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SOME NONLINEAR BOUNDARY PROBLEMS FOR THE SYSTEMS OF NONLINEAR WAVE EQUATIONS BY FINITE SLICE METHOD*

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§1. Introduction

1. The purpose of this work is to study some nonlinear boundary problems for the system

$$u_{tt}-u_{xx}+\operatorname{grad} F(u)=B(x, t, u)u_t+f(x, t, u, u_x, u_t) \tag{1}$$

of the nonlinear wave equations in the rectangular domain $Q_T=\{0\leqslant x\leqslant l, 0\leqslant t\leqslant T\}$, where $u(x, t)=(u_1(x, t), \cdots, u_m(x, t))$ is the m-dimensional unknown vector function, $f(x, t, u, p, q)$ is given m-dimensional vector function of $(x, t)\in Q_T$ and $u, p, q\in\mathbb{R}^m$, $B(x, t, u)$ is a $m\times m$ matrix of $(x, t)\in Q_T$ and $u\in\mathbb{R}^m$, $F(u)$ is a non-negative scalar function of $u\in\mathbb{R}^m$ and "grad" is the gradient operator with respect to $u\in\mathbb{R}^m$. The well-known Sine-Gordon equation

$$u_{tt}-u_{xx}=\sin u,$$

the nonlinear forced vibration equation

$$u_{tt}-u_{xx}+u^3=0$$

and the nonlinear wave equation

$$u_{tt}-u_{xx}+\sinh u=0$$

are the simple cases of the above mentioned system (1) of nonlinear wave equations. Many authors have paid great attention to the study of the various problems for these special nonlinear wave equations[1-17]. Some general systems of this type have been considered in [18—20].

At first we are going to consider the boundary problem for the system (1) with the fairly wide nonlinear mutual boundary conditions

$$\begin{aligned} u_x(0, t)&=\operatorname{grad}_0 \Phi(u(0, t), u(l, t), t),\\ -u_x(l, t)&=\operatorname{grad}_1 \Phi(u(0, t), u(l, t), t)\end{aligned} \tag{2}$$

and the initial conditions

$$\begin{aligned} u(x, 0)&=\varphi(x),\\ u_t(x, 0)&=\psi(x),\end{aligned} \tag{3}$$

where $\Phi(u_0, u_1, t)$ is a non-negative scalar function of $t\in[0, T]$ and $u_0, u_1\in\mathbb{R}^m$, "grad_0" and "grad_1" are the gradient operators with respect to the vector variables

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u_0 and u_1 respectively and $\varphi(x)$ and $\psi(x)$ are two m-dimensional initial vector functions.

When $\Phi(u_0, u_1, t)=\Phi_0(u_0, t)+\Phi_1(u_1, t)$, the boundary conditions (2) become the ordinary nonlinear (non-mutual) boundary conditions

$$\begin{aligned} u_x(0, t) &= \operatorname{grad} \Phi_0(u(0, t), t), \\ -u_x(l, t) &= \operatorname{grad} \Phi_1(u(l, t), t). \end{aligned} \tag{4}$$

If $\Phi(u_0, u_1, t)$ is a polynomial of $u_0, u_1 \in \mathbb{R}^m$ of the form

$$\begin{aligned} \Phi(u_0, u_1, t) = (u_0, B_{00}(t)u_0) + (u_0, B_{01}(t)u_1) + (u_1, B_{10}(t)u_0) \\ + (u_1, B_{11}(t)u_1) + (g_0(t), u_0) + (g_1(t), u_1), \end{aligned} \tag{5}$$

then (2) are simplified to a linear symmetric boundary conditions

$$\begin{aligned} u_x(0, t) &= (B_{00}(t)+B_{00}^*(t))u(0, t) + (B_{01}(t)+B_{10}^*(t))u(l, t) + g_0(t), \\ -u_x(l, t) &= (B_{01}^*(t)+B_{10}(t))u(0, t) + (B_{11}(t)+B_{11}^*(t))u(l, t) + g_1(t), \end{aligned} \tag{6}$$

where $B(t)$'s are the $m\times m$ matrices, "$*$" denotes the transpose of matrix and $g_0(t)$ and $g_1(t)$ are two m-dimensional vector functions of $t\in[0, T]$.

For the nonlinear partial differential equations and systems, it is natural to take into consideration of the nonlinear boundary problems both in theoretical and in pratical studies[21,22]. Hence the nonlinear boundary problems are of the number of the fundamental problems as the classical linear boundary problems.

In § 2 of the present work, we will give a series of a priori estimations for the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ of the nonlinear finite slice system. Then we will establish the existence of the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ for the nonlinear finite slice system by the fixed point method on the base of these estimations. By the limit process as $h\to 0$, we will obtain the generalized solution $u(x, t)$ of the ordinary boundary problem (2) and (3) for the system (1) of nonlinear wave equations. By this way the convergence behaviors of the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ of the nonlinear finite slice system are studied.

At the end of this work, we will take in consideration of some more general nonlinear boundary problems with the mixed conditions[23,24]

$$\begin{aligned} u_x(0, t) &= \operatorname{grad} \Phi(u(0, t), t), \\ -u_t(l, t) &= \Phi_1(u_x(l, t), u(0, t), u(l, t), t) \end{aligned} \tag{7}$$

by the finite slice method.

We adopt the similar notations and conventions as used in [18—20, 25, 26].

2. Suppose that for the system (1) of nonlinear wave equations, the nonlinear boundary conditions and the initial vector functions, $\varphi(x)$ and $\psi(x)$ the following conditions are satisfied:

(I) $F(u)\geqslant 0$ is a non-negative twice continuously differentiable scalar function of vector variables $u\in\mathbb{R}^m$.

(II) $B(x, t, u)$ is a $m\times m$ matrix, continuous for $(x, t)\in Q_T$ and $u\in\mathbb{R}^m$ and continuously differentiable with respect to x and u. $B(x, t, u)$ is semibound, i. e., for any $\xi\in\mathbb{R}^m$, $(\xi, B(x, t, u)\xi)\leqslant b|\xi|^2$, where b is a constant.

(III) $f(x, t, u, p, q)$ is a m-dimensional vector function of lower degree,