0 \leq \cdots \leq t_{10} = T\}$, $Y_T^\pi = \phi(X_T^\pi) = 1$ for put payoff (0 for call payoff). We then have the following time Euler discretisation of the 2BSDEs:

$$\begin{cases} Y_{t_i}^\pi = E_i^\pi[Y_{t_{i+1}}^\pi] + (Y_{t_i}^\pi + 2Z_{t_i}^\pi + 3\Gamma_{t_i}^\pi)(t_{i+1} - t_i), \\ Z_{t_i}^\pi = \dfrac{1}{(t_{i+1} - t_i)} E_i^\pi[Y_{t_{i+1}}^\pi (B_{t_{i+1}} - B_{t_i})], \\ \Gamma_{t_i}^\pi = \dfrac{1}{(t_{i+1} - t_i)} E_i^\pi[Z_{t_{i+1}}^\pi (B_{t_{i+1}} - B_{t_i})], \\ A_{t_i}^\pi = \dfrac{1}{(t_{i+1} - t_i)} E_i^\pi[Z_{t_{i+1}}^\pi]; i \in [0, 9]. \end{cases} \tag{5.3}$$

For the case of m-subdomain, the Schwarz method is defined on $[t_i, t_{i+1})$ through $n \geq 1$,

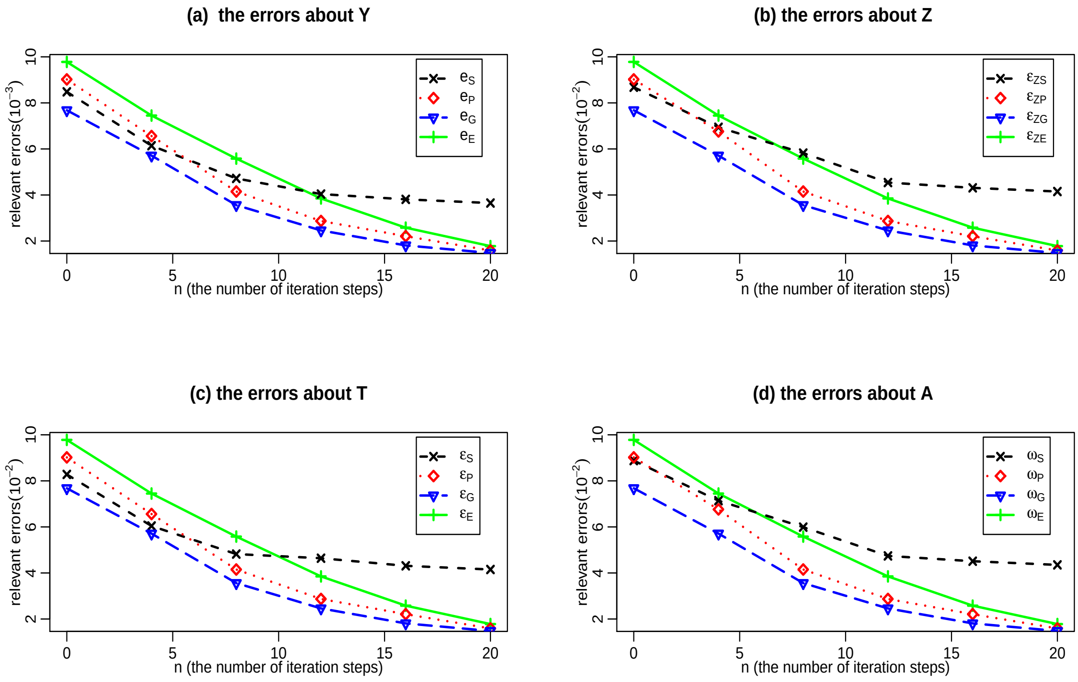

Fig. 5.1. Comparison results of the errors with call payoff in 2BSDEs.

and $\rho = 0.01$:

$$\begin{cases} Y_{t(j)}^{(n)} = \phi(X_T) + \int_t^T \left(Y_{s(j)}^{(n)} + 2Z_{s(j)}^{(n)} + 3\Gamma_{s(j)}^{(n)}\right) ds + \int_t^T \sum_{l=1}^{d} Z_{s(j)}^{l(n)} dB_s^l, \\ Z_{t(j)}^{l(n)} = z^l + \int_t^T \Gamma_{s(j)}^{l(n-1)} dB_s^l, A_{t(j)}^{l(n)} = \frac{1}{t_{i+1} - t_i} E\left(Z_{t(j)}^{l(n-1)}\right), \\ \Gamma_{t(j)}^{l(n)} = \frac{1}{t_{i+1} - t_i} E\left[Z_{t(j)}^{l(n-1)} (B_{t_{i+1}}^l - B_{t_i}^l)\right], \quad l \in [1,5], (t,x) \in U_j; \\ Y_{t(j)}^{(n)} = Y_{t(j+1)}^{(n-1)}, \quad Z_{t(j)}^{(n)} = Z_{t(j+1)}^{(n-1)}, \quad \Gamma_{t(j)}^{(n)} = \Gamma_{t(j+1)}^{(n-1)}, \\ A_{t(j)}^{(n)} = A_{t(j+1)}^{(n-1)}, \quad x \in \delta_j^+; \\ Y_{t(j)}^{(n-1)} = Y_{t(j+1)}^{(n)}, \quad Z_{t(j)}^{(n-1)} = Z_{t(j+1)}^{(n)}, \quad \Gamma_{t(j)}^{(n-1)} = \Gamma_{t(j+1)}^{(n)}, \\ A_{t(j)}^{(n-1)} = A_{t(j+1)}^{(n)}, \quad x \in \delta_j^-; \\ Y_{T(j)}^{(n)} = \phi(x), Z_{T(j)}^{(n)} = z, \quad x \in \Omega_j, \quad j = 0, 1, \cdots, m. \end{cases} \tag{5.4}$$

Here we set $Y_{t(j)}^0 = Z_{t(j)}^0 = \Gamma_{t(j)}^0 = 0$ and $Y^0 = Z^0 = \Gamma^0 = 0$ at step 0, $Y_{t(j)}^0 = Y^0 = 0, Z_{t(j)}^0 = Z^0 = 0, \Gamma_{t(j)}^0 = \Gamma^0$ on $\Omega_j \bigcap \Omega_{j+1}$.

To obtain the efficiencies of our proposed methods, we design a benchmark on fair comparison, and use the comparison errors as basis in the context of Monte-Carlo simulation. We also

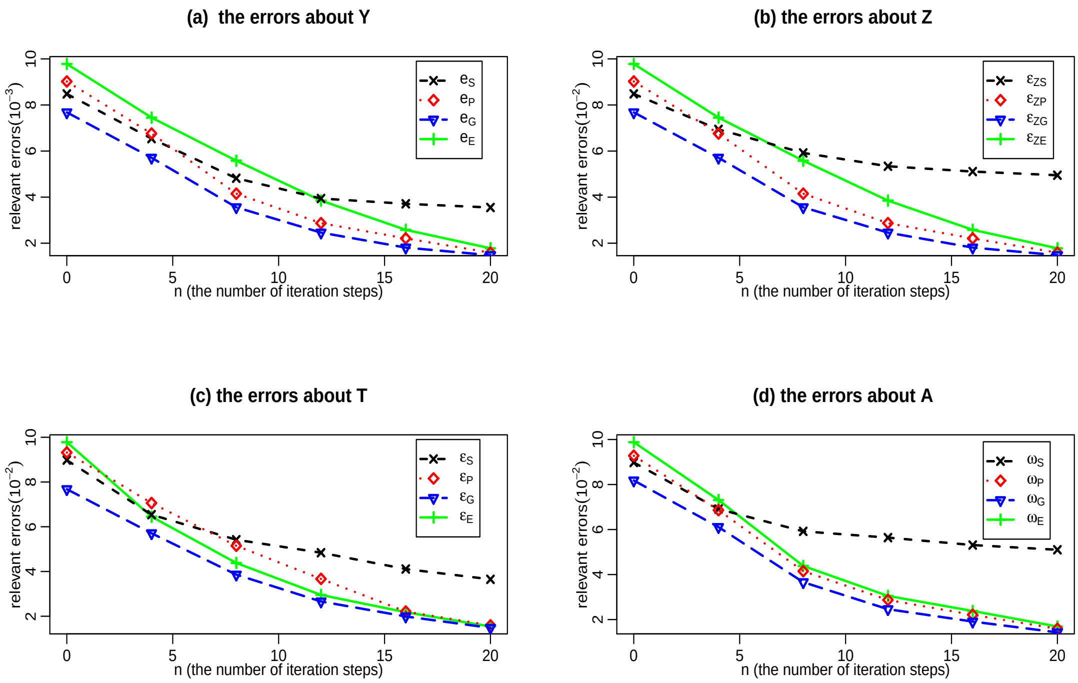

Fig. 5.2. Comparison results of the errors with put payoff in 2BSDEs.

define the following errors, in a similar way to Gobet and Labart (2007). For given n,

$$e_P = \max_{t,j} |u_j^{(n)} - Y_t|, e_E = \max_{t,\pi} |Y_{t_i}^{\pi} - Y_t|, e_S = \max_{t,j} |e_{t(j)}^{(n)}|; \tag{5.5a}$$

$$\varepsilon_{ZP} = \max_{t,j} |Du_j^{(n)} - Z_t|, \varepsilon_{ZE} = \max_{t,\pi} |Z_{t_i}^{\pi} - Z_t|, \varepsilon_{ZS} = \max_{t,j} \Big| \sum_{l=1}^{5} \varepsilon_{t(j)}^{l(n)} \Big|; \tag{5.5b}$$

$$\varepsilon_P = \max_{t,j} |D^2 u_j^{(n)} - \Gamma_t|, \varepsilon_E = \max_{t,\pi} |\Gamma_{t_i}^{\pi} - \Gamma_t|, \varepsilon_S = \max_{t,j} \Big| \sum_{l=1}^{5} \epsilon_{t(j)}^{l(n)} \Big|; \tag{5.5c}$$

$$\omega_P = \max_{t,j} |\mathcal{L} D u_j^{(n)} - A_t|, \omega_E = \max_{t,\pi} |A_{t_i}^{\pi} - A_t|, \omega_S = \max_{t,j} \Big| \sum_{l=1}^{5} \omega_{t(j)}^{l(n)} \Big|. \tag{5.5d}$$

Here $e_G, \varepsilon_{ZG}, \varepsilon_G$ and ω_G are the global optimal versions of $e_S, \varepsilon_{ZS}, \varepsilon_S$ and ω_S, respectively.

We examine the equations with call payoff and put payoff. Figs. 5.1 and 5.2 plot the related errors on n. It is shown that, both indirect method and optimal Schwarz method are more accurate than other methods.

In the setting of high dimension, Fig. 5.3 displays the related performance about the solution $(\hat{Z}_t, \hat{\Gamma}_t, \hat{A}_t)$. It is shown that they have small related errors on d, $\hat{A}_t$ is dependent on $(\hat{Z}_t, \hat{\Gamma}_t)$. When d increases, the errors increase.

To get the errors e_n between the Schwarz method (5.4) and the Euler method (5.3), we compute $e_n = \max\{E|Y_t^{(n)} - Y_{t_i}^{\pi}|^2, t \in [t_i, t_{i+1})\}$ on $m = 2, 4, 8, 16, 32$ and 64. Here

$$E|Y_t^{(n)} - Y_{t_i}^{\pi}|^2 = \frac{1}{500} \sum_{i=1}^{500} \big|Y_t^{(n)}(\omega_i) - Y_{t_i}^{\pi}(\omega_i)\big|^2,$$

Table 5.1: The errors between Schwarz method (3.2) and Euler method (1.3) for 2BSDEs.

(a) Call payoff							
m\n	1	2	3	4	5	6	7
2	3.659E+4	8.316E+3	21.59	1.759	4.372E-2	9.112E-4	5.486E-5
4	8.659E+4	1.897E+4	9.532E+2	1.321	2.153E-2	1.311E-3	2.537E-5
8	1.822E+5	4.125E+4	3.973E+3	64.51	8.321E-1	3.756E-3	4.957E-5
16	2.822E+6	6.732E+4	4.523E+3	34.51	4.763E-1	6.312E-3	8.397E-6
32	3.650E+8	8.597E+5	9.758E+3	67.52	2.376	4.561E-2	7.383E-4
64	2.822E+10	8.851E+6	7.365E+3	89.51	3.753	9.373E-2	4.679E-4
(b) Put payoff							
m\n	1	2	3	4	5	6	7
2	5.560E+5	6.781E+4	92.01	9.705	6.982E-1	4.883E-3	4.875E-5
4	9.071E+5	7.704E+4	4.511E+3	10.712	9.483E-2	4.916E-3	4.189E-5
8	5.967E+5	8.845E+4	1.008E+4	6.089E+3	18.90	8.321E-2	9.708E-4
16	7.706E+6	2.394E+5	9.970E+3	88.95	7.918E-1	6.879E-3	9.801E-6
32	7.109E+7	1.978E+6	3.551E+4	1.510E+3	9.789	8.141E-2	6.319E-4
64	8.572E+9	9.479E+7	6.178E+4	91.97	7.108	2.908E-3	8.097E-4

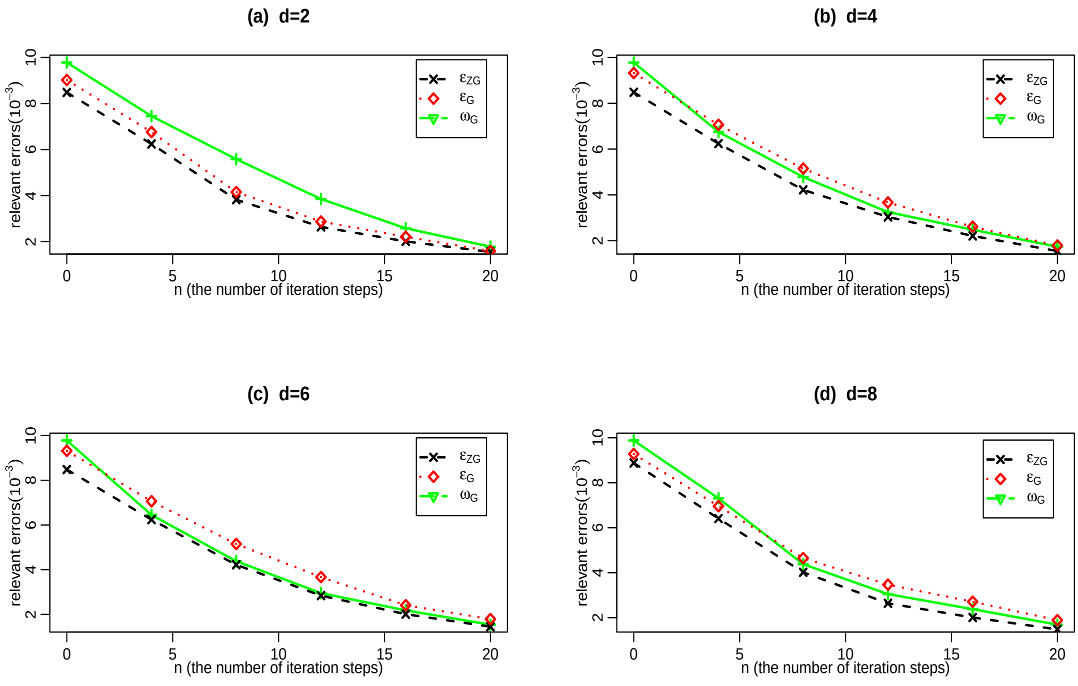

Fig. 5.3. Comparison results of the related errors in the high-dimensional case.

where $\omega_i \in \Omega$ and $i = 1, 2, \cdots, 500$.

Table 5.1 shows these comparison results on call payoff and put payoff. It is shown that, the errors decrease with increasing n, on different m. The error on call payoff, is more stable than that on put payoff. With given n, the errors increase when m increases. Thus, global Schwarz method is suited for the equations under several mild conditions. Although the Euler method becomes quite attractive, due to its simplicity and conditional stability, Schwarz method is more accurate, and faster. It also converges faster, and has less iteration to get accuracy solution.

6. Applications in Finance

We now consider an application of the 2BSDEs in finance. It is observed that, with a drift part of Z_t, the 2BSDEs are general stochastic volatility models (SVMs). We present the following Hagan-SVMs,

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t^S, \tag{6.1a}$$
$$d\sigma_t = \alpha_t \sigma_t dB_t^\sigma, \tag{6.1b}$$
$$\rho_t = E(B_t^S B_t^\sigma). \tag{6.1c}$$

where S_t is a price process, μ_t is a drift of the stock, and σ_t is a stochastic volatility. Its mean reversion parameter and diffusion coefficient are, zero and $\alpha_t\sigma_t$, respectively. α_t is another stochastic volatility for unaffected prices. B_t^S and B_t^σ are all Brownian motions. The setting of the SVMs is not mean reverting but a special case of Hull-White SVMs mentioned by Jackel (2005).

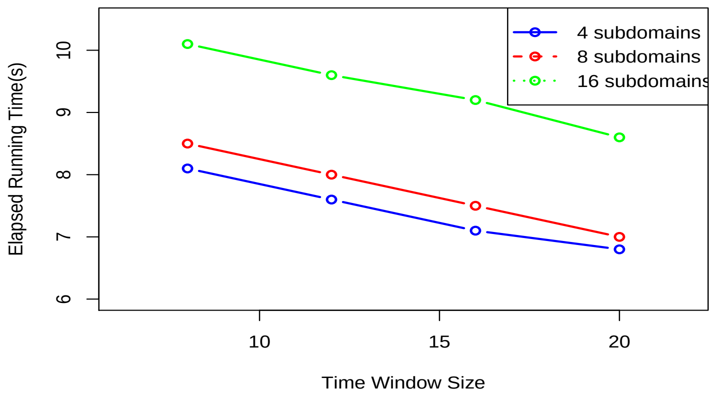

Fig. 6.1. Time performance of Schwarz method for the SVMs.

Suppose that $Y_t = \ln S_t, Z_t = \sigma_t, \Gamma_t = \alpha_t\sigma_t, f(\cdot) = -\mu_t$ and $B_t = B_t^S = B_t^\sigma$, the application yields the 2BSDEs (1.1). Now we have the following 2BSDE system.

$$\ln S_t = \phi(\cdot) - \int_t^T \mu_s ds - \int_t^T \sigma_s dB_s, \tag{6.2a}$$
$$\sigma_t = z + \int_t^T \alpha_s\sigma_s dB_s \text{ for } E(B_t^2) = \rho_t. \tag{6.2b}$$

In this case, the drift part of σ_t is a given constant, $\phi(\cdot)$ and z are all related constants.

We use the SVMs to fit the classic data sets of Crude Oil and Natural Gas Markets (1999-2011), which are derived from the Kiodex Global Market Data. We choose the data subsets on the prompt contract price for NYMEX Henry Hub (2000-2003). Fig. 6.1 represents execution times of our proposed method, on given domain sizes and fixed time window. Notably, a greater time window size results in lower communication time, and better execution time. Then the Schwarz method is computationally efficient. For example, only 6.8 second is required to solve the solution on four subdomains with 20 time steps.

Fig. 6.2 plots the comparison results of the errors of $(\hat{Z}_t, \hat{\Gamma}_t)$. It is shown that the proposed method is more accurate than other methods. In Tables 6.1 and 6.2, it is observed that the error e_n for NYMEX Henry Hu and NYMEX WTI, exhibits the same trend. When the size increases, the errors decrease. In addition, in 2000 to 2003, the errors are fewer than those in other years.

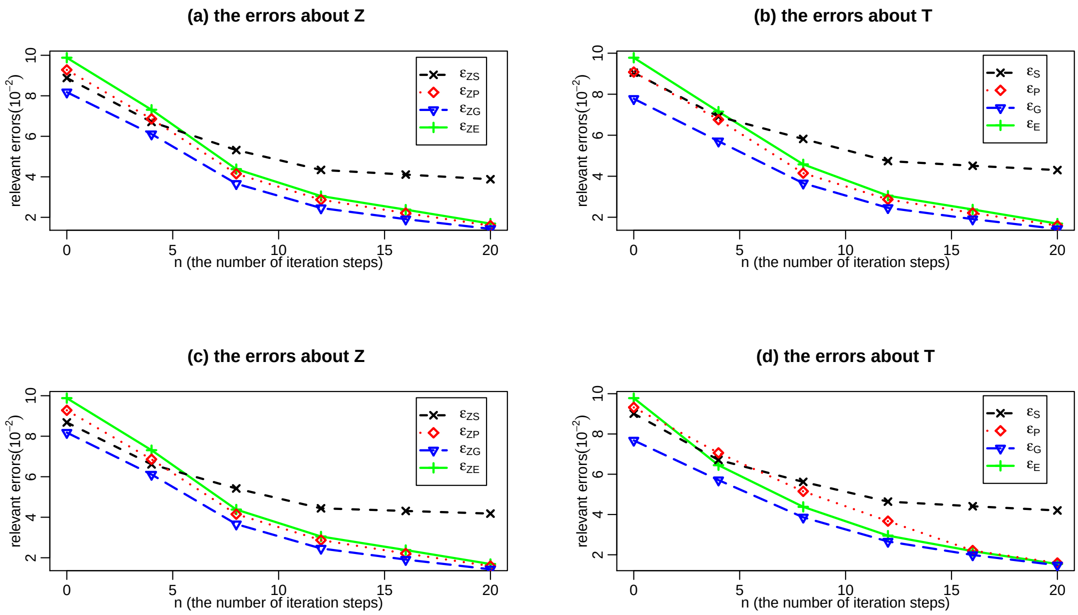

Fig. 6.2. Comparison results of the related errors for the data sets of NYMEX Henry Hu ((a)-(b)) and NYMEX WTI ((c)-(d)) (2008-2011).

Table 6.1: The errors between Schwarz method (3.2) and Euler method (1.3) for NYMEX Henry Hub(1999-2012).

years \ n	1	2	3	4	5	6	7
1997-2000	5.531E+2	7.511	66.59	7.331E-1	3.956E-2	6.733E-4	7.461E-5
2000-2003	2.312E+2	1.951	3.581E-1	5.741E-2	6.335E-3	4.112E-5	6.661E-6
2003-2006	1.131E+3	8.119E+2	67.31	8.39	6.311E-1	8.117E-3	9.113E-4
2006-2009	6.542E+4	3.271E+3	3.101E+2	27.31	1.933	7.391E-2	5.456E-3
2008-2011	3.312E+4	1.515E+3	86.53	12.12	1.053	6.511E-2	7.551E-3
	8	9	10	11	12	13	14
1997-2000	2.378E-6	1.951E-7	8.512E-8	3.111E-9	7.341E-10	4.781E-11	5.443E-12
2000-2003	5.434E-6	9.108E-7	8.706E-8	8.103E-9	9.402E-10	6.501E-11	5.312E-12
2003-2006	5.671E-5	6.712E-6	3.241E-7	7.121E-8	6.071E-10	7.301E-11	8.541E-12
2006-2009	4.501E-5	3.651E-6	4.501E-7	5.610E-8	3.332E-9	4.123E-10	6.071E-11
2008-2011	3.304E-5	5.012E-6	7.325E-7	5.633E-8	2.781E-9	8.532E-10	7.321E-12
	15	16	17	18	19	20	21
1997-2000	4.337E-13	8.167E-14	8.567E-15	1.234E-15	7.003E-16	6.503E-16	8.108E-17
2000-2003	5.671E-13	5.321E-15	6.121E-16	7.774E-17	4.904E-18	5.301E-19	4.303E-19
2003-2006	6.451E-13	7.112E-14	6.442E-15	5.445E-16	6.712E-17	5.431E-18	6.334E-19
2006-2009	5.061E-12	1.209E-13	2.371E-14	9.879E-14	7.657E-15	8.342E-16	6.124E-17
2008-2011	7.556E-13	3.441E-14	8.362E-15	7.452E-16	6.453E-16	7.124E-17	1.203E-17

Table 6.2: The errors between Schwarz method (3.2) and Euler method (1.3) for NYMEX WTI (1999-2011).

years \ n	1	2	3	4	5	6	7
1997-2000	4.776E+4	2.115E+3	89.97	9.912	2.981E-1	7.042E-2	5.902E-3
2000-2003	7.431E+5	5.906E+3	56.12	1.723	8.309E-2	5.341E-3	6.978E-4
2003-2006	3.341 E+5	4.561E+3	73.56	6.331	5.432E-2	6.001E-4	7.116E-5
2006-2009	2.781E+6	4.551E+4	83.31	7.505	7.881E-2	3.453E-3	9.987E-5
2008-2011	1.981E+5	5.664E+3	90.11	7.981	9.032E-2	1.567E-3	7.892E-5
	8	9	10	11	12	13	14
1997-2000	2.091E-5	5.782E-7	4.917E-9	5.321E-10	3.780E-11	7.564E-13	8.123E-15
2000-2003	8.678E-6	7.809E-8	6.564E-9	1.121E-11	5.778E-12	8.441E-14	8.453E-16
2003-2006	3.789E-6	7.112E-7	6.109E-9	1.980E-11	3.123E-12	1.341E-13	6.111E-16
2006-2009	8.001E-6	6.541E-8	6.113E-9	3.901E-11	7.006E-12	6.331E-14	7.009E-16
2008-2011	8.908E-6	4.331E-7	9.087E-9	5.321E-11	9.065E-12	3.337E-13	6.129E-15
	15	16	17	18	19	20	21
1997-2000	8.542E-17	7.034E-18	6.771E-19	9.507E-21	8.781E-22	6.110E-23	5.909E-24
2000-2003	3.120E-18	2.784E-19	7.551E-20	8.113E-21	9.447E-23	6.122E-24	2.899E-25
2003-2006	5.115E-17	6.587E-18	4.124E-20	7.457E-21	7.789E-22	3.445E-23	8.112E-23
2006-2009	3.012E-18	4.451E-19	7.451E-20	8.011E-21	9.771E-23	5.114E-24	2.711E-25
2008-2011	8.012E-17	6.144E-18	8.112E-19	9.047E-20	4.147E-22	8.011E-23	3.445E-24

7. Conclusion

Parallel and distributed computing was integrated due to the development of complex models. It is noting that, the 2BSDEs are also easy to perform in parallel. The development of parallel methods in 2BSDEs is, necessary and workable. For three other methods, the performance results of our proposed parallel method are superior. With fixed convergence rate, the iteration step n of the method is small and suitable.

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