Vol. 3 No. 3 JOURNAL OF COMPUTATIONAL MATHEMATICS July 1985

ON THE SOLVABILITY OF RATIONAL HERMITE-INTERPOLATION PROBLEM*

XU GUO-LIANG (徐国良), LI JIA-KAI (李家楷)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

The solvability of the rational Hermite-interpolation problem is investigated through an approach similar to that developed in an earlier paper [1] for the ordinary case. However, the subsequent deduction of analogous results involves much complications. The Quasi-Rational Hermite Interpolant r^*_{mn} is introduced. In the case of r^*_{mn} being nondegenerate, its explicit expression is given. Working with the notion of l-fold unattainable point and using algebraic elaboration, we have successively established several theorems concerning interpolating properties of r^*_{mn} and, in particular, obtained existence theorems for the solution of the proposed problem.

§ 1. Introduction

Let $y=f(x)$ be a bounded and real (or complex) valued function defined in a real domain $\mathscr{R}$ (or complex $\mathscr{C}$ as well), and as usual $y^{(j)}(x)$ be its j-th derivative. Suppose m and n are given non-negative integers, and

$$\boldsymbol{R}(m, n)=\{R: R=P/Q, P\in H_m, Q\in H_n\backslash\{0\}\},$$

where H_l denotes the class of all polynomials of degree at most l. In almost all of our considerations below, we have treated the polynomials P and Q to be of some more general form:

$$P(x)=\sum_{\alpha=0}^{m} a_\alpha g_\alpha(x), \quad Q(x)=\sum_{\beta=0}^{n} b_\beta h_\beta(x),$$

where $g_\rho(x)$ and $h_\rho(x)$ are simply given monic polynomials of degree ρ. For ease of writing we define in general

$$S=\{0, 1, \cdots, s\},$$

and likewise

$$K_i=\{0, 1, \cdots, k_i\}, \quad \text{while } i\in S.$$

The problem of rational Hermite-interpolation consists of finding a rational function $R=R(x)\in\boldsymbol{R}(m, n)$ on a given point set

$$X=\{\text{distinct } x_i: x_i\in\mathscr{R}(\text{say}), \quad \forall i\in S\},$$

such that

$$R^{(j)}(x_i)=y_i^{(j)}, \quad \forall i\in S, j\in K_i, \tag{1.1}$$

where $y_i^{(j)}=y^{(j)}(x_i)$ and m, n, s, k_j's are related by

$$m+n+1=\sum_{i\in S}(k_i+1). \tag{1.2}$$

* Received August 24, 1984.

A convenient approach for dealing with such a problem is to consider, just analogous to the case of the ordinary rational interpolation (cf. [1]), a closely related linearized one: namely, find $R=P/Q\in \boldsymbol{R}(m, n)$ such that

$$(P-yQ)^{(j)}(x_i)=0, \quad \forall i\in S, j\in K_i. \tag{1.3}$$

We can easily see that (1.3) turns out to be a system of $m+n+1$ homogeneous linear equations in $m+n+2$ unknowns a's and b's. Accordingly, it always possesses nontrivial solutions. Denote by $\boldsymbol{R}_0(m, n)$ the subset of $\boldsymbol{R}(m, n)$ consisting of all the elements which satisfy the system (1.3).

It is noteworthy to recall as in [1] that although the two problems (1.1) and (1.3) are not equivalent, yet they are intimately connected under certain conditions. On that account we narrate for self-containedness two comprehensible theorems as follows:

Theorem 1.1 (cf. [2]). *If $R\in \boldsymbol{R}(m, n)$ solves the problem (1.1), then $R\in \boldsymbol{R}_0(m, n)$ also. Conversely, if $R=P/Q\in \boldsymbol{R}_0(m, n)$ and $Q(x_i)\neq 0$, $\forall i\in S$, then R is a solution of (1.1).*

Theorem 1.2 ([3], [4, p. 21]). *There exists a rational function R satisfing the equations (1.1) if and only if the rational function $\widetilde{P}/\widetilde{Q}$, obtained by dividing out all the common factors in the numerator and denominator of $P/Q\in \boldsymbol{R}_0(m, n)$, remains to be in $\boldsymbol{R}_0(m, n)$.*

In this paper we make an effort to follow the same method of approach as demonstrated in the case of the ordinary rational interpolation and to carry on the study for the Hermite case in § 2 through some algebraic elaboration and complicated procedures, and finally to establish in § 3 the new, projected theorems 3.3 and 3.6 which are to give practical and useful results as corresponding to those theorems 2.2 and 2.4 in [1].

§ 2. Preliminary Lemmas, Definitions and Notations

First, let us quote the following

Lemma 2.1 ([3, p. 838], [4, p. 17]). *Let P/Q and $P_1/Q_1\in \boldsymbol{R}_0(m, n)$. Then $P_1Q=PQ_1$.*

Remark. We can deduce from this lemma that if $P_1=0$ for some $P_1/Q_1\in \boldsymbol{R}_0(m, n)$, then $P=0$ for any $P/Q\in \boldsymbol{R}_0(m, n)$. We further note from Theorem 1.2 that, in the case of $P=0$, the problem (1.1) is solvable if and only if $y_i^{(j)}=0$ for $\forall i\in S$ and $\forall j\in K_i$.

In the next lemma we shall precisely give the condition under which $P=0$.

Lemma 2.2. *For any $P/Q\in \boldsymbol{R}_0(m, n)$, $P=0$ if and only if $\sum\limits_{i\in S} m_i\geq m+1$, where*

$$m_i=\begin{cases} j, \ y_i^{(0)}=y_i^{(1)}=\cdots=y_i^{(j-1)}=0, \ y_i^{(j)}\neq 0, & j\leq k_i, \\ k_i+1, \ y_i^{(0)}=y_i^{(1)}=\cdots=y_i^{(k_i)}=0, & \end{cases} \quad i\in S. \tag{2.1}$$

Proof. From (1.3) we have that x_i is at least a m_i-fold zero of $P(x)$. Thus, $P(x)$ has at least $\sum\limits_{i\in S} m_i$ zeros (counting multiplicity). If $\sum\limits_{i\in S} m_i\geq m+1$, that is $P(x)$ has at least $m+1$ zeros, then $P=0$.