THE ITERATIVE ACCELERATIVE METHOD OF FINITE ELEMENT APPROXIMATION FOR THE SYSTEM $u=\sum u_j\frac{\partial u}{\partial x_j}+f$*

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Abstract

In this paper we use an iteration method[1] to get an approximate solution u^n and $\bar{u}^n$ which approximate the exact solution u with error estimates $\|u-u^n\|+ch|u-u^n|_1+|u-\bar{u}^n|_1\leqslant ch^{n+2}$.

Let us consider the following system:

$$\begin{cases}\Delta u=\sum_j u_j\frac{\partial u}{\partial x_j}+f,\\ u|_\Gamma=0\end{cases}\tag{1}$$

in two dimension, Ω is a bounded domain with boundary Γ sufficiently smooth. The weak form of (1) is:

$$\begin{cases}u\in(H_0^1(\Omega))^2=V,\\ (u,v)_1+\left(\sum_j u_j\frac{\partial u}{\partial x_j}+f,\ v\right)=0,\quad\forall v\in V.\end{cases}\tag{1$'$}$$

We assume that u is an isolated solution, i.e. the linear problem

$$\begin{cases}w\in V,\\ (w,v)_1+\left(\sum_j w_j\frac{\partial u}{\partial x_j},\ v\right)+\left(\sum_j u_j\frac{\partial w}{\partial x_j},\ v\right)=0,\quad\forall v\in V\end{cases}\tag{2}$$

has only a trivial solution $w=0$ in V.

Let $u^0\in S_h\subset V$ be the finite element solution[2] of the corresponding Galerkin problem, i.e.

$$\begin{cases}u^0\in S_h\subset V,\\ (u^0,v)_1+\left(\sum_j u_j^0\frac{\partial u^0}{\partial x_j}+f,\ v\right)=0,\quad\forall v\in S_h,\end{cases}\tag{3}$$

where S_h is finite element subspace with piecewise linear polynomial.

Let $\bar{u}^0$ be the solution of the following Poisson problem

$$\begin{cases}\bar{u}^0\in V,\\ (\bar{u}^0,v)_1+\left(\sum_j u_j^0\frac{\partial u^0}{\partial x_j}+f,\ v\right)=0,\quad\forall v\in V\end{cases}\tag{4}$$

and $u^1=\bar{u}^0+\varphi^0$ with $\varphi^0\in S_h$ and

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$$(\varphi^0, v)_1+\Big(\sum_j \frac{\partial u^0}{\partial x_j}(\varphi_j^0+\bar{u}_j^0-u_j^0)+\sum_j u_j^0\frac{\partial}{\partial x_j}(\varphi^0+\bar{u}^0-u^0), v\Big)=0, \quad \forall v\in S_h. \tag{5}$$

Problem (5) has a unique solution which will be proved below.

Let

$$u^{n+1}=\bar{u}^n+\varphi^n, \quad \bar{u}^n\in V \text{ such that} \tag{6}$$

$$(\bar{u}^n, v)_1+\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}+f, v\Big)=0, \quad \forall v\in V \tag{7}$$

and $\varphi^n\in S_h$ such that

$$(\varphi^n, v)_1+\Big(\sum_j \frac{\partial u^0}{\partial x_j}(\varphi_j^n+\bar{u}_j^n-u_j^n)+\sum_j u_j^0\frac{\partial}{\partial x_j}(\varphi^n+\bar{u}^n-u^n), v\Big)=0, \quad \forall v\in S_h. \tag{8}$$

Let us define the operator K, $w=Kg$ by

$$\begin{cases} w\in V, \\ (w, v)_1=(g, v), \quad \forall v\in V, g\in(L^2(\Omega))^2. \end{cases}$$

It is easy to see that K: $(L^2(\Omega))^2\to(H^2(\Omega)\cap H_0^1(\Omega))^2$ and

$$K: [(H_0^1(\Omega))^2]'=V'\to V.$$

Set

$$L\varphi=\sum_j \frac{\partial u^0}{\partial x_j}\varphi_j+\sum_j u_j^0\frac{\partial\varphi}{\partial x_j}.$$

So

$$L: (H^1(\Omega))^2\to(L^2(\Omega))^2$$

is linear continuous operator, we will prove in Lemma 3 that L: $(L^2(\Omega))^2\to[H_0^1(\Omega))^2]'=V'$ is linear continuous operator, i.e.,

$$\|L\varphi\|_{V'}=\sup_{v\in V}\frac{|\langle L\varphi, v\rangle|}{\|v\|_1}\leqslant C\|\varphi\|. \tag{9}$$

Problem (8) can be rewritten in operator form

$$\varphi^n+pKL\varphi^n+pKL(\bar{u}^n-u^n)=0, \tag{10}$$

where p is an orthogonal projection onto subspace S_h with the scalar product $(\ ,\ )_1$. Problem (2) can be rewritten in operator form

$$w+K\tilde{L}w=0, \tag{11}$$

where

$$\tilde{L}w+\sum_j\Big(\frac{\partial u}{\partial x_j}w_j+u_j\frac{\partial w}{\partial x_j}\Big). \tag{12}$$

As u is an isolated solution, $I+K\tilde{L}$ has bounded inverse $(I+K\tilde{L})^{-1}$ in $(H^1(\Omega))^2$. Since $u^0\to u$[2], we can prove that $(I+KL)$ has a bounded inverse operator $(I+KL)^{-1}$ and the norm $\|(I+KL)^{-1}\|$ is uniformly bounded. This conclusion will be proved in Lemma 2. By using the operator K and the projection operator p, the problem (1)′, (3), (4), (7) can be rewritten in operator form

$$u+K\Big(\sum_j u_j\frac{\partial u}{\partial x_j}+f\Big)=0, \tag{13}$$

$$u^0+pK\Big(\sum_j u_j^0\frac{\partial u^0}{\partial x_j}+f\Big)=0, \tag{14}$$

$$\bar{u}^0+K\Big(\sum_j u_j^0\frac{\partial u^0}{\partial x_j}+f\Big)=0, \tag{15}$$