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CONVERGENCE ANALYSIS ON SS-HOPM FOR BEC-LIKE NONLINEAR EIGENVALUE PROBLEMS*

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Abstract

Shifted symmetric higher-order power method (SS-HOPM) has been proved effective in solving the nonlinear eigenvalue problem oriented from the Bose-Einstein Condensation (BEC-like NEP for short) both theoretically and numerically. However, the convergence of the sequence generated by SS-HOPM is based on the assumption that the real eigenpairs of BEC-like NEP are finite. In this paper, we will establish the point-wise convergence via Łojasiewicz inequality by introducing a new related sequence.

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Key words: Nonlinear eigenvalues, Bose-Einstein Condensation, SS-HOPM, Point-wise convergence, Łojasiewicz inequality.

1. Introduction

Nonlinear eigenvalue problem originated from the Bose-Einstein condensates (BECs) [1–3] is recently well known to be an important and active field [4–8] in quantum physics. Its discrietized form can be described as

$$\begin{cases} \mathcal{A}\mathbf{x}^3 + \mathbf{B}\mathbf{x} = \lambda\mathbf{x} \\ ||\mathbf{x}||_2 = 1, \end{cases} \tag{1.1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n\times n\times n}$ is a symmetric 4th-order tensor, $\mathbf{B} \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\mathbf{x} \in \mathbb{R}^n$ is a vector. It can be easily verified that the nonlinear eigenvalue problem (1.1) can be viewed as the the KKT system or first-order necessary condition of the following nonconvex optimization problem

$$\begin{aligned} \min \quad & \frac{1}{2}\mathcal{A}\mathbf{x}^4 + \mathbf{x}^\top \mathbf{B}\mathbf{x} \\ s.t. \quad & ||\mathbf{x}||_2 = 1. \end{aligned} \tag{1.2}$$

Actually, (1.2) is a discrete form of the energy functional form of BECs [9]. From [10], we know that $(\lambda, \mathbf{x})$ is an eigenpair of nonlinear eigenvalue problem (1.1) if and only if $\mathbf{x}$ is a constraint stationary point of (1.2).

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Hu et al. [9] pointed out that it is a numerical challenge to solve (1.2) efficiently due to the large number of variables and the possible indefiniteness of the Hessian matrix. They have shown that the BEC problem is NP-hard via establishing its relation with the partition problem. So the general BEC-like NEP is NP-hard.

Generally, the approaches for solving BEC-like problem can be divided into numerical methods [11, 12] and optimization methods [7–9, 13]. The shifted symmetric higher-order power method (SS-HOPM) has been proved effective for solving such nonconvex optimization problem in [14]. However the point-wise convergence of SS-HOPM for BEC-like NEP has not been proven yet.

Łojasiewicz inequality [15] has been proven to be an useful tool in analyzing the local convergence of nonconvex optimization problems [16–19]. Based on the Łojasiewicz gradient inequality, Uschmajew has given the point-wise convergence of Gauss-Seidel higher-order power method [20] with results in [21]. The key is to transform the constrained optimization to a proper unconstrained optimization as in [21]. Luo and Yang proved the point-wise convergence of SS-HOPM for high-order tensor eigenpairs with the similar approach [22]. Specifically, they first defined a new sequence based on the sequence generated by SS-HOPM and corresponding analytic function. Then with Łojasiewicz inequality, they showed the global convergence of the new established sequence, which in return ensured the point-wise convergence of the original sequence. Motivated by the work mentioned above, we intend to enhance the convergence result in [14], i.e, to prove the point-wise convergence of SS-HOPM for BEC-like NEP.

The rest of this paper is outlined as follows. Section 2 presents some preliminaries about the basic notation and SS-HOPM. In Section 3, we establish the point-wise convergence of $\{\mathbf{x}_k\}$ generated by SS-HOPM for BEC-like NEP. Conclusions are provided in Section 4.

2. Notation and Preliminaries

Throughout this paper, we exclusively consider the tensor notation introduced in [23]. In particularly, vectors are denoted by boldface lowercase letters, e.g., $\mathbf{a}$. Matrices are denoted by boldface capital letters, e.g., $\mathbf{A}$. Higher-order tensors are denoted by Euler script letters, e.g., $\mathcal{A}$. Scalars are denoted by lowercase letters, e.g., a. Let Σ denote the unit sphere on $\mathbb{R}^n$, i.e., $\Sigma = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| = 1\}$.

2.1. Tensors and tensor eigenpair

A *tensor* is a multidimensional array. Its order is the number of its dimensions. An Nth-order tensor is denoted as $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, whose $(i_1, \cdots, i_N)$ element is $a_{i_1 \cdots i_N}, 1 \le i_k \le I_k, k = 1, \cdots, N$. Specifically, for $N = 1$ and $N = 2$, tensors are vectors and matrices respectively.

Definition 2.1 (Symmetric tensor [24]). *A tensor* $\mathcal{A} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{m}}$ *is symmetric if*

$$a_{i_{p(1)} \cdots i_{p(m)}} = a_{i_1 \cdots i_m} \quad \text{for all} \ \ i_1 \cdots i_m \in \{1, \cdots, n\} \ \ \text{and} \ \ p \in \Pi_m,$$

where Π_m *denotes the set of all permutations of* $(1, \cdots, m)$.

Definition 2.2 (Symmetric tensor-vector multiply). *Let* $\mathcal{A} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{m}}$ *be symmetric and* $\mathbf{x} \in \mathbb{R}^n$. *Then for* $0 \le r \le m - 1$, *the* $(m - r)$*-times product of tensor* $\mathcal{A}$ *with the vector* $\mathbf{x}$

is denoted by $\mathcal{A}\mathbf{x}^{m-r} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{r}}$ *and defined by*

$$(\mathcal{A}\mathbf{x}^{m-r})_{i_1 \cdots i_r} \equiv \sum_{i_{r+1},\cdots,i_m} a_{i_1\cdots i_m} x_{r+1} \cdots x_{i_m} \quad \textit{for all} \;\; i_1 \cdots i_r \in \{1, \cdots, n\}.$$

Definition 2.3 (Identical tensor [25, 26]**).** *If a symmetric tensor* $\mathcal{E} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{m}}$ *satisfies*

$$\mathcal{E}\mathbf{x}^{m-1} = \mathbf{x} \quad \textit{for all} \;\; \mathbf{x} \in \Sigma.$$

For m even, the identity tensor $\mathcal{E}$ for all $\mathbf{x} \in \Sigma$ is given by

$$\mathcal{E}_{i_1\cdots i_m} = \frac{1}{m} \sum_{p \in \Pi_m} \delta_{i_{p(1)} i_{p(2)}} \delta_{i_{p(3)} i_{p(4)}} \cdots \delta_{i_{p(m-1)} i_{p(m)}}$$

for $i_1 \cdots i_m \in \{1, \cdots, n\}$, where δ is the standard *Kronecker* delta, i.e.,

$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

We remark here that there is no identity tensor for m odd.

Definition 2.4 ([25, 27]). *Assume that* $\mathcal{A}$ *is a symmetric mth-order n-dimensional real-valued tensor.* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $\mathcal{A}$*, if there exists an* $\mathbf{x} \in \mathbb{C}^n$ *such that*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x} \quad \textit{and} \;\; \mathbf{x}^\dagger \mathbf{x} = 1,$$

where "†" denotes complex conjugate transposition. The vector $\mathbf{x}$ *is a corresponding eigenvector, and* $(\lambda, \mathbf{x})$ *is called an eigenpair.*

2.2. SS-HOPM for BEC-like NEP

Based on Kolda and Mayo's work [26], the SS-HOPM was generated to solve BEC-like NEP (1.2) in [14]. $\mathbf{B}$ was assumed to be positive definite there. Adding a suitable shift α to the objective function $f = \frac{1}{2}\mathcal{A}\mathbf{x}^4 + \mathbf{x}^\top \mathbf{B}\mathbf{x}$ in (1.2), then it transformed into an equal problem

$$\begin{aligned} \min \quad & \frac{1}{2}(\mathcal{A} + \alpha\mathcal{E})\mathbf{x}^4 + \mathbf{x}^\top \mathbf{B}\mathbf{x} \\ s.t. \quad & \|\mathbf{x}\|_2 = 1, \end{aligned}$$

where $\mathcal{E}$ is the identity tensor as defined previously. Denote

$$\hat{f}(\mathbf{x}) = \frac{1}{2}(\mathcal{A} + \alpha\mathcal{E})\mathbf{x}^4 + \mathbf{x}^\top \mathbf{B}\mathbf{x}.$$

The SS-HOPM algorithm for the BEC-like NEP may be stated as following.

Require:
1. Given a symmetric tensor $\mathcal{A} \in \mathbb{R}^{n\times n\times n\times n}$ and positive definite matrix $\mathbf{B}$.
2. Shift α;

Ensure:
Eigenvalue λ and the corresponding eigenvectors $\mathbf{x}$.

1: Given $\mathbf{x}_0 \in \mathbb{R}^n$ with $||\mathbf{x}_0||_2 = 1$. Let $\lambda_0 = \mathcal{A}\mathbf{x}_0^4 + \mathbf{x}_0^\top \mathbf{B}\mathbf{x}_0$
2: **for** $k = 0, 1, \cdots,$ **do**
3: **if** $\alpha \geq 0$ **then**
4: $\hat{\mathbf{x}}_{k+1} \leftarrow \mathcal{A}\mathbf{x}_k^3 + \mathbf{B}\mathbf{x}_k + \alpha\mathbf{x}_k$
5: **else**
6: $\hat{\mathbf{x}}_{k+1} \leftarrow -(\mathcal{A}\mathbf{x}_k^3 + \mathbf{B}\mathbf{x}_k + \alpha\mathbf{x}_k)$
7: **end if**
8: $\mathbf{x}_{k+1} \leftarrow \hat{\mathbf{x}}_{k+1}/||\hat{\mathbf{x}}_{k+1}||_2$
9: $\lambda_{k+1} = \mathcal{A}\mathbf{x}_{k+1}^4 + \mathbf{x}_{k+1}^\top \mathbf{B}\mathbf{x}_{k+1}$
10: **end for**
11: **return** $\lambda = \lambda_{k+1}, \mathbf{x} = \mathbf{x}_{k+1}$.

Algorithm 2.1. The shifted symmetric higher-order power method (SS-HOPM).

In Algorithm SS-HOPM, if α is chosen appropriately, $\hat{f}(\mathbf{x})$ can be ensured convex on the entire sphere. In order to deduce the shifted term α, define β as

$$\beta \equiv \max_{\mathbf{x}\in\Sigma} \rho(3\mathcal{A}\mathbf{x}^2 + \mathbf{B}), \tag{2.1}$$

where "ρ" denotes spectral radius of matrix, and $\mathcal{A}\mathbf{x}^2$ is a matrix with $(\mathcal{A}\mathbf{x}^2)_{ij} = \sum_{k,l=1}^n a_{ijkl}x_k x_l$, for any $i, j = 1, \cdots, n$.

In each iteration of SS-HOPM for BEC-like NEP, we calculate the gradient of $\hat{f}(\mathbf{x})$, and project it to Σ. Since $\mathcal{A}$ is symmetric, we can easily verify that the gradient and Hessian of the modified function $\hat{f}(\mathbf{x})$ are

$$\begin{aligned} \hat{g}(\mathbf{x}) &\equiv \nabla \hat{f}(\mathbf{x}) = g(\mathbf{x}) + 2\alpha\left(\mathbf{x}^T\mathbf{x}\right)\mathbf{x} \\ \hat{\mathbf{H}}(\mathbf{x}) &\equiv \nabla^2 \hat{f}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) + 2\alpha\left(\mathbf{x}^T\mathbf{x}\right)\mathbf{I} + 4\alpha\mathbf{x}\mathbf{x}^T, \end{aligned}$$

where g and $\mathbf{H}$ are the gradient and Hessian of f, respectively. When $\mathbf{x} = 0$, $\hat{\mathbf{H}}(\mathbf{x}) = \hat{\mathbf{H}}(\mathbf{0}) = \mathbf{H}(\mathbf{0}) = \mathbf{B} \succ 0$. For any $\mathbf{x} \neq 0$, let $\bar{\mathbf{x}} = \mathbf{x}/||\mathbf{x}|| \in \Sigma$. For any $\mathbf{y} \in \Sigma$,

$$\mathbf{y}^T\hat{\mathbf{H}}(\mathbf{x})\mathbf{y} = \|\mathbf{x}\|^2\left(\mathbf{y}^T(6\mathcal{A}\overline{\mathbf{x}}^2)\mathbf{y} + 2\alpha + 4\alpha\left(\overline{\mathbf{x}}^T\mathbf{y}\right)^2\right) + 2\mathbf{y}^T\mathbf{B}\mathbf{y}. \tag{2.2}$$

When $\alpha > \beta$,

$$\mathbf{y}^T\hat{\mathbf{H}}(\mathbf{x})\mathbf{y} \geq \|\mathbf{x}\|^2(-2\beta + 2\alpha + 0) > 0 \quad \text{for all} \quad \mathbf{y} \in \Sigma$$

so the shift term $\alpha > \beta$ makes the Hessian of $\hat{f}(\mathbf{x})$ positive definite, thus $\hat{f}$ is convex on $\mathbb{R}^n$ and convergence of $\{\lambda_k\}$ follows. Without loss of generality, we consider the convex case only.

In [14], the authors have proven the convergence of Algorithm SS-HOPM with a similar scheme in [26]. We invoke this convergence theorem here without proof.

Theorem 2.1 ([14]). *Let $\mathcal{A} \in \mathbb{R}^{n\times n\times n\times n}$ be a symmetric tensor,* $\mathbf{B}$ *be a positive semidefinite symmetric matrix. For $\alpha > \beta$, where β is defined in* (2.1)*. The iterates $\{\lambda_k, \mathbf{x}_k\}$ produced by Algorithm SS-HOPM satisfy the following properties.*
(a) *The sequence $\{\mathbf{x}_k\}$ has an accumulation point $\mathbf{x}_*$.*
(b) *The sequence $\{\lambda_k\}$ is nondecreasing, and there exists λ_* such that $\lambda_k \to \lambda_*$.*
(c) *For each such accumulation point $\mathbf{x}_*$, the pair $(\lambda_*, \mathbf{x}_*)$ is an eigenpair of* (1.1)*.*
(d) *If* (1.1) *has finitely many real eigenpairs, then there exists $\mathbf{x}_*$ such that $\mathbf{x}_k \to \mathbf{x}_*$.*

The convergence of $\{\mathbf{x}_k\}$ in the above theorem is based on the condition that the BEC-like NEP has finite many real eigenvectors. But we can not guarantee that. For example, let $\mathcal{A}$ be a symmetric 4th order 3 dimensional tensor, $\mathbf{B}$ be a 3 dimensional positive semidefinite matrix, and $a_{1111} = 1, b_{11} = 1$, the rest elements of $\mathcal{A}, \mathbf{B}$ are all zeros. It is easy to verify that any $(0, y, z)$ with $y^2 + z^2 = 1$ is an eigenvector corresponding to eigenvalue 0. Thus, the BEC-like NEP has infinite real eigenvectors. One of our purposes is to prove the point-wise convergence of SS-HOPM algorithm for BEC-like NEP hold unconditionally. We refer the point-wise convergence of SS-HOPM as the convergence of $\{\mathbf{x}_k\}$.

3. Point-wise Convergence Analysis on SS-HOPM for BEC-like NEP

3.1. Some properties of SS-HOPM for BEC-like NEP

We induce some propositions on Algorithm SS-HOPM for BEC-like NEP which are useful for us to establish the point-wise convergence of the algorithm.

We introduce a lemma of convex function in [28] that will be used in the proof of a property below.

Lemma 3.1 ([28]). *Let f be a function that is convex and continuously differentiable on $\mathbb{B}$. If $\mathbf{v}, \mathbf{w} \in \Sigma$ with $\nabla f(\mathbf{w})/||\nabla f(\mathbf{w})|| \neq \mathbf{w}$, then $f(\mathbf{v}) - f(\mathbf{w}) > 0$.*

Proposition 3.1. *Let $\mathcal{A} \in \mathbb{R}^{n\times n\times n\times n}$ be a symmetric tensor,* $\mathbf{B}$ *be a positive definite symmetric matrix, and $\alpha > \beta$, the sequence $\{\lambda_k, \hat{\mathbf{x}}_k, \mathbf{x}_k\}$ generated by Algorithm SS-HOPM for BEC-like NEP satisfies the following properties:*
(1) $\lambda_k + \alpha > 0$;
(2) $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \to 0$;
(3) $\lambda_{k+1} - \lambda_k \geq \gamma\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2$, *for large enough k and $\gamma > 0$;*
(4) $\|\hat{\mathbf{x}}_k\| \to \lambda_* + \alpha$, *where λ_* is the limit point of $\{\lambda_k\}$.*

Proof. (1) From the Algorithm SS-HOPM, we can achieve $\lambda_k = \mathcal{A}\mathbf{x}_k^4 + \mathbf{x}_k^\top \mathbf{B}\mathbf{x}_k$, and

$$\mathcal{A}\mathbf{x}_k^4 + \mathbf{x}_k^\top \mathbf{B}\mathbf{x}_k = \mathbf{x}_k^\top(\mathcal{A}\mathbf{x}_k^2 + \mathbf{B})\mathbf{x}_k.$$

From the definition of β and the assumption that $\mathbf{B}$ is positive semidefinite, we can see that if $3\mathcal{A}\mathbf{x}_k^2 + \mathbf{B} \succeq 0$, then $2\mathcal{A}\mathbf{x}_k^2 + \mathbf{B} \succeq 0$. Since $\alpha > \beta$, we have

$$\lambda_k + \alpha = \mathbf{x}_k^\top(\mathcal{A}\mathbf{x}_k^2 + \mathbf{B} + \alpha\mathbf{I})\mathbf{x}_k > 2\mathbf{x}_k^\top(2\mathcal{A}\mathbf{x}_k^2 + \mathbf{B})\mathbf{x}_k \geq 0.$$

If $3\mathcal{A}\mathbf{x}_k^2 + \mathbf{B} \preceq 0$, then $\mathcal{A}\mathbf{x}_k^2 \preceq 0$. Since $\alpha > \beta$, so

$$\mathcal{A}\mathbf{x}_k^2 + \mathbf{B} + \alpha\mathbf{I} \succ \mathcal{A}\mathbf{x}_k^2 + \mathbf{B} - 3\mathcal{A}\mathbf{x}_k^2 - \mathbf{B} = -2\mathcal{A}\mathbf{x}_k^2 \succeq 0.$$

Consequently,

$$\lambda_k + \alpha = \mathbf{x}_k^\top (\mathcal{A}\mathbf{x}_k^2 + \mathbf{B} + \alpha \mathbf{I})\mathbf{x}_k > 0.$$

(2) From the Algorithm SS-HOPM, we know that

$$\mathbf{x}_{k+1} = \frac{\hat{g}(\mathbf{x}_k)}{\|\hat{g}(\mathbf{x}_k)\|}.$$

By Lemma 3.1, we can obtain

$$\hat{f}(\mathbf{x}_{k+1}) - \hat{f}(\mathbf{x}_k) \geq 0.$$

Since $\{\hat{f}(\mathbf{x}_{k+1})\}$ is a nondecreasing and bounded sequence, then we have

$$\hat{f}(\mathbf{x}_{k+1}) - \hat{f}(\mathbf{x}_k) \to 0.$$

Because $\mathbf{x}_{k+1} = \hat{g}(\mathbf{x}_k) / \|\hat{g}(\mathbf{x}_k)\| \in \Sigma$ and $\mathbf{x}_k \in \Sigma$, so

$$\hat{g}(\mathbf{x}_k)^T (\mathbf{x}_{k+1} - \mathbf{x}_k) = \|\hat{g}(\mathbf{x}_k)\| \mathbf{x}_{k+1}^T (\mathbf{x}_{k+1} - \mathbf{x}_k) \geq 0.$$

Noting that

$$\hat{g}(\mathbf{x}_k)^T (\mathbf{x}_{k+1} - \mathbf{x}_k) \leq \hat{f}(\mathbf{x}_{k+1}) - \hat{f}(\mathbf{x}_k),$$

we have

$$\hat{g}(\mathbf{x}_k)^T (\mathbf{x}_{k+1} - \mathbf{x}_k) \to 0.$$

Because $\|\hat{g}(\mathbf{x}_k)\|$ is bounded away from 0 and $\mathbf{x}_k, \mathbf{x}_{k+1} \in \Sigma$, so

$$\|\mathbf{x}_k - \mathbf{x}_{k+1}\| \to 0.$$

(3) Since $\|\mathbf{x}_k - \mathbf{x}_{k+1}\| \to 0$, so there exists a K such that for all $k > K$, $\|\mathbf{x}_k - \mathbf{x}_{k+1}\| \leq 1$, i.e., $\mathbf{x}_{k+1}^\top \mathbf{x}_k \geq \frac{1}{2}$. Thus for any $\theta \in (0,1)$, we have

$$\begin{aligned}
&\|\theta \mathbf{x}_k + (1-\theta)\mathbf{x}_{k+1}\| \\
=&\sqrt{\|\theta \mathbf{x}_k + (1-\theta)\mathbf{x}_{k+1}\|^2} = \sqrt{\theta^2 + (1-\theta)^2 + 2\theta(1-\theta)\mathbf{x}_{k+1}^\top \mathbf{x}_k} \\
\geq&\sqrt{\theta^2 + (1-\theta)^2 + \theta(1-\theta)} \geq \sqrt{1 - \theta(1-\theta)} \geq \frac{\sqrt{3}}{2}.
\end{aligned}$$

Based on Algorithm SS-HOPM, we can express $\lambda_{k+1} - \lambda_k$ as below

$$\begin{aligned}
\lambda_{k+1} - \lambda_k &= \mathcal{A}\mathbf{x}_{k+1}^4 + \mathbf{x}_{k+1}^\top \mathbf{B}\mathbf{x}_{k+1} - (\mathcal{A}\mathbf{x}_k^4 + \mathbf{x}_k^\top \mathbf{B}\mathbf{x}_k) \\
&= \hat{f}(\mathbf{x}_{k+1}) - \hat{f}(\mathbf{x}_k) + \frac{1}{2}\left(\mathcal{A}\mathbf{x}_{k+1}^4 - \mathcal{A}\mathbf{x}_k^4\right) + \frac{\alpha}{2}\Big((\mathbf{x}_{k+1}^\top \mathbf{x}_{k+1})^2 - (\mathbf{x}_k^\top \mathbf{x}_k)^2\Big).
\end{aligned}$$

Since $\|x_k - x_{k+1}\| \to 0$, it is not hard to verify that

$$\mathcal{A}\mathbf{x}_{k+1}^4 - \mathcal{A}\mathbf{x}_k^4 \to 0.$$

Noting that $\mathbf{x}_k^\top \mathbf{x}_k = \mathbf{x}_{k+1}^\top \mathbf{x}_{k+1} = 1$ gives

$$\lambda_{k+1} - \lambda_k \to \hat{f}(\mathbf{x}_{k+1}) - \hat{f}(\mathbf{x}_k).$$

Combining with Taylor expansion of $\hat{f}(\mathbf{x})$ and (2.2), for sufficiently large k and some $\theta \in (0,1)$, we can achieve

$$\begin{aligned}
&\hat{f}\left(\mathbf{x}_{k+1}\right)-\hat{f}\left(\mathbf{x}_{k}\right)\\
=&\left\langle\nabla \hat{f}\left(\mathbf{x}_{k}\right), \mathbf{x}_{k+1}-\mathbf{x}_{k}\right\rangle+\frac{1}{2}\left(\mathbf{x}_{k+1}-\mathbf{x}_{k}\right)^{\top} \nabla^{2} \hat{f}\left(\theta \mathbf{x}_{k}+(1-\theta) \mathbf{x}_{k+1}\right)\left(\mathbf{x}_{k+1}-\mathbf{x}_{k}\right)\\
\geq& 2\left\langle\left\|\hat{\mathbf{x}}_{k+1}\right\| \mathbf{x}_{k+1}, \mathbf{x}_{k+1}-\mathbf{x}_{k}\right\rangle+\frac{3}{4}(\alpha-\beta)\left\|\mathbf{x}_{k+1}-\mathbf{x}_{k}\right\|^{2}\\
\geq& \frac{3}{4}(\alpha-\beta)\left\|\mathbf{x}_{k+1}-\mathbf{x}_{k}\right\|^{2}.
\end{aligned}$$

Consequently,

$$\lambda_{k+1}-\lambda_{k} \geq \frac{3}{4}(\alpha-\beta)\left\|\mathbf{x}_{k+1}-\mathbf{x}_{k}\right\|^{2}.$$

Denoting $\gamma \equiv \frac{3}{4}(\alpha-\beta)>0$ gives

$$\lambda_{k+1}-\lambda_{k} \geq \gamma\left\|\mathbf{x}_{k+1}-\mathbf{x}_{k}\right\|^{2}. \tag{3.1}$$

(4) From the Algorithm SS-HOPM, we can obtain

$$\hat{\mathbf{x}}_{k+1}^{\top} \mathbf{x}_{k}=\mathcal{A} \mathbf{x}_{k}^{4}+\mathbf{x}_{k}^{\top} \mathbf{B} \mathbf{x}_{k}+\alpha\left\|\mathbf{x}_{k}\right\|^{2}=\lambda_{k}+\alpha=\left\|\hat{\mathbf{x}}_{k+1}\right\| \mathbf{x}_{k+1}^{\top} \mathbf{x}_{k}.$$

Since $\left\|\mathbf{x}_{k}-\mathbf{x}_{k+1}\right\| \rightarrow 0$, and $\left\|\mathbf{x}_{k}\right\|=\left\|\mathbf{x}_{k+1}\right\|=1$, we have $\mathbf{x}_{k+1}^{\top} \mathbf{x}_{k} \rightarrow 1$. Hence there exists a λ_{*} such that $\lambda_{k} \rightarrow \lambda_{*}$, and we achieve $\left\|\hat{\mathbf{x}}_{k}\right\| \rightarrow \lambda_{*}+\alpha$. □

3.2. Point-wise convergence analysis

In order to prove the point-wise convergence of Algorithm SS-HOPM, we first define a sequence $\{\mathbf{y}_k\}$ related to $\{\lambda_k, \mathbf{x}_k\}$ as $\mathbf{y}_k \equiv (\lambda_k+\alpha)^{\frac{1}{4}}\mathbf{x}_k$. If $\{\mathbf{y}_k\}$ is convergent, it follows obviously that $\{\mathbf{x}_k\}$ converges. So our goal is to prove that $\{\mathbf{y}_k\}$ converges in the sequel.

Theorem 3.1 ([21]). *Let $f: V \rightarrow \mathbb{R}$ be a real-analytic function on a finite dimensional real vector space V, and let $\{\mathbf{y}_k\} \subset \mathbb{R}^n$ be a sequence satisfying*

$$f(\mathbf{y}_k)-f(\mathbf{y}_{k+1}) \geq \sigma\|\nabla f(\mathbf{y}_k)\|\ \|\mathbf{y}_k-\mathbf{y}_{k+1}\|$$

for all large enough k and some $\sigma>0$. Assume further that the implication

$$[f(\mathbf{y}_{k+1})=f(\mathbf{y}_k)] \implies \mathbf{y}_{k+1}=\mathbf{y}_k$$

holds. Then a cluster point $\mathbf{y}_*$ of the sequence $\{\mathbf{y}_k\}$ must be its limit. In particular, if the sequence is bounded, it is convergent.

In order to use the above theorem, we define a function $\tilde{f}$ for convex case on $\mathbf{y}$ as below.

$$\tilde{f}(\mathbf{y}) \equiv-2\left(\mathcal{A} \mathbf{y}^{4}+\mathbf{y}^{\top} \mathbf{y} \mathbf{y}^{\top} \mathbf{B} \mathbf{y}+\alpha\left(\mathbf{y}^{\top} \mathbf{y}\right)^{2}\right)+\left(\mathbf{y}^{\top} \mathbf{y}\right)^{4} \tag{3.2}$$

With the basic assumption $\mathbf{x}^{\top}\mathbf{x}=1$ and $\lambda_k=\mathcal{A}\mathbf{x}_k^4+\mathbf{x}_k^{\top}\mathbf{B}\mathbf{x}_k$, we obtain

$$\begin{aligned}
\tilde{f}\left(\mathbf{y}_{k}\right) &=-2\left(\left(\lambda_{k}+\alpha\right)\left(\mathcal{A} \mathbf{x}_{k}^{4}+\mathbf{x}_{k}^{\top} \mathbf{x}_{k} \mathbf{x}_{k}^{\top} \mathbf{B} \mathbf{x}_{k}\right)+\alpha\left(\lambda_{k}+\alpha\right)\right)+\left(\lambda_{k}+\alpha\right)^{2} \\
&=-2\left(\left(\lambda_{k}+\alpha\right)\left(\mathcal{A} \mathbf{x}_{k}^{4}+\mathbf{x}_{k}^{\top} \mathbf{B} \mathbf{x}_{k}\right)+\alpha\left(\lambda_{k}+\alpha\right)\right)+\left(\lambda_{k}+\alpha\right)^{2} \\
&=-2\left(\left(\lambda_{k}+\alpha\right) \lambda_{k}+\alpha\left(\lambda_{k}+\alpha\right)\right)+\left(\lambda_{k}+\alpha\right)^{2} \\
&=-\left(\lambda_{k}+\alpha\right)^{2}
\end{aligned} \tag{3.3}$$

From (3.1), we know that $\lambda_{k+1} - \lambda_k > 0$. Combining with $\lambda_k + \alpha > 0$, we can obtain

$$\tilde{f}(\mathbf{y}_k) - \tilde{f}(\mathbf{y}_{k+1}) = (\lambda_{k+1} + \alpha)^2 - (\lambda_k + \alpha)^2 = (\lambda_{k+1} + \lambda_k + 2\alpha)(\lambda_{k+1} - \lambda_k) > 0. \tag{3.4}$$

So the sequence $\{\tilde{f}(\mathbf{y}_k)\}$ is monotonic decreasing with sequence $\{\mathbf{y}_k\}$, which satisfies the sufficient decrease condition in Theorem 3.1.

We propose some properties of $\{\mathbf{y}_k\}$ and then prove the convergence of $\{\mathbf{y}_k\}$ with the Theorem 3.1. Finally we can achieve the point-wise convergence of SS-HOPM for BEC-like NEP.

Proposition 3.2. *Let $\alpha > \beta$, $\{\lambda_k, \mathbf{x}_k\}$ is generated by Algorithm SS-HOPM, λ_* is the limit of $\{\lambda_k\}$, $\tilde{f}$ is defined as (3.2), $\mathbf{y}_k \equiv (\lambda_k + \alpha)^{\frac{1}{4}} \mathbf{x}_k$. Then for sufficiently large k, there exists a triplet $(\sigma,\ \tau,\ \kappa) > 0$ such that*

(1) $\tilde{f}(\mathbf{y}_k) - \tilde{f}(\mathbf{y}_{k+1}) \geq \sigma \|\mathbf{y}_k - \mathbf{y}_{k+1}\|^2$, (3.5a)

(2) $\|\mathbf{y}_{k+1} - \mathbf{y}_k\| \geq \tau \|\mathbf{x}_{k+1} - \mathbf{x}_k\|$, (3.5b)

(3) $\|\mathbf{y}_k - \mathbf{y}_{k+1}\| \geq \kappa \|\nabla \tilde{f}(\mathbf{y}_k)\|$. (3.5c)

Proof. (1) It is not hard to verify that $\rho(\hat{\mathbf{H}}(\mathbf{x}))$ and $\nabla \hat{f}(\mathbf{x})$ are bounded for any bounded $\mathbf{x}$. Since $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \to 0$ and $\|\mathbf{x}_{k+1}\| = \|\mathbf{x}_k\| = 1$, then $\theta \mathbf{x}_k + (1-\theta)\mathbf{x}_{k+1}$, with $\theta \in (0,1)$ is bounded. Thus, due to Lipschitz continuity of $\nabla \hat{f}(\mathbf{x})$, there exists an $R > 0$ for sufficiently large k such that

$$\begin{aligned} \lambda_{k+1} - \lambda_k &\leq \left\langle \nabla \hat{f}(\mathbf{x}_k), \mathbf{x}_{k+1} - \mathbf{x}_k \right\rangle + \frac{R}{2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2 \\ &\leq \left(\left\| \nabla \hat{f}(\mathbf{x}_k) \right\| + \frac{R}{2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \right) \|\mathbf{x}_{k+1} - \mathbf{x}_k\|, \end{aligned} \tag{3.6}$$

i.e., there exist a large enough $M > 0$ such that

$$\lambda_{k+1} - \lambda_k \leq M \|\mathbf{x}_{k+1} - \mathbf{x}_k\|.$$

Noting that $\{\lambda_k\}$ is monotonically increasing to λ_* and $\lambda_k + \alpha > 0$, so for all sufficiently large k, combining with (3.4), we have

$$\tilde{f}(\mathbf{y}_k) - \tilde{f}(\mathbf{y}_{k+1}) \geq (\lambda_* + \alpha)(\lambda_{k+1} - \lambda_k).$$

From Proposition 3.1, for all sufficiently large k, there exists an $\gamma > 0$ such that

$$\lambda_{k+1} - \lambda_k \geq \gamma \|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2.$$

According to the definition of $\mathbf{y}_k$, we have

$$\begin{aligned} \|\mathbf{y}_{k+1} - \mathbf{y}_k\| &= \left\| (\lambda_{k+1} + \alpha)^{\frac{1}{4}} \mathbf{x}_{k+1} - (\lambda_k + \alpha)^{\frac{1}{4}} \mathbf{x}_k \right\| \\ &= (\lambda_{k+1} + \alpha)^{\frac{1}{4}} \left\| \mathbf{x}_{k+1} - \left(\frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}\right)^{\frac{1}{4}} \mathbf{x}_k \right\| \\ &\leq (\lambda_* + \alpha)^{\frac{1}{4}} \left\| \mathbf{x}_{k+1} - \left(\frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}\right)^{\frac{1}{4}} \mathbf{x}_k \right\| \end{aligned} \tag{3.7}$$

It is not hard to verify that

$$0 \le 1 - \left(\frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}\right)^{\frac{1}{4}} \le 1 - \frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}. \tag{3.8}$$

So for sufficiently large $k > 0$, we have

$$\begin{aligned}
\|\mathbf{y}_{k+1} - \mathbf{y}_k\| &\le (\lambda_* + \alpha)^{\frac{1}{4}} \left(\|\mathbf{x}_{k+1} - \mathbf{x}_k\| + \left\| \mathbf{x}_k \left(1 - \frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}\right) \right\| \right) \\
&\le (\lambda_* + \alpha)^{\frac{1}{4}} \left(\|\mathbf{x}_{k+1} - \mathbf{x}_k\| + 1 - \frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha} \right) \\
&\le (\lambda_* + \alpha)^{\frac{1}{4}} \left(\|\mathbf{x}_{k+1} - \mathbf{x}_k\| + \frac{\lambda_{k+1} - \lambda_k}{\lambda_{k+1} + \alpha} \right) \\
&\le (\lambda_* + \alpha)^{\frac{1}{4}} \left(1 + \frac{M}{\lambda_{k+1} + \alpha} \right) \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \\
&\le (\lambda_* + \alpha)^{\frac{1}{4}} \left(1 + \frac{2M}{\lambda_* + \alpha} \right) \|\mathbf{x}_{k+1} - \mathbf{x}_k\|
\end{aligned} \tag{3.9}$$

Consequently,

$$\begin{aligned}
\tilde{f}(\mathbf{y}_k) - \tilde{f}(\mathbf{y}_{k+1}) &\ge (\lambda_* + \alpha)(\lambda_{k+1} - \lambda_k) \\
&\ge (\lambda_* + \alpha)\gamma \|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2 \\
&\ge \frac{(\lambda_* + \alpha)\gamma}{\left((\lambda_* + \alpha)^{\frac{1}{4}} \left(1 + \frac{2M}{\lambda_* + \alpha}\right)\right)^2} \|\mathbf{y}_{k+1} - \mathbf{y}_k\|^2.
\end{aligned} \tag{3.10}$$

Denote $\sigma = (\lambda_* + \alpha)\gamma / ((\lambda_* + \alpha)^{\frac{1}{4}}(1 + \frac{2M}{\lambda_* + \alpha}))^2$. Then $\sigma > 0$. So for sufficiently large k and some σ, we have (3.5a).

(2) Observe that

$$\begin{aligned}
\|\mathbf{y}_{k+1} - \mathbf{y}_k\| &= \|(\lambda_{k+1} + \alpha)^{\frac{1}{4}} \mathbf{x}_{k+1} - (\lambda_k + \alpha)^{\frac{1}{4}} \mathbf{x}_k\| \\
&= (\lambda_{k+1} + \alpha)^{\frac{1}{4}} \left\| \mathbf{x}_{k+1} - \left(\frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}\right)^{\frac{1}{4}} \mathbf{x}_k \right\|.
\end{aligned} \tag{3.11}$$

From Proposition 3.1, $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \to 0$ and note that $\{\lambda_k\}$ is increasing monotonically to λ_*, then there exists an K such that for all $k \ge K$, $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \le 1$. From triangle view, we have

$$\left\| \mathbf{x}_{k+1} - \left(\frac{\lambda_k + \alpha}{\lambda_{k+1} + \alpha}\right)^{\frac{1}{4}} \mathbf{x}_k \right\| \ge \frac{\sqrt{3}}{2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|. \tag{3.12}$$

Consequently,

$$\|\mathbf{y}_{k+1} - \mathbf{y}_k\| \ge \frac{\sqrt{3}}{2} (\lambda_{k+1} + \alpha)^{\frac{1}{4}} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \ge \frac{\sqrt{3}}{4} (\lambda_* + \alpha)^{\frac{1}{4}} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \tag{3.13}$$

for sufficiently large k. Denote $\tau = \frac{\sqrt{3}}{4}(\lambda_* + \alpha)^{\frac{1}{4}} > 0$, then for sufficiently large k, we have (3.5b).

(3) Observe that

$$\begin{aligned}\nabla \tilde{f}(\mathbf{y}_k) &= -8\left(\mathcal{A}\mathbf{y}_k^3 + \mathbf{y}_k^\top \mathbf{y}_k \mathbf{B}\mathbf{y}_k + \alpha\left(\mathbf{y}_k^\top \mathbf{y}_k\right) y_k\right) + 8\left(y_k^\top \mathbf{y}_k\right)^3 y_k \\ &= -8\left((\lambda_k+\alpha)^{\frac{3}{4}} \mathcal{A}\mathbf{x}_k^3 + \mathbf{B}\mathbf{x}_k + \alpha(\lambda_k+\alpha)^{\frac{3}{4}}\mathbf{x}_k\right) + 8(\lambda_k+\alpha)^{\frac{7}{4}}\mathbf{x}_k \\ &= -8(\lambda_k+\alpha)^{\frac{3}{4}}\left(\mathcal{A}\mathbf{x}_k^3 + \mathbf{B}\mathbf{x}_k - \lambda_k\mathbf{x}_k\right). \end{aligned} \tag{3.14}$$

In convex case, $\hat{\mathbf{x}}_{k+1} = \mathcal{A}\mathbf{x}_k^3 + \mathbf{B}\mathbf{x}_k + \alpha\mathbf{x}_k, \mathbf{x}_{k+1} = \hat{\mathbf{x}}_{k+1}/\|\hat{\mathbf{x}}_{k+1}\|$, then

$$\begin{aligned}\mathcal{A}\mathbf{x}_k^3 + \mathbf{B}\mathbf{x}_k - \lambda_k\mathbf{x}_k &= \|\hat{\mathbf{x}}_{k+1}\|\mathbf{x}_{k+1} - \alpha\mathbf{x}_k - \lambda_k\mathbf{x}_k \\ &= \|\hat{\mathbf{x}}_{k+1}\|\left(\mathbf{x}_{k+1} - \frac{\alpha+\lambda_k}{\|\hat{\mathbf{x}}_{k+1}\|}\mathbf{x}_k\right).\end{aligned} \tag{3.15}$$

Since $\lambda_k + \alpha = \|\hat{\mathbf{x}}_{k+1}\|\mathbf{x}_{k+1}^\top\mathbf{x}_k$, we have

$$\left(\mathbf{x}_{k+1} - \frac{\lambda_k+\alpha}{\|\hat{\mathbf{x}}_{k+1}\|}\mathbf{x}_k\right)^\top \mathbf{x}_k = \mathbf{x}_{k+1}^\top\mathbf{x}_k - \frac{\alpha+\lambda_k}{\|\hat{\mathbf{x}}_{k+1}\|} = 0. \tag{3.16}$$

Consequently,

$$\left\|\mathbf{x}_{k+1} - \frac{\lambda_k+\alpha}{\|\hat{\mathbf{x}}_{k+1}\|}\mathbf{x}_k\right\| \le \|\mathbf{x}_{k+1} - \mathbf{x}_k\|. \tag{3.17}$$

From (2), for all sufficiently large k, there exists a $\tau > 0$ such that (3.5b) holds. Then we achieve

$$\begin{aligned}\|\mathbf{y}_{k+1} - y_k\| &\ge \tau\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \ge \tau\left\|\mathbf{x}_{k+1} - \frac{\alpha+\lambda_k}{\|\hat{\mathbf{x}}_{k+1}\|}\mathbf{x}_k\right\| \\ &\ge \frac{\tau}{8\|\hat{\mathbf{x}}_{k+1}\|(\lambda_k+\alpha)^{\frac{3}{4}}}\left\|\nabla\tilde{f}(\mathbf{y}_k)\right\|\end{aligned} \tag{3.18}$$

for all sufficiently large k. From Proposition 3.1, we know that $\|\hat{\mathbf{x}}_{k+1}\| \to \lambda_* + \alpha$. And $\lambda_k + \alpha > 0$ is increasing monotonically to its limit $\lambda_* + \alpha$. Thus

$$\|\mathbf{y}_{k+1} - y_k\| \ge \frac{\tau}{16(\lambda_*+\alpha)^{\frac{7}{4}}}\left\|\nabla\tilde{f}(\mathbf{y}_k)\right\|. \tag{3.19}$$

Denote $\kappa = \frac{\tau}{16(\lambda_*+\alpha)^{\frac{7}{4}}} > 0$. Thus for all sufficiently large k, we have (3.5c). □

Based on the propositions above, we can achieve our final theorem about the point-wise convergence of SS-HOPM algorithm for BEC-like NEP.

Theorem 3.2. *Let* $\alpha > \beta$, $\{\lambda_k, \mathbf{x}_k\}$ *is generated by Algorithm SS-HOPM,* λ_* *is the limit of* $\{\lambda_k\}$, $\{\mathbf{y}_k\}$ *is defined as* $\mathbf{y}_k \equiv (\lambda_k+\alpha)^{\frac{1}{4}}\mathbf{x}_k$, $\tilde{f}(\mathbf{y})$ *is defined as* (3.2). $\{\mathbf{y}_k\}$ *converges to a point* $\mathbf{y}_*$ *with* $\nabla\tilde{f}(\mathbf{y}_*) = 0$. *And the sequence* $\{\mathbf{x}_k\}$ *converges to* $(\lambda_*+\alpha)^{-\frac{1}{4}}\mathbf{y}_*$.

For the concave case, i.e.,$\alpha < -\beta$, the similar results can be obtained. We should claim that it is a challenging job to choose a proper α to ensure convexity or concavity for Algorithm SS-HOPM.

4. Conclusion

We consider the convergence of an alternative sequence which is closely related to the sequence generated by Algorithm SS-HOPM for BEC-like NEP, while not proving the point-wise convergence of the sequence generated by the algorithm directly. The BEC-like NEP we solved by SS-HOPM is a spherical constrained optimization problem, and we convert it into an unconstrained optimization problem in some sense. The strong convexity or concavity hidden in SS-HOPM for BEC-like NEP makes it possible to use the well established results based on Łojasiewicz inequality to prove the point-wise convergence of the new sequence. From the point-wise convergence of the new established sequence, we gained the point-wise convergence of our algorithm.

The analysis of convergence rate is an important research topic in the algorithm analysis. In [29], Hu and Li gave a wonderful result of convergence rate for the HOPM in best rank-one approximation of tensors. It seems that their result for convergence rate applies the convergence rate of SS-HOPM. We will study this problem in a separate paper.

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