EIGENVALUES AND EIGENVECTORS OF A MATRIX DEPENDENT ON SEVERAL PARAMETERS*

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Abstract

This paper describes a method for investigating the analyticity and for obtaining perturbation expansions of eigenvalues and eigenvectors of a matrix dependent on several parameters. Some of results of this paper provide justification of the applications of the Newton method for inverse matrix eigenvalue problems.

§ 1. Introduction

Although investigation for the analyticity of eigenvalues and eigenvectors has a long history[4,6,8,10], relatively little attention has been paid to the analyticity and perturbation expansion of eigenvalues and eigenvectors when the matrix depends analytically on several parameters, and we feel that this problem should be discussed whenever one is trying to treat inverse matrix eigenvalue problems (ref. [2, 5, 9]). The object of this paper is to describe a method for investigating the analyticity and for obtaining perturbation expansions of eigenvalues and eigenvectors of a matrix dependent on several complex or real parameters. Our approach is on the basis of the theory of implicit functions and matrix operations. Some of our results provide justification of the applications of the Newton method for inverse matrix eigenvalue problems.

Notation. The symbol $\mathbb{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices and $\mathbb{R}^{m\times n}$ the set of real $m\times n$ matrices, $\mathbb{C}^n=\mathbb{C}^{n\times 1}$, $\mathbb{C}=\mathbb{C}^1$, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. $\lambda(A)$ stand for the set of eigenvalues of a metrix A. $I^{(n)}$ is the $n\times n$ identity matrix, and O is the null matrix. The superscript H is for conjugate transpose, and T for transpose. $\|x\|$ denotes the usual Euclidean vector norm of x and $\|A\|$ denotes the spectral norm of A.

Before all we cite the following implicit function theorems.

Theorem 1.1[1, p. 89]. *If the complex-value functions*

$$f_i(\xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_t), \quad i=1, \cdots, s$$

are analytic functions of $s+t$ complex variables in some neighbourhood of the origin of $\mathbb{C}^{s+t}$, if $f_i(0, 0)=0$, $i=1, \cdots, s$, and if

$$\det \frac{\partial(f_1, \cdots, f_s)}{\partial(\xi_1, \cdots, \xi_s)}\neq 0 \quad \textit{for } \xi_1=\cdots=\xi_s=\eta_1=\cdots=\eta_t=0,$$

then the equations

$$f_i(\xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_t)=0, \quad i=1, \cdots, s$$

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have a unique solution

$$\xi_i = g_i(\eta_1, \cdots, \eta_t), \quad i=1, \cdots, s$$

vanishing for $\eta_1 = \cdots = \eta_t = 0$ *and analytic in some neighbourhood of the origin of* $\mathbb{C}^t$.

Theorem 1.2[3, p. 277]. *If the real-value functions*

$$f_i(\xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_t), \quad i=1, \cdots, s$$

are real analytic functions of $s+t$ *real variables in some neighbourhood of the origin of* $\mathbb{R}^{s+t}$, *if* $f_i(0, 0)=0$, $i=1, \cdots, s$, *and if*

$$\det \frac{\partial(f_1, \cdots, f_s)}{\partial(\xi_1, \cdots, \xi_s)} \neq 0 \quad \text{for } \xi_1 = \cdots = \xi_s = \eta_1 = \cdots = \eta_t = 0,$$

then the equations

$$f_i(\xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_t) = 0, \quad i=1, \cdots, s$$

have a unique solution

$$\xi_i = g_i(\eta_1, \cdots, \eta_t), \quad i=1, \cdots, s$$

vanishing for $\eta_1 = \cdots = \eta_t = 0$ *and real analytic in some neighbourhood of the origin of* $\mathbb{R}^t$.

§ 2. Simple Eigenvalues and Associated Eigenvectors

Let $p = (p_1, \cdots, p_N)^T \in \mathbb{C}^N$ (or $\mathbb{R}^N$), and $A(p) = (a_{ij}(p)) \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$) be an analytic (or real analytic) function in some neighbourhood $B(0)$ of the origin. i.e.,

$$A(p) = A(0) + E(p), \quad E(p) = (\varepsilon_{ij}(p)),$$

where

$$\varepsilon_{ij}(p) = \sum_{r=1}^{\infty} \sum_{\Sigma t=r} \alpha_{t_1, \cdots, t_N}^{(i,j)} p_1^{t_1} \cdots p_N^{t_N}, \quad 1 \leqslant i, j \leqslant N, \ p \in B(0)$$

and $\sum t = t_1 + \cdots + t_N$.

Suppose that λ is an eigenvalue of $A(0)$, then there exist vectors x, $y \in \mathbb{C}^n$ (or $\mathbb{R}^n$) such that

$$A(0)x = \lambda x, \quad y^T A(0) = \lambda y^T. \tag{2.1}$$

Such vectors, x, y will be called right and left eigenvectors of $A(0)$ corresponding to the eigenvalue λ respectively.

First applying Theorem 1.1 we prove the following theorem.

Theorem 2.1. *Let* $p \in \mathbb{C}^N$, *and* $A(p) \in \mathbb{C}^{n\times n}$ *be an analytic function of* p *in some neighbourhood* $B(0)$ *of the origin. Suppose that* λ_1 *is a simple eigenvalue of* $A(0)$, *and* x_1, y_1 *are associated eigenvectors satisfying the relations* (2.1) *and* $\|x_1\| = 1$, $y_1^T x_1 = 1$. *Then*

1) *there exists a simple eigenvalue* $\lambda_1(p)$ *of* $A(p)$ *which is an analytic function of* p *in some neighbourhood* B_0 *of the origin, and* $\lambda_1(0) = \lambda_1$;

2) *the right eigenvector* $x_1(p)$ *and left eigenvector* $y_1(p)$ *of* $A(p)$ *corresponding to* $\lambda_1(p)$ *may be defined to be analytic functions of* p *in* B_0, *and* $x_1(0) = x_1$, $y_1(0) = y_1$.

Proof. By the hypotheses there exist X_2, $Y_2 \in \mathbb{C}^{n\times(n-1)}$ such that