ON THE ERROR ESTIMATE FOR THE ISOPARAMETRIC FINITE ELEMENT METHOD*

WANG LIE-HENG (王烈衡)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

In this paper, the isoparametric element of 2-degree Lagrange type for second order elliptic P. D. E. with nonhomogeneous Dirichlet boundary value is considered. We prove

$$\|u-u_h\|_{1,\Omega}=O(h^2),$$

which is the same as $\|u-u_h\|_{1,\Omega_h}=O(h^2)$, where Ω and Ω_h in R^2 are the domain of the boundary value problem and the isoparametric triangulation domain respectively.

§ 1. Introduction

We consider the nonhomogeneous Dirichlet problem:

$$\begin{cases}\text{find } u\in V, \text{ such that}\\ a(u, v)=\langle f, v\rangle, \quad \forall v\in V,\end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^2 with a sufficiently smooth boundary $\partial\Omega$,

$$a(u, v)=\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x)\partial_i u\partial_j v\,dx, \tag{1.2}$$

$$\langle f, v\rangle=\int_\Omega f\cdot v\,dx, \tag{1.3}$$

$a_{ij}\in W^{2,\infty}(\Omega)$, $f\in W^{2,q}(\Omega)(q\geq 2)$, g is restriction of a function in $H^3(\Omega)$, $\beta=\text{const.}>0$, such that

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j\geq\beta\sum_{i=1}^{2}\xi_i^2, \qquad \forall x\in\Omega,\ \xi_i\in\mathbb{R},\ i=1, 2, \tag{1.4}$$

and

$$V=\{v\in H^1(\Omega):\ v=g \text{ on } \partial\Omega\}, \tag{1.5}$$

$$V=H_0^1(\Omega). \tag{1.6}$$

The notations above are introduced from [1].

In [2], [3], Ciarlet and Raviart studied isoparametric finite element approximation of the problem (1.1), in 2-degree Lagrange type, or the so-called type (2), as follows:

Let $(\mathscr{T}_h)_{h>0}$ be a family of regular isoparametric triangulations of type (2), F_K: $\hat{K}\to K$ $\forall K\in\mathscr{T}_h$ be the isoparametric mapping of type (2) (cf. Fig. 1), $\Omega_h=\bigcup_{K\in\mathscr{T}_h} K$, in general $\Omega_h\not\subset\Omega$. Let X_h be the isoparametric finite element space of type (2), and

$$V_h=\{v_h\in X_h:\ v_h=g \text{ at the boundary nodes on } \partial\Omega\}, \tag{1.7}$$

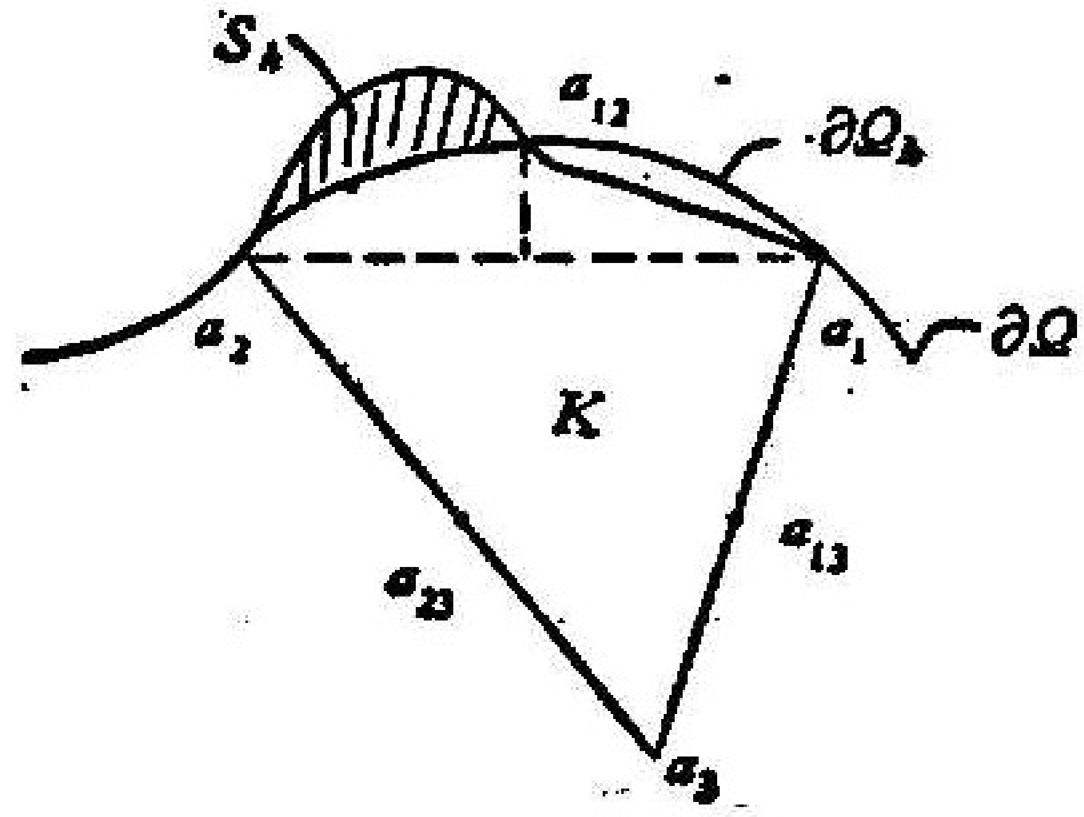
$$V_{0h}=\{v_h\in X_h:\ v_h=0 \text{ at the boundary nodes on } \partial\Omega\}. \tag{1.8}$$

* Received July, 3, 1984.

Then the approximation problem with numerical integration is the following:

$$\begin{cases} \text{find } u_h \in V_h, \text{ such that} \\ a_h(u_h, v_h) = \langle f, v_h \rangle_h, \quad \forall v_h \in V_{0h}, \end{cases} \tag{1.9}$$

where $a_h(u_h, v_h)$ and $\langle f, v_h \rangle_h$ are formulas of numerical integrations (cf. [2], [3]). As to the isoparametric finite element approximation without numerical integrations, it is not available in practice.



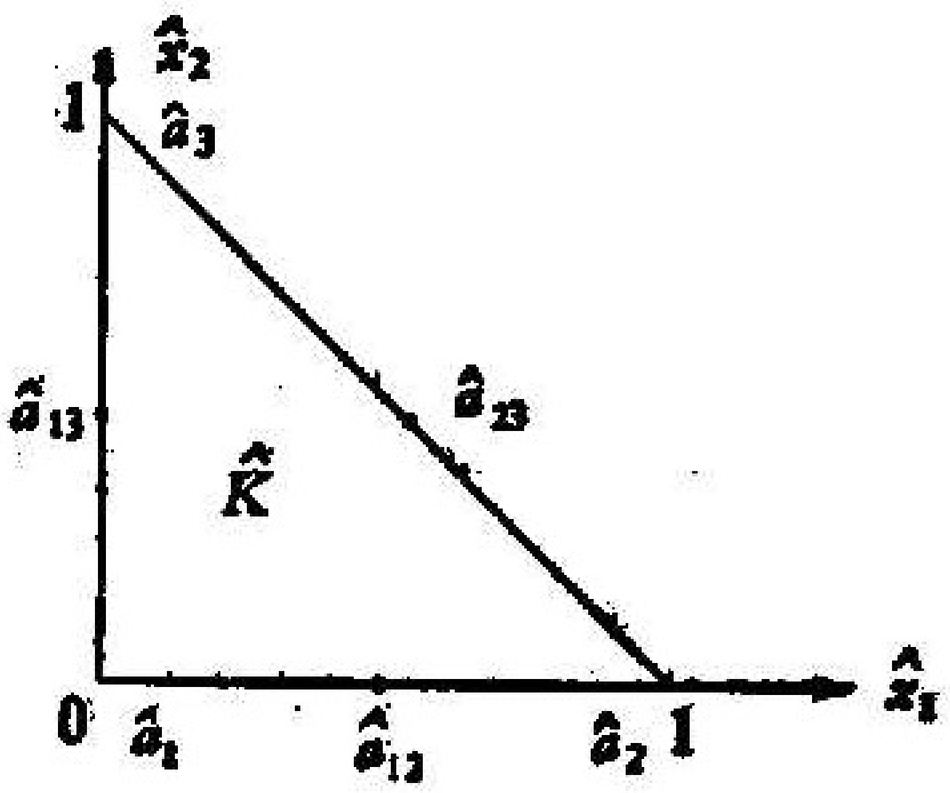


Fig. 1

Ciarlet and Raviart have shown the error estimate $\|u-u_h\|_{1,\Omega_h}=O(h^2)$ on the isoparametric triangulation domain Ω_h. In [4], in order to obtain the error estimate $\|u-u_h\|_{1,\Omega}=O(h^2)$ on the domain Ω, Li Li-kang reformed the isoparametric element approximation, which is not of the standard isoparametric type as in [2], [3], and in which it is necessary to have an expression of the boundary $\partial\Omega$ in x.

In this paper, the standard isoparametric element is considered. Section 2 contains the extension of the isoparametric finite element solution u_h. In section 3, we prove the error estimate $\|u-u_h\|_{1,\Omega}=O(h^2)$ on the domain Ω for the extension of u_h as in section 2.

§ 2. Extension of u_h

In order to estimate the error between the solutions u of (1.1) and u_h of (1.9), it is necessary to extend u_h from Ω_h onto $\Omega \cup \Omega_h$, since the solution u_h of (1.9) is defined on Ω_h only.

As well known, the estimation of $\|u-u_h\|_{1,\Omega_h}$ on the isoparametric triangulation domain Ω_h is independent of the way the solution u of (1.1) is extended from Ω onto $\Omega \cup \Omega_h$ in the case u remains in H^3 space. However the case is different for the estimation of $\|u-u_h\|_{1,\Omega}$ on the domain Ω if the solution u_h of (1.9) is extended from Ω_h onto $\Omega \cup \Omega_h$. Let us show some examples.

Example 1. Let $\mathscr{T}'_h$ be the set of boundary isoparametric triangles, and K' be a curve triangle of reflection of $K \in \mathscr{T}'_h$ with $\overline{a_1a_2}$, i.e. K consists of $\overline{a_3a_1}$, $\overline{a_3a_2}$ and $\widehat{a_1a_{12}a_2}$, and K' consists of $\overline{a'_3a_1}$, $\overline{a'_3a_2}$ and $\widehat{a_1a_{12}a_2}$ (cf. Fig. 2). Let $\tilde{\Omega} \supseteq \Omega_h \cup (\bigcup_{K \in \mathscr{T}'_h} K') \supset \Omega$, and since the boundary $\partial\Omega$ is sufficiently smooth, let $\tilde{u} \in H^3(\tilde{\Omega})$ be the extension of u. If the extension $\tilde{u}_h$ of u_h is defined as the interpolation $\tilde{u}^I$ of $\tilde{u}$ on K', then it is easy to estimate $\|u-\tilde{u}_h\|_{1,\Omega}=O(h^2)$. But it is not available, since the solution u of (1.1) is unknown.