ON THE CONVERGENCE OF DIAGONAL ELEMENTS AND ASYMPTOTIC CONVERGENCE RATES FOR THE SHIFTED TRIDIAGONAL QL ALGORITHM*

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Abstract

The convergence of diagonal elements of an irreducible symmetric tridiagonal matrix under QL algorithm with some kinds of shift is discussed. It is proved that if $\alpha_1-\sigma\to 0$ and $\beta_j\to 0$, $j=1, 2, \cdots, m$, then $\alpha_j\to\lambda_j$, $j=1, 2, \cdots, m$, where $\lambda_j (j=1, 2, \cdots, m)$ are m eigenvalues of the matrix, and σ is the origin shift. The asymptotic convergence rates of three kinds of shift, Rayleigh quotient shift, Wilkinson's shift and RW shift, are analysed.

§ 1. Introduction

The shifted QL algorithm is a very efficient algorithm for finding all eigenvalues of a symmetric tridiagonal matrix. The global convergence of the QL algorithm with Wilkinson's shift is proved in [1], [2]. The asymptotic convergence rate of this case is at least quadratic[1], and is often cubic or better than cubic except for special bizarre matrices if they exist[2]. The RW shift is proposed in [3]. The global convergence and at least cubic aymptotic convergence rate for the case of RW shift are proved in [3].

We apply the shifted QL algorithm to a symmetric tridiagonal matrix $T=T^{(1)}$. Let the k-th iteration matrix be

$$T^{(k)}=\begin{pmatrix} \alpha_1^{(k)} & \beta_1^{(k)} & & & \\ \beta_1^{(k)} & \alpha_2^{(k)} & \beta_2^{(k)} & & \\ & \beta_2^{(k)} & & & \\ & & & & \beta_{n-1}^{(k)} \\ & & & \beta_{n-1}^{(k)} & \alpha_n^{(k)} \end{pmatrix}.$$

The global convergence means that $\beta_1^{(k)}\to 0$. Does $\alpha_1^{(k)}$ converge at the same time? Although we know there is an eigenvalue $\lambda_1^{(k)}$ of $T^{(k)}$ such that

$$|\alpha_1^{(k)}-\lambda_1^{(k)}|<|\beta_1^{(k)}|, \tag{1}$$

it seems that no one has proved that for large enough k, $\lambda_1^{(k)}$ is independent of k.

Furthermore, if $\beta_i^{(k)}\to 0$ $(i=1, 2, \cdots, j)$ can we say $\alpha_i^{(k)}$ $(i=1, 2, \cdots, j)$ are convergent?

In this paper the following theorem is proved:

Theorem. *Let $T=T^{(1)}$ be an irreducible symmetric tridiagonal matrix. The QL algorithm with shift $\{\sigma_k\}$ is applied to $T^{(1)}$. If $\alpha_1^{(k)}-\sigma_k\to 0$ and $\beta_i^{(k)}\to 0 (i=1, 2, \cdots, j)$,*

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then $\alpha_s^{(k)} \to \lambda_s$ $(s=1, 2, \cdots, j)$, *where* $\lambda_1, \lambda_2, \cdots, \lambda_j$ *are* j *different eigenvalues of* T.

Using the above theorem, we can give an improvement on Theorem 8.11 of [4] as follows:

Theorem. *Let the QL algorithm with Wilkinson's shift be applied to an unreduced tridiagonal matrix* T. *Then as* $k \to \infty$, $\beta_1 \to 0$. *If, in addition,* $\beta_2 \to 0$, $\beta_3 \to 0$, *then as* $k \to \infty$,

$$|\hat{\beta}_1/\beta_1^3\beta_2^2| \to |\lambda_2-\lambda_1|^{-3}|\lambda_3-\lambda_1|^{-1} \neq 0,$$

where $\lambda_1, \lambda_2, \lambda_3$ *are the limits of* $\alpha_1, \alpha_2, \alpha_3$.

There is also a dicussion on the asymptotic convergence rate in the case of the Rayleigh quotient shift and the RW shift.

§ 2. Some Basic Theorems

Let

$$T=\begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

be a real tridiagonal symmetric matrix. Given a scalar σ, called the shift, consider the orthogonal-lower triangular factorization

$$T-\sigma I = QL, \tag{2}$$

where I is the identity matrix, Q is an $n\times n$ orthogonal matrix

$$Q=(q_1, q_2, \cdots, q_n),$$

$$q_i=(q_{1i}, q_{2i}, \cdots, q_{ni})^T,$$

and L is a lower triangular matrix

$$L=(l_{ij}),\ l_{ij}=0 \text{ when } j>i.$$

Let

$$\hat{T}=LQ+\sigma I. \tag{3}$$

Obviously $\hat{T}$ is a symmetric tridiagonal matrix too. Denote

$$\hat{T}=\begin{pmatrix} \hat{\alpha}_1 & \hat{\beta}_1 & & & 0 \\ \hat{\beta}_1 & \hat{\alpha}_2 & \hat{\beta}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \hat{\beta}_{n-1} \\ 0 & & & \hat{\beta}_{n-1} & \hat{\alpha}_n \end{pmatrix}$$

and there is a relationship between T and $\hat{T}$, namely

$$\hat{T}=Q^T T Q. \tag{4}$$

The transformation from T to $\hat{T}$ is a QL transformation with shift σ.

Given a symmetric tridiagonal matrix T, let $T^{(1)}=T$. We do QL transformation with shift σ_k to $T^{(k)}$ successively and get a matrix-sequence $\{T^{(k)}\}$, such that

$$T^{(k)}-\sigma_k I = Q_k L_k,$$