MULTIGRID AND MGR[ν] METHODS FOR DIFFUSION EQUATIONS*[1)]

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Abstract

The MGR[ν] algorithm of Ries, Trottenberg and Winter with $\nu=0$ and the Algorithm 2.1 of Braess are essentially the same multigrid algorithm for the discrete Poisson equation: $-\Delta_h U=f$. In this report we consider the extension to the general diffusion equation $-\nabla\cdot p\nabla u=f$, $p=p(x,y)\geqslant p_0>0$. In particular, we indicate the proof of the basic result $\rho\leqslant\frac{1}{2}(1+Kh)$, thus extending the results of Braess and Ries, Trottenberg and Winter. In addition to this theoretical result we present computational results which indicate that other constant coefficient estimates carry over to this case.

§ 1. Introduction

Multigrid methods are proving themselves to be successful tools for the solution of the algebraic equations associated with the discretization of elliptic boundary-value problems. Nevertheless, it seems we are just beginning to understand this powerful idea. Hence there is a need for continued probing, experimentation and new proofs——less for the sake of proof and more for the sake of insight.

Let X_n be a finite dimensional vector space of dimension n. Let A_n be a non-singular linear operator mapping X_n onto X_n. We are concerned with the problem

$$A_n U=f. \tag{1.1}$$

Multigrid methods for the solution of (1.1) are based on the following set of ideas. Suppose that (1.1) arises from the discretization of an elliptic boundary value problem. Then U is an approximation to a "smooth function" $U(x, y)$. Moreover $U(x, y)$ can also be approximated by other approximants $\{U_m\}\in\{X_m\}$——with X_m a finite dimensional vector space of dimension m. Thus U can be approximated by such a U_m with $m<n$. At the same time, most of the classical iterative methods for the solution of (1.1) converge very slowly. For these methods the spectral radius of the iteration matrix is of the form

$$\rho\sim 1-c/n. \tag{1.2}$$

Indeed, *ADI* and *SOR* methods are considered exceptionally good because

$$\rho\sim 1-c/\sqrt{n}. \tag{1.3}$$

The same analysis which yields (1.2) also shows that the eigenvectors associated

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with this slow rate of convergence are converging (as $n\to\infty$) to a very smooth function. That is, these (not *all*) classical iterative schemes have the effect of "smoothing" the error.

A multigrid method for the solution of (1.1) is based on the following entities:

(a) *A smoothing operator* $S: X_n\to X_n$

S is an affine operator of the form

$$Sv=Gv+Kf, \tag{1.4}$$

where G and K are linear operators. And, if u is the unique solution of (1) then u is a fixed point of S, i.e.

$$Su=Gu+Kf=u. \tag{1.5}$$

(b) *A subspace* X_m with

$$\dim X_m=m\ll\dim X_n=n. \tag{1.6}$$

(c) *Two linear "communication" operators:*

$$I_n^m: X_n\to X_m, \tag{1.7}$$

$$I_m^n: X_m\to X_n. \tag{1.8}$$

(d) *A course grid operator*: a nonsingular operator A_m,

$$A_m: X_m\to X_m. \tag{1.9}$$

Having listed these ingredients let us describe the multigrid iterative scheme for the solution of (1.1).

Step 1. Let u^0 be a first guess.

Step 2. $Su^0=\tilde{u}$, $r=f-A\tilde{u}$.

Step 3. $r_m=I_n^m r$.

Step 4. Solve

$$A_m\hat{u}=r_m.$$

Step 5. $u^1=\tilde{u}+I_m^n\hat{u}$.

Remark. It might appear that we have (merely) described a "two grid" iterative method. However, true "multigrid" iterative schemes are described by this outline. The operator A_m may require the use of other spaces $X_{m'}$.

In our discussion of these methods we follow a basic observation of S. McCormick and J. Ruge[2]; *we should focus our attention on the two basic spaces*

$$R:=\text{Range } I_m^n, \tag{1.10}$$

$$N:=\text{Nullspace } I_n^m A_n. \tag{1.11}$$

A basic result is

Theorem 1. *Suppose* $X_n=R\oplus N$ *and*

$$A_m=\hat{A}_m:=I_n^m A_n I_m^n. \tag{1.12}$$

Suppose $\hat{A}_m$ *is nonsingular, and*

$$\tilde{\varepsilon}:=U-\tilde{u}=\eta+I_m^n w, \tag{1.13}$$

where

$$\eta\in N,\quad w\in X_m. \tag{1.14}$$

Then

$$\varepsilon^1:=U-u^1=\eta. \tag{1.15}$$