THE SPECTRAL VARIATION OF PENCILS OF MATRICES*

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Abstract

Perturbation theorems for the spectrum of a regular matrix pencil $\lambda A-B$ are given. As it may include points near or at infinity the Euclidean distance is not appropriate. We use the chordal metric and the distances $w(\lambda,\tilde{\lambda})=\min\{|\lambda-\tilde{\lambda}|, |\lambda^{-1}-\tilde{\lambda}^{-1}|\}$ and $v(\lambda,\tilde{\lambda})=\{|\lambda-\tilde{\lambda}|$ if $|\lambda|\leq 1$ and $|\lambda^{-1}-\tilde{\lambda}^{-1}|$ if $|\lambda|>1\}$. For those purposes we develop here an algebraic treatment of matrix pairs, with special reference to diagonable and definite pairs, using ideas from the theory of matrix polynomials.

§ 1. Introduction

Throughout this paper (A, B) will denote a regular pair of $n\times n$ complex matrices. That is $\det(\lambda A-B)\not\equiv 0$, where λ is a complex parameter. Thus, there is a discrete set of complex numbers, the eigenvalues of the pair, for which $\det(\lambda A-B)=0$. Denote this set by $\sigma(\lambda A-B)$, the spectrum of the pencil $\lambda A-B$, and note that it may include the point at infinity.

We are concerned with a perturbation problem: To find bounds for the variation in the eigenvalues when (A, B) is perturbed to $(A+E, B+F)$ in terms of norms of E and F. It is well recognized that, when A is singular, or "nearly" so, the Euclidean metric is not appropriate for measuring the eigenvalue variations. This has led to investigations in terms of a homogeneous problem: Consider the set of complex pairs λ, μ for which $\det(\lambda A-\mu B)=0$ and measure the distance between pairs in terms of the chordal metric, ρ. This is because the chordal metric has the homogeneity property:

$$\rho((k\lambda, k\mu), (\alpha, \beta))=\rho((\lambda, \mu), (\alpha, \beta)).$$

This formulation also has the merit of treating A and B in a symmetrical way. This line of attack has been studied by Stewart[7], Elsner and Sun[2], et al.

The investigations of this paper are based on a rather different idea. The chordal metric suggests that an eigenvalue λ is better viewed as a representative $(\lambda, 1)$ of a class of equivalent number pairs for the eigenvalue problem in homogeneous form. Since the difficulties of perturbation theory arise when $|\lambda|$ is large compared to 1, we propose that, when $|\lambda|>1$ we consider the "reversed" eigenvalue pair $(1, \lambda^{-1})$. In this way we retain the symmetrical treatment of A and B and avoid

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some of the complications associated with working with the chordal metric.

When considering perturbation of an eigenvalue λ of (A, B) with $|\lambda| \leqslant 1$ it is convenient to measure the distance from $\sigma(\lambda\tilde{A}-\tilde{B})$ by also writing all eigenvalues of $(\tilde{A}, \tilde{B})$ in the direct form $(\tilde{\lambda}, 1)$. Similarly, if $|\lambda|>1$, $\tilde{\lambda}$ is written in the reverse form $(1, \tilde{\lambda}^{-1})$ for all $\tilde{\lambda}\in\sigma(\lambda\tilde{A}-\tilde{B})$. Thus, we are led to the measure of distance

$$v(\lambda, \tilde{\lambda})=\begin{cases}|\lambda-\tilde{\lambda}| & \text{if } |\lambda|\leqslant 1,\\ |\lambda^{-1}-\tilde{\lambda}^{-1}| & \text{if } |\lambda|>1.\end{cases} \tag{1.1}$$

Note that v is not a metric. For example, $v(\lambda, \tilde{\lambda})\neq v(\tilde{\lambda}, \lambda)$, in general. Nevertheless, it is a useful and convenient measure in this context.

We shall also employ the following measures of distance (the latter is the chordal metric and will admit comparisons with the analysis of [2] and [7], for example):

$$w(\lambda, \tilde{\lambda})=\min(|\lambda-\tilde{\lambda}|,\ |\lambda^{-1}-\tilde{\lambda}^{-1}|), \tag{1.2}$$

$$\rho(\lambda, \tilde{\lambda})=\frac{|\lambda-\tilde{\lambda}|}{(1+|\lambda|^2)^{1/2}(1+|\tilde{\lambda}|^2)^{1/2}}. \tag{1.3}$$

Note that $w(\lambda, \tilde{\lambda})=w(\lambda^{-1}, \tilde{\lambda}^{-1})$ and $\rho(\lambda, \tilde{\lambda})=\rho(\lambda^{-1}, \tilde{\lambda}^{-1})$. Also, these measures of distance are related as follows:

$$\rho\leqslant w\leqslant v, \quad w\leqslant\frac{2\rho}{1-\rho} \tag{1.4}$$

and, if $w<1$, then $v\leqslant w/(1-w)$.

Let (A, B) and $(\tilde{A}, \tilde{B})$ be regular pairs of $n\times n$ matrices with (possibly infinite) eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n$, respectively. A spectral variation is defined for these pairs in terms of v, w or ρ by

$$S_{(A,B)}(\tilde{A}, \tilde{B})=\max_j \min_i v(\tilde{\lambda}_j, \lambda_i), \tag{1.5}$$

$$S^{(1)}_{(A,B)}(\tilde{A}, \tilde{B})=\max_j \min_i w(\tilde{\lambda}_j, \lambda_i), \tag{1.6}$$

$$S^{(2)}_{(A,B)}(\tilde{A}, \tilde{B})=\max_j \min_i \rho(\tilde{\lambda}_j, \lambda_i). \tag{1.7}$$

The primary objective of this paper is to obtain bounds for these spectral variations in terms of $\|\tilde{A}-A\|$ and $\|\tilde{B}-B\|$. This is achieved in Theorem 3.3 for diagonable pairs and in Theorem 5.3 for definite hermitian pairs. In Theorem 5.2 we have a result for a more general class of hermitian pairs. In the case of $S^{(2)}$ our results give some improvement on the results of Elsner and Sun[2]. The contributions of this paper also include improved proofs, and (possibly more convenient) measures S and $S^{(1)}$ of the spectral variation, and the more general result on hermitian pairs just cited.

The crux of our analysis is the division of the eigenvalues of (A, B) into those which are "small" and "large" in an appropriate sense. This is suggestive of an algebraic analysis of matrix polynomials presented by Gohberg, Lancaster and Rodman in Chapter 7 of [3]. Taking advantage of these ideas we need, and develop here, an algebraic treatment of matrix pairs, with special reference to diagonable and definite pairs; a treatment that seems to be missing in the literature on this problem area.