A-STABLE AND L-STABLE BLOCK IMPLICIT ONE-STEP METHODS*

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Abstract

A class of methods for solving the initial problem for ordinary differential equations are studied. We develop k-block implicit one-step methods whose nodes in a block are nonequidistant. When the components of the node vector are related to the zeros of Jacobi's orthogonal polynomials $P_{k-1}^{(1,1)}(u)$ or $P_{k-1}^{(1,0)}(u)$, we can derive a subclass of formulas which are A- or L-stable. The order can be arbitrarily high with A- or L-stability. We suggest a modified algorithm which avoids the inversion of a $km\times km$ matrix during Newton-Raphson iterations, where m is the number of differential equations. When $k=4$, for example, only a couple of $m\times m$ matrices have to be inversed, but four values can be obtained at one time.

§ 1. Introduction

We shall study a class of methods for solving numerically the initial value problem for ordinary differential equations. These procedures, termed k-block implicit one-step methods, advance the numerical solution by a block of k new solution values at one time. The nodes of a block can be nonequidistant.

Because implicit one-step methods have many merits, such as self-starting, easy change of steplength, high accuracy and good stability, they have attracted much attention from a number of authors, e.g., Butcher[3,4], Shampine and Watts[10,11], Williams and Frand de Hoog[12] and Bichart and Picel[1]. However, the block methods with nonequidistant nodes have not received as much attention. Shampine and Watts[10,11] presented a different approach based on interpolatory formulas of Newton-Cotes type, whose block methods for sizes $k=1, 2, \cdots, 8$ are A-stable, but for $k=9$, 10 are not. Bichart and Picel[1] also had a detailed study of block implicit methods which are stiffly stable at least through order 25.

In this paper, we continue the study of general k-block implicit methods with nonequidistant nodes. The formulas developed by Shampine and Watts[11] are involved. If the components of a node vector are related to the zeros of Jacobi's orthogonal polynomials $P_{k-1}^{(1,1)}(u)$ or $P_{k-1}^{(1,0)}(u)$, we can derive a subclass of formulas which are A- or L-stable for arbitrary sizes k. The A-stable formulas are of order $k+2$ and the L-stable formulas are of order $k+1$ for $k\geqslant 2$.

The fatal defect of the implicit one-step block methods is inversion of large matrices during Newton-Raphson iterations. In this paper, we present a modified algorithm, which comes from a 4-block implicit method, and only two ordinary matrices need to be inversed for four new values.

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Some comparative numerical results are presented to show the efficiency of the modified algorithm.

§ 2. Block Implicit Methods with Nonequidistant Nodes

We shall be interested in obtaining a numerical solution of

$$y'(x)=f(x, y), \quad y(\alpha)=\eta, \quad \alpha\leqslant x\leqslant\beta, \tag{2.1}$$

where we make the usual assumptions that f is continuous and satisfies

$$|f(x, y)-f(x, z)|\leqslant L|y-z| \tag{2.2}$$

and on $[\alpha, \beta]\times(-\infty, \infty)$ the existence of a unique solution $y(x)\in C'[\alpha, \beta]$ is guaranteed. We shall assume that y has continuous derivatives on $[\alpha, \beta]$ of any order needed rather than make specific differentiability assumptions.

Now, let $x_{n+i}=x_n+\alpha_i h$, where $n=0, k, 2k, \cdots, 0<\alpha_i<k, i=1, \cdots, k-1, \alpha_k=k$ and $\alpha_i\neq\alpha_j$ when $i\neq j$. Define $a=(\alpha_1, \alpha_2, \cdots, \alpha_k)^T$ as a node vector. Let y_j denote the approximation of $y(x_j)$. The formulas we shall study may be put in the form

$$Y_{n,a}=y_n a^0+hBF(Y_{n,a})+hf_n b, \quad n=0, k, 2k, \cdots, \tag{2.3}$$

where $f_j=f(x_j, y_j)$, $a^0=(1, 1, \cdots, 1)^T$, $B=(b_{ij})_{k\times k}$, $b=(b_{10}, \cdots, b_{k0})^T$, $Y_{n,a}=(y_{n+1}, \cdots, y_{n+k})^T$, $F(Y_{n,a})=(f_{n+1}, \cdots, f_{n+k})^T$ and the initial value $y_0=\eta$. Equation (2.3) represents a system of non-linear equations for the new values which can be shown to have a unique solution if h is suitably small. In practice we may have to presume the existence of a solution.

With the block implict method (2.3) we associate a linear difference operator vector $\mathscr{L}$ defined by

$$\mathscr{L}[Y(x; a); h]=Y(x; a)-y_n a^0-hBY'(x; a)-hy'(x)b, \tag{2.4}$$

where $Y(x, a)=(y(x+\alpha_1 h), \cdots, y(x+\alpha_k h))^T$. Expanding the function $y(x+\alpha_j h)$ and its derivative $y'(x+\alpha_j h)$ as Taylor series about x and collecting terms in (2.4) give

$$\mathscr{L}[Y(x; a); h]=y(x)c_0+hy'(x)c_1+\cdots+h^q y^{(q)}(x)c_q+\cdots, \tag{2.5}$$

where c_q are constant vectors. A simple calculation yields the following formulas for the constant vectors c_q in terms of the coefficients a, B and b

$$\begin{cases} c_0=0, \\ c_1=a^1-Ba^0-b, \\ c_q=\dfrac{1}{q!}a^q-\dfrac{1}{(q-1)!}Ba^{q-1}, \quad q=2, 3, \cdots, n, \end{cases} \tag{2.6}$$

where $a^s=(\alpha_1^s, \cdots, \alpha_k^s)^T$.

For formula (2.3), we can state a convergence theorem.

Theorem 1. *Suppose we have a k-block implicit one-step method defined by (2.3), and let us assume the existence of v and $0<q\leqslant v$ such that the linear difference operator vector $\mathscr{L}$ satisfies $\|\mathscr{L}\|=O(h^{q+1})$ and $|(\mathscr{L})_k|=O(h^{v+1})$, where $(\mathscr{L})_k$ is the k-th component of $\mathscr{L}$. Then the method is convergent with global error of order h^p where $p=\min(v, q+1)$, that is $\|Y_{n,a}-Y(x; a)\|=O(h^p)$ for each $n=0, k, 2k, \cdots$, such that $x_{n+k}\leqslant\beta$, and the method is said to be of order p.*