$W^{1,\infty}$-INTERIOR ESTIMATES FOR FINITE ELEMENT METHOD ON REGULAR MESH*

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Abstract

For a large class of piecewise polynomial subspaces S^h defined on the regular mesh, $W^{1,\infty}$-interior estimate $\|u_h\|_{1,\infty,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1}$, $u_h\in S^h(\Omega_1)$ satisfying the interior Ritz equation $B(u_h,\varphi)=0$, $\forall\varphi\in \mathring{S}^h(\Omega_1)$, $\Omega_0\subset\subset\Omega_1\subset\subset\Omega$, is proved. For the finite element approximation u_h (of degree $r-1$) to u, we have $W^{1,\infty}$-interior error estimate $\|u-u_h\|_{1,\infty,\Omega_0}\leqslant ch^{r-1}(\|u\|_{r,\infty,\Omega_1}+\|u\|_{1,\Omega})$. If the triangulation is strongly regular in Ω_1 and $r=2$ we obtain $W^{1,\infty}$-interior superconvergence

$$\max_{x\in X}|D(u-\bar{u}_h)(x)|\leqslant ch^2(|\ln h|\,\|u\|_{3,\infty,\Omega_1}+\|u\|_{2,\Omega}).$$

§ 1. Introduction

Let Ω be an n-dimensional bounded domain with the boundary $\partial\Omega$. Denote the norm and semi-norm of the Sobolev space $W^{k,p}(\Omega)$, $1\leqslant p\leqslant\infty$, respectively, by

$$\|u\|_{k,p,\Omega}=\sum_{|\alpha|\leqslant k}\|D^\alpha u\|_{L^p(\Omega)},\quad |u|_{k,p,\Omega}=\sum_{|\alpha|=k}\|D^\alpha u\|_{L^p(\Omega)}.$$

We simply write $W^{k,2}=H^k$, $\|u\|_{k,2,\Omega}=\|u\|_{k,\Omega}$ if $p=2$.

We consider the elliptic boundary value problem

$$\begin{cases} Lu=-D_j(a_{ij}D_iu+a_{0j}u)+a_{i0}D_iu+a_{00}u=f, & \text{in } \Omega\\ u=0, & \text{on } \partial\Omega\end{cases}\tag{1.1}$$

and a bilinear form

$$B(u,v)=\int_\Omega\sum_{i,j=0}^n a_{ij}D_iuD_jv\,dx,\quad D_0u=u,$$

where the coefficients a_{ij} are suitably smooth in $\bar{\Omega}$. Suppose that

$$B(v,v)\geqslant c\|v\|^2_{1,\Omega},\quad c>0,\ \forall v\in\mathring{H}^1(\Omega).\tag{1.2}$$

On a regular (i. e. quasi-uniform) mesh-domain Ω_h of Ω we give a finite dimensional subspace $S^h\subset C(\bar{\Omega})$, consisting of piecewise polynomials of degree $r-1$, and

$$\mathring{S}^h(\Omega_1)=\{\varphi\in S^h(\Omega)\,|\,\mathrm{supp}\,\varphi\subseteq\bar{\Omega}_1\},\quad \Omega_1\subset\subset\Omega.$$

An approximate solution $u_h\in S^h(\Omega)$ to u satisfies the interior Ritz equation

$$B(u-u_h,\varphi)=0,\quad \forall\varphi\in\mathring{S}^h(\Omega_1).\tag{1.3}$$

An important special case occurs when $Lu=0$. Then $u_h\in S^h(\Omega)$ satisfies[1]

$$B(u_h,\varphi)=0,\quad \forall\varphi\in\mathring{S}^h(\Omega_1).\tag{1.4}$$

Such u_h will play a central role in deriving the interior error estimates. For the regular mesh in Ω_1, J. Nitsche and A. Schatz[1] first proved L^2-interior estimate

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$$\|u_h\|_{1,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1},\quad \Omega_0\subset\subset\Omega_1, \tag{1.5}$$

where $s\geqslant 0$ is an integer, arbitrary but fixed, and $\|u_h\|_{-s,\Omega}$ negative norm. For the uniform mesh, J. Bramble, J. Nitsche and A. Schatz[2] later proved L^∞-interior estimate

$$\|u_h\|_{0,\infty,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1}. \tag{1.6}$$

For the regular mesh, A. Schatz and L. Wahlbin[8] also proved it by the technique of estimating derivatives on annuluses. The present paper extends these results and proves the following

Fundamental Lemma. *Suppose that the triangulation is regular in $\Omega_1\subset\subset\Omega$, and $u_h\in S^h$ satisfies* (1.3). *Then*

$$\|u_h\|_{1,\infty,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1}. \tag{1.7}$$

Using the lemma, we may derive $W^{1,\infty}$-interior error estimate (Theorem 1) and $W^{1,\infty}$-interior superconvergence (Theorem 2) for the general problem (1.1).

§ 2. Some Assumptions

[7] and [8] discussed a priori estimate and the solvability of solution $u\in W^{2,p}(\Omega)$, $1<p<\infty$, for the problem (1.1). We obtained

Lemma 1[9]. *Let $\Omega\in C^{1,1}$, $a_{ij}\in W^{1,\infty}(\Omega)$, $i+j\neq 0$, $a_{00}\in L^\infty(\Omega)$, $f\in L^p(\Omega)$, $1<p<\infty$, and $u\in W^{2,p}(\Omega)$ is a unique solution of* (1.1). *Then*

$$\|u\|_{2,p,\Omega}\leqslant c\tilde{p}^\lambda\|f\|_{0,p,\Omega}, \tag{2.1}$$

where $\tilde{p}=\max(p, p')$, $p'=p/(p-1)$, and the constants λ and c are independent of p and f.

Let $\Omega=G$ be a sphere with radius R suitably small. Suppose that the Green function $g(x, y)$ for (1.1) exists such that

$$|D^\alpha g(x, y)|\leqslant\begin{cases} c(|\ln|x-y||+1), & n=2 \text{ and } \alpha=0,\\ c|x-y|^{2-n-|\alpha|}, & n>2 \text{ or } |\alpha|=1.\end{cases} \tag{2.2}$$

By the Green function $g(x, y)$, the solution u of (1.1) can be expressed by

$$u(x)=\int_G g(x, y)f(y)dy. \tag{2.3}$$

If $1\leqslant q<n/(n-1)$, we have

$$|u|_{1,q,G}\leqslant c\left(\int_G\int_G|x-y|^{(1-n)q}|f(y)|dx\,dy\right)^{1/q}\left(\int_G|f(y)|dy\right)^{1/q'}\leqslant c\|f\|_{0,1,G}. \tag{2.4}$$

We now turn to the finite dimensional subspace $S^h(\Omega)$[1] and make the following assumptions (for $1\leqslant p\leqslant\infty$):

A1. For each $u\in W^{1,p}(G)$, $1\leqslant t\leqslant r$, there exists a $\varphi\in S^h(\Omega_1)$ such that

$$\|u-\varphi\|_{s,p,G}\leqslant ch^{t-s}\|u\|_{t,p,G},\quad s=0, 1. \tag{2.5}$$

A2. Let $\omega\in C_0^\infty(G_0)$ and $u_h\in S^h(G)$, $G_0\subset\subset G\subset\subset\Omega$. Then there exists $\varphi\in \mathring{S}^h(G)$ such that

$$\|\omega u_h-\varphi\|_{1,p,G}\leqslant ch\|u_h\|_{1,p,G}. \tag{2.6}$$

A3. For each $h\in(0, 1]$, there exists a mesh-domain G_1, $G_0\subset\subset G_1\subset\subset G$, such that, for all $\varphi\in S^h(\Omega_1)$,