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A CLASS OF NONLINEAR METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS*

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Consider the initial value problem

$$\frac{dy}{dx}=f(x, y), \quad y(x^{(0)})=y^{(0)}, \tag{1}$$

where $x\in R$, $y, f\in R^n$. If the solution of the differential equation is approximated by polynomials, then general linear methods, such as linear multistep methods and Runge–Kutta methods, can be constructed. When one approximates the solution by rational fractions, there are some nonlinear methods[1,2].

In this paper, we propose a new class of nonlinear methods. Set

$$y_0=x, \quad f_0(y_0, y_1, \cdots, y_n)\equiv 1, \tag{2}$$

$$Y=(y_0, y_1, \cdots, y_n)^T, \quad F=(f_0, f_1, \cdots, f_n)^T.$$

Then (1) is converted to the following initial value problem

$$\frac{dY}{dx}=F(Y), \quad Y(x^{(0)})=Y^{(0)}. \tag{3}$$

Obviously, the solution $Y(x)$ is a curve in R^{n+1}. By means of Frenet frame and the normal representation of curves, we construct a class of one-step multistage nonlinear methods. According to the absolute stability, a stepsize criterion is obtained. It shows that the stepsize should be in inverse proportion to the curvature of the solution curve. It reflects the geometric nature of the solution curve computed. The stepsize criterion applies to nonstiff problems, especially to stiff problems. Numerical experiments for a stiff problem in reaction dynamics have demonstrated the efficiency of this class of nonlinear methods.

§ 1. Normal Representation of Curves

As in differential geometry, the normal representation of curves usually does not exceed the third order[3]. To obtain fourth order nonlinear methods, one must expand the curves further. Let the curve Y be parametrized by the arc length s: $Y(x)=Y(x(s))$, which will be denoted as $Y(s)$ too. Let the arc start from the point where the Frenet frame is established. Then, in the neighborhood of the starting point of the arc, we have

$$Y(s)=Y(0)+s\dot{Y}(0)+\frac{s^2}{2}\ddot{Y}(0)+\frac{s^3}{6}\dddot{Y}(0)+\frac{s^4}{24}\ddddot{Y}(0)+O(s^5). \tag{4}$$

By the Frenet equation

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$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix}=\begin{bmatrix}0 & \varkappa & 0 & 0\\ -\varkappa & 0 & \tau & 0\\ 0 & -\tau & 0 & \sigma\\ 0 & 0 & -\sigma & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix},$$

the unit tangent vector

$$e_1=\dot{Y}=\frac{dY}{dx}\,\frac{dx}{ds}=\frac{1}{l}F,$$

where

$$l=\Big\{\sum_{j=0}^{n}f_j^2\Big\}^{1/2}; \tag{5}$$

the unit principal normal vector satisfies

$$\varkappa e_2=\dot{e}_1=\frac{1}{l^2}\Big\{U-\frac{q}{l^2}F\Big\},$$

where

$$u_i=\sum_{j=0}^{n}\frac{\partial f_i}{\partial y_j}f_j,\quad U=(u_0,u_1,\cdots,u_n)^T,$$

$$p=\Big\{\sum_{j=0}^{n}u_j^2\Big\}^{1/2},\quad q=\sum_{i=0}^{n}f_iu_i,$$

$$\varkappa=\sqrt{l^2p^2-q^2}/l^3. \tag{6}$$

The expresions for unit vectors e_3 and e_4 are not listed here as τ and σ do not appear in the ensuing computations. Thus,

$$-\varkappa e_1+\tau e_3=\dot{e}_2=-\dot{\varkappa}/\varkappa\; e_2+1/\varkappa\;\dddot{Y}$$

and

$$-\tau e_2+\sigma e_4=\dot{e}_3=\frac{1}{\varkappa\tau}\{\ddddot{Y}+3\varkappa\dot{\varkappa}e_1+(\varkappa^3-\ddot{\varkappa})e_2-(2\dot{\varkappa}\tau+\varkappa\dot{\tau})e_3\}.$$

Substituting the above expressions into (4) we obtain the following result.

Theorem 1. *Let $Y(s)$ be a curve in $R^{n+1}(n\geq 3)$, parametrized by arc length s. Then, in the neighborhood of $s=0$, the curve has the following normal representation*

$$Y(s)=Y(0)+\Big[s-\frac{s^3}{6}\varkappa^2-\frac{s^4}{8}\varkappa\dot{\varkappa}\Big]e_1+\Big[\frac{s^2}{2}\varkappa+\frac{s^3}{6}\dot{\varkappa}+\frac{s^4}{24}(\ddot{\varkappa}-\varkappa^3-\varkappa\tau^2)\Big]e_2$$
$$+\Big[\frac{s^3}{6}\varkappa\tau+\frac{s^4}{24}(2\dot{\varkappa}\tau+\varkappa\dot{\tau})\Big]e_3+\frac{s^4}{24}\varkappa\tau\sigma e_4+O(s^5). \tag{7}$$

§ 2. First to Third Order Schemes

1) Omitting s^2 and higher order terms in (7) and substituting h for s, one gets the first order scheme

$$\begin{bmatrix}y_0+\Delta y_0\\ y_1+\Delta y_1\\ \vdots\\ y_n+\Delta y_n\end{bmatrix}=\begin{bmatrix}y_0\\ y_1\\ \vdots\\ y_n\end{bmatrix}+\frac{h}{\sqrt{f_0^2+f_1^2+\cdots+f_n^2}}\begin{bmatrix}f_0\\ f_1\\ \vdots\\ f_n\end{bmatrix}. \tag{8}$$

This is an analogue of the Euler polygon method, but here h, instead of being an increment in x as usual, is the stepsize of movement along the tangent to the