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BOUNDARY INTEGRAL EQUATIONS FOR ISOTROPIC LINEAR ELASTICITY*

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Abstract

This articles first investigates boundary integral operators for the three-dimensional isotropic linear elasticity of a biphasic model with piecewise constant Lamé coefficients in the form of a bounded domain of arbitrary shape surrounded by a background material. In the simple case of a spherical inclusion, the vector spherical harmonics consist of eigenfunctions of the single and double layer boundary operators and we provide their spectra. Further, in the case of many spherical inclusions with isotropic materials, each with its own set of Lamé parameters, we propose an integral equation and a subsequent Galerkin discretization using the vector spherical harmonics and apply the discretization to several numerical test cases.

Mathematics subject classification: 65R20, 65N38, 74B05.
Key words: Isotropic elasticity, Boundary integral equation, Spherical inclusions, Vector spherical harmonics, Layer potentials.

1. Introduction

We consider three-dimensional boundary value or interface problems of the isotropic elasticity equation related to the following operator:

$$\mathbf{L}u := -\operatorname{div}\Big(2\mu e(u) + \lambda \operatorname{Tr}\big(e(u)\big)\operatorname{Id}\Big), \tag{1.1}$$

where the strain tensor reads $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$. It is obvious to see that the operator $\mathbf{L}$ is self-adjoint on $L^2(\mathbb{R}^3)^3$.

In the definition of the operator (1.1), $\mu, \lambda \in \mathbb{R}, \mu > 0, 2\mu + 3\lambda > 0$ are the so-called (constant) Lamé parameters. The parameter μ denotes the shear modulus which describes the tendency of the object to deform at a constant volume when being imposed with opposing forces. The other Lamé parameter λ has no physical meanings but is introduced to simplify the definition of the operator (1.1). Indeed, it is related to the bulk modulus K through the relation

$$\lambda = K - \frac{2}{3}\mu,$$

where the bulk modulus K represents the object's tendency to deform in all directions when acted on by opposing force from all directions. We refer to [12] for more detailed descriptions

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of the Lamé parameters. It is sometimes useful to introduce Poisson's ratio ν which is defined by

$$\nu = \frac{\lambda}{2(\mu + \lambda)}, \tag{1.2}$$

and whose admissible range is $(-1, 1/2)$. The material is extremely compressible in the limit $\nu \to -1$ while extremely incompressible in the other limit $\nu \to 1/2$ [18].

A model of linear elasticity with appropriate boundary conditions can be approximated by the classic finite element method, see for example [15,19] just to name a few contributions from an abundant body of literature, for the general case with non-homogeneous source term. On the other hand, displacement fields u being homogeneous solutions, i.e., $\mathbf{L}u = 0$ within a given domain, can also be represented by isotropic elastic potentials [3,13] and elasticity in piecewise constant isotropic media can then be treated as integral equations for specified interface conditions. At the origin of the integral formulation lies the definitions of layer potentials and their corresponding integral operators [3,20] based on the Green's function [1] in the context of the isotropic linear elasticity.

In particular, on a unit sphere, one can introduce the vector spherical harmonics forming an orthonormal basis of $[L^2(\mathbb{S}^2)]^3$ and which are eigenfunctions of the corresponding double and single layer boundary operators based on the Green's function [1] of isotropic linear elasticity. The vector spherical harmonics were introduced in [9,10] as an extension of the scalar spherical harmonics [16,23] to the vectorial case. They were further used in the discretization of different physical models such as the Navier-Stokes equations [8] or Maxwell's equations [2,6]. However, they are not widely used and only sparely reported in literature, in particular in the context of isotropic elasticity. We demonstrate in this article that the corresponding integral operators have interesting spectral properties which can be made explicit by employing the vector spherical harmonics.

Our main motivation for this work is the derivation of an integral equation to model elastic materials represented by piecewise constant Lamé constants with spherical inclusions following similar principles that were presented in [4,5,14] in the case of scalar diffusion. The particular choice of the vector spherical harmonics as basis functions for a Galerkin discretization thereof leads then to an efficient and stable numerical scheme by exploiting the spectral properties of the involved integral operators. A similar physical model was introduced in [22] with an algebraic formula of the approximate solution. However, with the spectral properties of the layer potentials and integral operators at hand, our approach first introduces an integral formulation for the exact solution and thus a rigorous mathematical framework. In a second step, we then propose the Galerkin discretization. The mathematical framework lays out the basis to derive a rigorous error analysis which we plan in the future.

We summarize the main contributions and organization of this work as follows:

- In Sections 2 and 3, we give an introduction and overview of the layer potentials and corresponding boundary integral operators of the isotropic linear elasticity operator (1.1) on an arbitrary bounded domain with Lipschitz boundary which are sparely reported in the literature.

- Analytical properties of layer potentials and boundary integral operators are presented and proven in Section 3.4.

- On the unit sphere, we introduce the vector spherical harmonics in Section 4 and prove spectral properties of the boundary operators and layer potentials of this particular basis.

- As an application, we consider a piecewise constant elastic model with spherical inclusions and derive a integral equation in Section 6 that is then discretized by means of the vector spherical harmonics and tested numerically in Section 7.

2. Preliminaries

Denote $\mathbb{S}^2$ the unit sphere and B the unit ball in $\mathbb{R}^3$. Let throughout this paper $\Omega^- \subset \mathbb{R}^3$ denote a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega^-$ and outward pointing normal vector field $\mathbf{n} : \Gamma \to \mathbb{S}^2$. Further, we denote by Ω^+ the unbounded set $\mathbb{R}^3 \backslash \overline{\Omega^-}$.

2.1. Notations

We will first introduce some standard notions in the context of integral equations which can be found in standard textbooks (see, e.g., [17, 20, 21]).

Let Ω be a domain with Lipschitz boundary, e.g., $\Omega = \Omega^-$ or $\Omega = \Omega^+$ (unbounded). Following the conventions and notation of [20], we define for $s \in \mathbb{R}$

$$H^s_{\mathrm{loc}}(\Omega) = \left\{ u \in \left(C^\infty_{\mathrm{comp}}(\Omega)\right)^* \,\middle|\, \forall \chi \in C^\infty_{\mathrm{comp}}(\Omega) : \chi u \in H^\ell(\Omega) \right\}, \tag{2.1}$$

see Definition 2.6.1 in [20], and note that this consist of a slightly unconventional definition of $H^\ell_{\mathrm{loc}}(\Omega)$, see also Remark 2.6.2. We further define, see Definition 2.6.5 in [20], for $s \in \mathbb{R}$

$$H^s_{\mathrm{comp}}(\Omega) = \bigcup_K \left\{ u \in H^s_{\mathrm{loc}}(\Omega) \mid \mathrm{supp}(u) \subset K \right\}, \tag{2.2}$$

where the union is taken over all relatively compact subsets $K \subset \Omega$, and introduce

$$V_0(\Omega^-) = \left\{ v \in H^1(\Omega^-) \,\middle|\, \int_{\Omega^-} v = 0 \right\}. \tag{2.3}$$

Next, we denote by $H^{\frac{1}{2}}(\Gamma)^3$ the Sobolev space of order $\frac{1}{2}$ with the usual Sobolev-Slobodeckij norm $\|\lambda\|^2_{H^{\frac{1}{2}}(\Gamma)} := \sum_{k=1}^3 \|\lambda_k\|^2_{H^{\frac{1}{2}}(\Gamma)}$ for $\lambda = (\lambda_1, \lambda_2, \lambda_3)^\top$ and with

$$\|\lambda_k\|^2_{H^{\frac{1}{2}}(\Gamma)} := \|\lambda_k\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|\lambda_k(x) - \lambda_k(y)|^2}{|x-y|^3}\, dxdy.$$

Moreover, we define $H^{-\frac{1}{2}}(\Gamma)^3 := \left(H^{\frac{1}{2}}(\Gamma)^3\right)^*$ and we equip this Sobolev space with the canonical dual norm $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma)}$. We introduce

$$\gamma^\mp : H^1_{\mathrm{loc}}(\Omega^\mp)^3 \to H^{\frac{1}{2}}(\Gamma)^3 \tag{2.4}$$

as the continuous, linear and surjective interior and exterior Dirichlet trace operators respectively, see Theorem 2.6.8 [20], and define the jump operator by

$$[\![\varphi]\!] = \gamma^-\varphi - \gamma^+\varphi. \tag{2.5}$$

Further, let $\gamma : H^1_{\mathrm{loc}}(\mathbb{R}^3)^3 \to H^{\frac{1}{2}}(\Gamma)^3$ be given by $\gamma\varphi = \gamma^-\varphi = \gamma^+\varphi$ almost everywhere.

Consider now the stress tensor $\mathcal{T}$ associate with $\mathbf{L}$, as is defined by (1.1), reading

$$\mathcal{T}\varphi := 2\mu e(\varphi) + \lambda \operatorname{Tr} e(\varphi)\, \mathrm{Id}, \quad \varphi \in H^1_{\mathrm{loc}}(\Omega^\mp)^3. \tag{2.6}$$

For the domains $\Omega^{\mp}$, the classical normal derivative operator, satisfying

$$\mathcal{T}_{\mathbf{n}}^{\mp}\varphi := \gamma^{\mp}(\mathcal{T}\varphi\mathbf{n}), \tag{2.7}$$

for regular φ, can be extended to an operator $\mathcal{T}_{\mathbf{n}}^{\mp} : H^1_{\mathbf{L}}(\Omega^{\mp})^3 \to H^{-\frac{1}{2}}(\Gamma)^3$, with $H^1_{\mathbf{L}}(\Omega)^3 = \{u \in H^1(\Omega)^3 \mid \mathbf{L}u \in L^2_{\mathrm{loc}}(\Omega)^3\}$, based on Green's first identity. We then define the corresponding jump operator by

$$[\![\mathcal{T}\varphi]\!] = \mathcal{T}_{\mathbf{n}}^{-}\varphi - \mathcal{T}_{\mathbf{n}}^{+}\varphi. \tag{2.8}$$

Further, define $\mathcal{T}_{\mathbf{n}} : H^1_{\mathbf{L}}(\mathbb{R}^3)^3 \to H^{-\frac{1}{2}}(\Gamma)^3$ the global normal derivative operator given by $\mathcal{T}_{\mathbf{n}}\varphi = \mathcal{T}_{\mathbf{n}}^{-}\varphi = \mathcal{T}_{\mathbf{n}}^{+}\varphi$.

2.2. Fundamental solutions

Consider the matrix-valued fundamental solution $G = (G_{ij})_{ij}$ to the linear isotropic elasticity equation such that G_i, the i-th column of the matrix G satisfies the following identity:

$$\mathbf{L}G_i(x) = \delta(x)\,\mathbf{e}_i, \tag{2.9}$$

with $\mathbf{L}$ defined by (1.1), δ being the Dirac distribution at the origin and $\mathbf{e}_i$ the canonical basis in $\mathbb{R}^3$. The Green's function G is given by [17, 21]:

$$G_{ij}(x) := \frac{1}{8\pi\mu|x|}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\delta_{ij} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{x_i x_j}{|x|^2}\right), \tag{2.10}$$

where we recall that μ, λ are the Lamé constants and δ_{ij} is the Kronecker symbol.

2.3. Rigid displacement

For a given domain $\Omega \subset \mathbb{R}^3$, we consider the following problem: find $u \in H^1_{\mathrm{loc}}(\Omega)^3$ such that

$$\mathbf{L}u = 0, \quad \text{in } H^{-1}(\Omega)^3 \tag{2.11}$$

with appropriate boundary conditions. Equation (2.11) holds obviously if $e(u) = 0$. Indeed, we call the displacement $u \in H^1_{\mathrm{loc}}(\Omega)^3$ a *rigid displacement* if $e(u) = 0$. It is well-known that the displacement u is a rigid displacement if and only if it has the form $u = Ax + b$ where $A \in \mathbb{R}^{3\times 3}$ is a constant skew matrix and $b \in \mathbb{R}^3$ a constant vector (see, e.g., [11, 17]).

3. Layer Potentials

In this section, we introduce the layer potentials and associated boundary operators which only have been sparsely reported in the literature for the operator $\mathbf{L}$. We therefore provide a complete overview.

3.1. Single layer potentials

Using the fundamental solution (2.10), we can now define the single layer potential $\mathcal{S}$: $H^{-\frac{1}{2}}(\Gamma)^3 \to H^1(\mathbb{R}^3 n\Gamma)^3$ associated to the isotropic elasticity operator $\mathbf{L}$:

$$(\mathcal{S}\phi)(x) := \int_{\Gamma} G(x-y)\,\phi(y)\,dy, \quad x \in \mathbb{R}^3 n\Gamma. \tag{3.1}$$

Further, see e.g., [3], such function $\mathcal{S}\phi$ defined on $\mathbb{R}^3 n\Gamma$ is continuous across the interface Γ, i.e. $[\![\mathcal{S}\phi]\!] = 0$ and a single layer boundary operator $\mathcal{V} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3$ can be defined by restricting the single layer potential to the boundary Γ:

$$(\mathcal{V}\phi)(x) := \int_\Gamma G(x-y)\,\phi(y)\,dy, \quad x \in \Gamma, \tag{3.2}$$

so that $\gamma\mathcal{S}\phi = \mathcal{V}\phi$. The following result is obvious:

Lemma 3.1. *For $\phi \in H^{-\frac{1}{2}}(\Gamma)^3$ and $\mathcal{S}\phi$ defined by* (3.1)*, let $\mathbf{L}$ be the isotropic elasticity operator* (1.1) *and we have*

$$\mathbf{L}\mathcal{S}\phi = 0 \quad in\ \mathbb{R}^3 n\Gamma.$$

3.2. Double layer potential

We introduce the double layer potential $\mathcal{D} : H^{\frac{1}{2}}(\Gamma)^3 \to H^1(\mathbb{R}^3\backslash\Gamma)^3$, by

$$\mathcal{D}\varphi(x) = \int_\Gamma \mathcal{T}_{\mathbf{n},y}(G)(x-y)\,\varphi(y)\,dy, \quad x \in \mathbb{R}^3\backslash\Gamma, \tag{3.3}$$

where the subscript y means that the normal derivative operator $\mathcal{T}_{\mathbf{n}}$, defined in Section 2.1, is taken with respect to the y-variable. We define the double layer boundary operator $\mathcal{K}$: $H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3$ by

$$(\mathcal{K}\varphi)(x) = \int_\Gamma \mathcal{T}_{\mathbf{n},y}(G)(x-y)\,\varphi(y)\,dy, \quad x \in \Gamma, \tag{3.4}$$

in the sense of principal value. Further, the adjoint double layer boundary operator $\mathcal{K}^*$: $H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3$ is given as

$$(\mathcal{K}^*\phi)(x) = \int_\Gamma \left(\mathcal{T}_{\mathbf{n},x}(G)\right)^\top (x-y)\,\phi(y)\,dy, \quad x \in \Gamma, \tag{3.5}$$

Similar to Lemma 3.1, the following result is obvious:

Lemma 3.2. *For $\varphi \in H^{\frac{1}{2}}(\Gamma)^3$ and $\mathcal{D}\varphi$ defined by* (3.3)*, we have*

$$\mathbf{L}\mathcal{D}\varphi = 0 \quad in\ \mathbb{R}^3\backslash\Gamma,$$

where $\mathbf{L}$ is the isotropic elasticity operator (1.1).

3.3. Newton potential

Finally, for sake of completeness, we also give the Newton potential associated to the isotropic elasticity operator (1.1). Define $\mathcal{N} : H^s_{\text{comp}}(\mathbb{R}^3)^3 \to H^{s+2}_{\text{loc}}(\mathbb{R}^3)^3$ for $s \in \mathbb{R}$:

$$\mathcal{N}\psi(x) = \int_{\mathbb{R}^3} G(x-y)\psi(y)\,dy, \quad x \in \mathbb{R}^3, \tag{3.6}$$

where G is the Green's function defined by (2.10). Following the definition for the elasticity operator $\mathbf{L}$ and all $\psi \in \mathcal{D}'(\mathbb{R}^3)^3$, we have

$$\psi = \mathbf{L}\mathcal{N}\psi = \mathcal{N}\mathbf{L}\psi, \quad \text{in } \mathcal{D}'(\mathbb{R}^3)^3. \tag{3.7}$$

Let γ^* : $H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-1}_{\text{comp}}(\mathbb{R}^3)^3$, $\mathcal{T}_{\mathbf{n}}{}^*$: $H^{\frac{1}{2}}(\Gamma)^3 \to H^{-1}_{\text{comp}}(\mathbb{R}^3\backslash\Gamma)^3$ be the adjoint of the trace operator γ and the adjoint of the normal derivative operator $\mathcal{T}_{\mathbf{n}}$ respectively, defined in Section 2.1. We then give an equivalent definition of the single and double layer potential:

$$\mathcal{S} = \mathcal{N}\gamma^*, \qquad \mathcal{D} = \mathcal{N}\mathcal{T}_{\mathbf{n}}^*. \tag{3.8}$$

3.4. Properties of layer potentials

We are now listing a selection of known results of layer potentials that will be used in the following. Let us first recall the following theorem given in [3] (see also [21, Section 6.7]):

Theorem 3.1. *Let $\phi \in H^{-\frac{1}{2}}(\Gamma)^3$ and the single layer potential $\mathcal{S}$, the adjoint double layer boundary operator $\mathcal{K}^*$ be defined by* (3.1) *and* (3.5) *respectively. Then the interior and exterior normal traces of the stress tensor satisfy*

$$\mathcal{T}_{\mathbf{n}}^{-}\mathcal{S}\phi = \frac{1}{2}\phi + \mathcal{K}^*\phi, \qquad \mathcal{T}_{\mathbf{n}}^{+}\mathcal{S}\phi = -\frac{1}{2}\phi + \mathcal{K}^*\phi, \qquad \text{on } H^{-\frac{1}{2}}(\Gamma)^3. \tag{3.9}$$

We now show several jump conditions relating to the boundary layer potentials above which can be found, for example, in [17, Theorem 6.10].

Theorem 3.2. *Let $\Omega^- \in \mathbb{R}^3$ be a bounded Lipschitz domain with boundary Γ. Consider the single and double layer potentials defined by* (3.1) *and* (3.3) *respectively. Then it holds*

$$\begin{array}{lll} [\![\mathcal{S}\phi]\!] = 0, & [\![\mathcal{D}\varphi]\!] = -\varphi, & \text{on } H^{\frac{1}{2}}(\Gamma)^3, \\ [\![\mathcal{T}\mathcal{S}\phi]\!] = \phi, & [\![\mathcal{T}\mathcal{D}\varphi]\!] = 0, & \text{on } H^{-\frac{1}{2}}(\Gamma)^3 \end{array} \tag{3.10}$$

for all $\phi \in H^{-\frac{1}{2}}(\Gamma)^3, \varphi \in H^{\frac{1}{2}}(\Gamma)^3$.

We now consider the invertibility of the single layer boundary operator (3.2) (see [17, Theorem 10.7] or [21, Theorem 6.36].

Lemma 3.3. *Let $\Omega^- \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ. If $\mu > 0$ and $\lambda \geq 0$, the single layer boundary operator $\mathcal{V}$ defined by* (3.2) *is coercive, i.e.*

$$\langle \mathcal{V}\phi, \phi \rangle_{H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)} > c\,\|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \qquad \forall \phi \in H^{-\frac{1}{2}}(\Gamma)^3.$$

Corollary 3.1 (Invertibility of the single layer boundary operator). *Let $\Omega^- \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ and $\mu > 0$ and $\lambda \geq 0$. Then, the single layer boundary operator $\mathcal{V} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3$ is invertible.*

4. Real Vector Spherical Harmonics

4.1. Surface gradient

In the following, we introduce the real vector spherical harmonics. We begin with some conventions of the gradient. On a given domain Ω, consider a scalar valued function $f : \Omega \to \mathbb{R}^3$ and a column-vector valued function $F : \Omega \to \mathbb{R}^3$, we define their gradients by

$$\nabla f(x) \in \mathbb{R}^3, \text{ with } (\nabla f)_i = \frac{\partial f}{\partial x_i}, \qquad \nabla F(x) \in \mathbb{R}^{3\times 3}, \text{ with } (\nabla F)_{ij} = \frac{\partial F_i}{\partial x_j}. \tag{4.1}$$

Note in particular that ∇f is a column-vector while ∇F are row-wise gradients for each component F_i.

Restricting the considerations to the unit ball $\Omega = B$ and it surface $\partial\Omega = \mathbb{S}^2$, we denote by

$$\nabla_{\mathrm{s}} = \hat{\theta}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \tag{4.2}$$

the surface gradient operator and $\hat{r}, \hat{\theta}, \hat{\phi}$ are radial, polar and azimuthal unit vectors which are supposed to be row vectors. Let f denote a scalar function and F a vector-valued function, i.e. $f \in \mathbb{R}$ and $F \in \mathbb{R}^3$ and with the convention of the gradient field, the surface gradient (4.2) can alternatively be written as

$$\begin{aligned} \nabla_{\mathrm{s}} f &= \nabla f - \mathbf{n}(\mathbf{n}^\top \nabla f), \\ \nabla_{\mathrm{s}} F &= \nabla F - (\nabla F \mathbf{n})\mathbf{n}^\top, \end{aligned} \tag{4.3}$$

where ∇ is the gradient in $\mathbb{R}^3$ based on the convention (4.1). It is immediate to verify that

$$\nabla_{\mathrm{s}} f^\top \mathbf{n} = 0, \quad \nabla_{\mathrm{s}} F \mathbf{n} = 0.$$

4.2. Definition of vector spherical harmonics

The construction of the vector spherical harmonics is based on the scalar real spherical harmonics defined on the unit sphere $\mathbb{S}^2$ denoted by $(Y_{\ell m})_{l\geq 0}^{|m|\leq \ell}$ which are normalized such that

$$\langle Y_{\ell m}, Y_{\ell' m'} \rangle_{\mathbb{S}^2} = \int_{\mathbb{S}^2} Y_{\ell m} Y_{\ell' m'} = \delta_{\ell\ell'} \delta_{m' m'}.$$

The vector spherical harmonics $V_{\ell m}, W_{\ell m}, X_{\ell m} : \mathbb{S}^2 \to \mathbb{R}^3$ of degree $\ell \geq 0$ and order m, $|m| \leq \ell$ are given by

$$\begin{aligned} V_{\ell m} &:= \nabla_{\mathrm{s}} Y_{\ell m}(\theta, \phi) - (\ell + 1) Y_{\ell m}(\theta, \phi)\hat{r}, \\ W_{\ell m} &:= \nabla_{\mathrm{s}} Y_{\ell m}(\theta, \phi) + \ell Y_{\ell m}(\theta, \phi)\hat{r}, \\ X_{\ell m} &:= \hat{r} \times \nabla_{\mathrm{s}} Y_{\ell m}(\theta, \phi). \end{aligned} \tag{4.4}$$

The symbol $\times$ represents the cross product in $\mathbb{R}^3$. We refer to Appendix A for some explicit expressions of the vector spherical harmonics for the first few degrees. The vector spherical harmonics satisfy the following orthogonal properties:

$$\begin{aligned} &\int_{\mathbb{S}^2} V_{\ell m} \cdot W_{\ell' m'} = 0, \ \int_{\mathbb{S}^2} W_{\ell m} \cdot X_{\ell' m'} = 0, \ \int_{\mathbb{S}^2} X_{\ell m} \cdot V_{\ell' m'} = 0, \\ &\int_{\mathbb{S}^2} V_{\ell m} \cdot V_{\ell' m'} = \delta_{\ell\ell'} \delta_{m' m'} (\ell + 1)(2\ell + 1), \ \int_{\mathbb{S}^2} W_{\ell m} \cdot W_{\ell' m'} = \delta_{\ell\ell'} \delta_{m' m'} \ell(2\ell + 1), \\ &\int_{\mathbb{S}^2} X_{\ell m} \cdot X_{\ell' m'} = \delta_{\ell\ell'} \delta_{m' m'} \ell(\ell + 1). \end{aligned} \tag{4.5}$$

The scalar spherical harmonics (and thus the vector spherical harmonics) can be extended to any sphere $\Gamma_r(x_0) = \partial B_r(x_0)$ by translation and scaling. We will introduce the following scaled scalar product on $\Gamma_r(x_0)$ given by

$$\langle u, v \rangle_{\Gamma_r(x_0)} = \frac{1}{r^2} \int_{\Gamma_r(x_0)} u(s) \cdot v(s) ds = \int_{\mathbb{S}^2} u(x_0 + rs') \cdot v(x_0 + rs') ds'. \tag{4.6}$$

In practice, the exact value of the scalar product (4.6) cannot be computed explicitly in general. With a set $\{s_t, w_t\}_{t=1}^{T_g}$ of integration points and weights on the unit sphere, the scalar product is approximated by the quadrature rule

$$\langle u, v \rangle_{\Gamma_r(x_0),t} = \sum_{t=1}^{T_g} w_t \, u(x_0 + rs_t) \cdot v(x_0 + rs_t). \tag{4.7}$$

In the numerical tests below in Section 7, we will use the Lebedev quadrature points [7], which have the property that scalar spherical harmonics up to a certain degree N_g are integrated exactly. This relationship is displayed in Table 4.1. It can be noticed that the number of points increases quadratically with N_g.

Further, the family of vector spherical harmonics gives a complete basis of $L^2(\Gamma_r(x_0))^3$ and any real function $f \in L^2(\Gamma_r(x_0))^3$ can be represented as

$$f(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [v]_{\ell m} V_{\ell m}\Big(\frac{x-x_0}{r}\Big) + [w]_{\ell m} W_{\ell m}\Big(\frac{x-x_0}{r}\Big) + [x]_{\ell m} X_{\ell m}\Big(\frac{x-x_0}{r}\Big), \tag{4.8}$$

where $[v]_{\ell m}, [w]_{\ell m}, [x]_{\ell m} \in \mathbb{R}$.

Table 4.1: Degree N_g and number of points T_g of Lebedev quadrature rules such that spherical haromonics up to degree N_g are integrated exactly with T_g points.

N_g	3	5	7	9	11	13	15	17	19	21	23	25	27	29
T_g	6	14	26	38	50	74	86	110	146	170	194	230	266	302
N_g	31	35	41	47	53	59	65	71	77	83	89	95	101	107
T_g	350	434	590	770	974	1202	1454	1730	2030	2354	2702	3074	3470	3890

4.3. Properties of the derivatives

We give some derivative properties of the surface gradient (4.3) that shall be useful in the upcoming analysis. In the following, let u be the a scalar-valued function, F, G be vector-valued functions and $\mathbf{A}$ be a matrix-valued function. We have the following product rule:

$$\begin{aligned} \nabla_{\mathrm{s}}(uF) &= F\nabla_{\mathrm{s}}u^\top + u\nabla_{\mathrm{s}}F, \\ \nabla_{\mathrm{s}}(F^\top G) &= \nabla_{\mathrm{s}}F^\top G + \nabla_{\mathrm{s}}G^\top F. \end{aligned} \tag{4.9}$$

We also have the property for the cross product:

$$\nabla_{\mathrm{s}}(F \times G) = \nabla_{\mathrm{s}}F \times G - \nabla_{\mathrm{s}}G \times F. \tag{4.10}$$

Later proof also requires the triple product

$$\big(\mathbf{A} \times F\big)^\top G = \mathbf{A}^\top\big(F \times G\big) = \big(G \times \mathbf{A}\big)^\top F. \tag{4.11}$$

Now let $h = h(r)$ be a scalar function which does not depend on the polar angles and $u = u(\theta, \phi), H = H(\theta, \phi)$ a scalar and a vector valued function respectively depending only on the polar angles. Then there holds

$$\begin{aligned} \nabla(hu) &= h_r u\hat{r} + \tfrac{1}{r}h\nabla_{\mathrm{s}}u, \\ \nabla(hH) &= h_r H\hat{r}^\top + \tfrac{1}{r}h\nabla_{\mathrm{s}}H. \end{aligned} \tag{4.12}$$

For the scalar function $h = h(r)$, denote by h_r, h_{rr} the first and second derivative. Then we have

$$\begin{aligned} \mathrm{div}(hV_{\ell m}) &= -(\ell+1)\Big(h_r + \frac{\ell+2}{r}h\Big)Y_{\ell m}, \\ \mathrm{div}(hW_{\ell m}) &= \ell\Big(h_r - \frac{\ell-1}{r}h\Big)Y_{\ell m}, \\ \mathrm{div}(hX_{\ell m}) &= 0, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\Delta(hV_{\ell m}) &= \Big(h_{rr} + \frac{2}{r}h_r - \frac{(\ell+1)(\ell+2)}{r^2}h\Big)V_{\ell m},\\ \Delta(hW_{\ell m}) &= \Big(h_{rr} + \frac{2}{r}h_r - \frac{(\ell-1)\ell}{r^2}h\Big)W_{\ell m},\\ \Delta(hX_{\ell m}) &= \Big(h_{rr} + \frac{2}{r}h_r - \frac{\ell(\ell+1)}{r^2}h\Big)X_{\ell m}.\end{aligned} \tag{4.14}$$

Eqs. (4.13) and (4.14) are given in [9]. Finally, the following identities hold, resulting directly from the definition of the vector spherical harmonics:

$$\nabla_{\rm s} Y_{\ell m} = \frac{1}{2\ell+1}(\ell V_{\ell m} + (\ell+1)W_{\ell m}), \quad Y_{\ell m}\hat{r} = \frac{1}{2\ell+1}(W_{\ell m} - V_{\ell m}). \tag{4.15}$$

5. Spectral Properties of the Layer Potentials

We will give the main results in Section 5.1, prepare some preliminary results in Section 5.2 and finally provide the proofs in Section 5.3.

5.1. Main results

Consider the single layer potential $\mathcal{S}$ and the single layer boundary operator $\mathcal{V}$ defined by (3.1) and (3.2), we have the following result.

Theorem 5.1. *Let $\underline{Y_{\ell m}}$ be the matrix such that*

$$\underline{Y_{\ell m}} = (V_{\ell m}|W_{\ell m}|X_{\ell m}) := (Y^1_{\ell m}|Y^2_{\ell m}|Y^3_{\ell m}). \tag{5.1}$$

Then we have:

1. *On the unit sphere $\mathbb{S}^2$, $\mathcal{V}\underline{Y_{\ell m}}(x) = \underline{Y_{\ell m}}A_{\mathcal{V},\ell}$, where $\mathcal{V}$ is the single layer boundary operator defined by* (3.2) *and $A_{\mathcal{V},\ell}$ is a constant matrix given by*

$$\begin{aligned} A_{\mathcal{V},\ell} &= \begin{bmatrix} \frac{(3\ell+1)\mu+\ell\lambda}{(2\ell+3)(2\ell+1)\mu(2\mu+\lambda)} & 0 & 0 \\ 0 & \frac{(3\ell+2)\mu+(\ell+1)\lambda}{(2\ell-1)(2\ell+1)\mu(2\mu+\lambda)} & 0 \\ 0 & 0 & \frac{1}{\mu(2\ell+1)} \end{bmatrix} \\ &= \mathrm{diag}(\tau^1_{\mathcal{V},\ell}, \tau^2_{\mathcal{V},\ell}, \tau^3_{\mathcal{V},\ell}). \end{aligned} \tag{5.2}$$

2. *When $|x| < 1$, we have*

$$(\mathcal{S}\underline{Y_{\ell m}})(x) = \underline{Y_{\ell m}}\left(\frac{x}{|x|}\right) A^{\rm in}_{\mathcal{S},\ell}(x),$$

where $\mathcal{S}$ is the single layer potential given by (3.1) *and the matrix $A^{\rm in}_{\mathcal{S},\ell}(x)$ has the form*

$$A^{\rm in}_{\mathcal{S},\ell}(x) = \begin{bmatrix} \frac{(3\ell+1)\mu+\ell\lambda}{(2\ell+3)(2\ell+1)\mu(2\mu+\lambda)}|x|^{\ell+1} & 0 & 0 \\ \frac{(\ell+1)(\mu+\lambda)}{2(2\ell+1)\mu(2\mu+\lambda)}(|x|^{\ell+1} - |x|^{\ell-1}) & \frac{(3\ell+2)\mu+(\ell+1)\lambda}{(2\ell-1)(2\ell+1)\mu(2\mu+\lambda)}|x|^{\ell-1} & 0 \\ 0 & 0 & \frac{1}{(2\ell+1)\mu}|x|^{\ell} \end{bmatrix}. \tag{5.3}$$

3. *When* $|x| > 1$, *we have*

$$(\mathcal{S}\underline{Y_{\ell m}})(x) = \underline{Y_{\ell m}}\left(\frac{x}{|x|}\right) A^{\text{out}}_{\mathcal{S},\ell}(x),$$

where the matrix $A^{\text{out}}_{\mathcal{S},\ell}(x)$ *is given by*

$$A^{\text{out}}_{\mathcal{S},\ell}(x) = \begin{bmatrix} \frac{(3\ell+1)\mu+\ell\lambda}{(2\ell+3)(2\ell+1)\mu(2\mu+\lambda)}|x|^{-\ell-2} & \frac{\ell(\mu+\lambda)}{2(2\ell+1)\mu(2\mu+\lambda)}(|x|^{-\ell-2}-|x|^{-\ell}) & 0 \\ 0 & \frac{(3\ell+2)\mu+(\ell+1)\lambda}{(2\ell-1)(2\ell+1)\mu(2\mu+\lambda)}|x|^{-\ell} & 0 \\ 0 & 0 & \frac{1}{(2\ell+1)\mu}|x|^{-\ell-1} \end{bmatrix}. \tag{5.4}$$

The following result is a corollary of Theorem 5.1.

Corollary 5.1. *Let* $\underline{Y_{\ell m}}$ *be the a matrix defined by* (5.1)*. Then, on the unit sphere* $\mathbb{S}^2$*, there holds*

$$\mathcal{K}^*\underline{Y_{\ell m}} = \mathcal{K}\underline{Y_{\ell m}} = \underline{Y_{\ell m}}A_{\mathcal{K}^*,\ell},$$

where $\mathcal{K}$ *is the double layer boundary operator* (3.5) *with its adjoint* $\mathcal{K}^*$ *and* $A_{\mathcal{K}^*,\ell}$ *is a constant and diagonal matrix:*

$$\begin{aligned} A_{\mathcal{K}^*,\ell} &= \begin{bmatrix} -\frac{2(2\ell^2+6\ell+1)\mu-3\lambda}{2(2\ell+1)(2\ell+3)(2\mu+\lambda)} & 0 & 0 \\ 0 & \frac{2(2\ell^2-2\ell-3)\mu-3\lambda}{2(2\ell+1)(2\ell-1)(2\mu+\lambda)} & 0 \\ 0 & 0 & \frac{1}{2\mu(2\ell+1)} \end{bmatrix} \\ &= \operatorname{diag}(\tau^1_{\mathcal{K}^*,\ell}, \tau^2_{\mathcal{K}^*,\ell}, \tau^3_{\mathcal{K}^*,\ell}). \end{aligned} \tag{5.5}$$

Remark 5.1. Recall that the Lamé constats μ, λ satisfy $\mu > 0, 2\mu + 3\lambda > 0$, we can verify that the eigenvalue $\tau^k_{\mathcal{K}^*,\ell}$ of the adjoint double layer boundary operator is $-\frac{1}{2}$ if and only if $\ell = 1, k = 2$. And the eigenvectors associated with the eigenvalue $-\frac{1}{2}$ are W_{1m}, $m = \pm 1, 0$.

The following theorem gives explicit expressions of the double layer potential.

Theorem 5.2. *Let* $\underline{Y_{\ell m}}$ *be given by* (5.1) *and* $\mathcal{D}$ *the double layer potential on the unit sphere introduced in* (3.3)*, we have:*

1. *For* $|x| < 1$,

$$(\mathcal{D}\underline{Y_{\ell m}})(x) = \underline{Y_{\ell m}}\Big(\frac{x}{|x|}\Big)A^{\text{in}}_{\mathcal{D},\ell}(x)$$

where

$$A^{\text{in}}_{\mathcal{D},\ell}(x) = \begin{bmatrix} a^{\text{in},\mathcal{D},\ell}_{11}|x|^{\ell+1} & a^{\text{in},\mathcal{D},\ell}_{12}|x|^{\ell+1} & 0 \\ a^{\text{in},\mathcal{D},\ell}_{21,1}|x|^{\ell+1} + a^{\text{in},\mathcal{D},\ell}_{21,2}|x|^{\ell-1} & a^{\text{in},\mathcal{D},\ell}_{22,1}|x|^{\ell+1} + a^{\text{in},\mathcal{D},\ell}_{22,2}|x|^{\ell-1} & 0 \\ 0 & 0 & -\frac{\ell+1}{(2\ell+1)\mu}|x|^{\ell} \end{bmatrix}. \tag{5.6}$$

2. *For* $|x| > 1$,

$$(\mathcal{D}\underline{Y_{\ell m}})(x) = \underline{Y_{\ell m}}\Big(\frac{x}{|x|}\Big)A^{\text{out}}_{\mathcal{D},\ell}(x)$$

where

$$A^{\rm out}_{\mathcal{D},\ell}(x) = \begin{bmatrix} a^{{\rm out},\mathcal{D},\ell}_{11,1}|x|^{-\ell-2} + a^{{\rm out},\mathcal{D},\ell}_{11,2}|x|^{-\ell} & a^{{\rm out},\mathcal{D},\ell}_{12,1}|x|^{-\ell-2}+a^{{\rm out},\mathcal{D},\ell}_{12,2}|x|^{-\ell} & 0 \\ a^{{\rm out},\mathcal{D},\ell}_{21}|x|^{-\ell} & a^{{\rm out},\mathcal{D},\ell}_{22}|x|^{-\ell} & 0 \\ 0 & 0 & \frac{\ell}{(2\ell+1)\mu}|x|^{-\ell-1} \end{bmatrix}. \tag{5.7}$$

The constants $a^{{\rm in},\mathcal{D},\ell}_{ij}, a^{{\rm out},\mathcal{D},\ell}_{ij}$ *are listed in Appendix B.*

The results of Theorems 5.1, 5.2 and Corollary 5.1 can be extended to any sphere $\Gamma_r(x_0) = \partial B_r(x_0)$, which is, a sphere centered at x_0 with radius r_0, by setting $\underline{Y_{\ell m}}\Big(\frac{x-x_0}{|x-x_0|}\Big)$ and using the following scaling in r:

$$\begin{aligned} (\mathcal{S}\underline{Y_{\ell m}})(x) &= r\underline{Y_{\ell m}}\Big(\frac{x-x_0}{|x-x_0|}\Big)A^{\rm in/out}_{\mathcal{S},\ell}\Big(\frac{x-x_0}{r}\Big), \\ (\mathcal{D}\underline{Y_{\ell m}})(x) &= \underline{Y_{\ell m}}\Big(\frac{x-x_0}{|x-x_0|}\Big)A^{\rm in/out}_{\mathcal{D},\ell}\Big(\frac{x-x_0}{r}\Big), \\ (\mathcal{V}\underline{Y_{\ell m}})(x) &= r\underline{Y_{\ell m}}\Big(\frac{x-x_0}{|x-x_0|}\Big)A_{\mathcal{V},\ell}, \quad (\mathcal{K}^*\underline{Y_{\ell m}})(x) = (\mathcal{K}\underline{Y_{\ell m}})(x) = \underline{Y_{\ell m}}\Big(\frac{x-x_0}{|x-x_0|}\Big)A_{\mathcal{K}^*,\ell}. \end{aligned} \tag{5.8}$$

5.2. Preliminary lemmas

To prove the results of Section 5.1 in the upcoming Section 5.3, we derive first several preliminary lemma.

Lemma 5.1. *For a scalar function* $h = h(r)$, *we have the following identities*

$$\begin{aligned} 2e\Big(h(r)V_{\ell m}\Big)\mathbf{n}|_{\mathbb{S}^2} &= \Big(\frac{(3\ell+2)h_r(1)-\ell(\ell+2)h(1)}{2\ell+1}\Big)V_{\ell m} + \Big(\frac{-(\ell+1)h_r(1)-(\ell+1)(\ell+2)h(1)}{2\ell+1}\Big)W_{\ell m}, \\ 2e\Big(h(r)W_{\ell m}\Big)\mathbf{n}|_{\mathbb{S}^2} &= \Big(\frac{-\ell h_r(1)+\ell(\ell-1)h(1)}{2\ell+1}\Big)V_{\ell m} + \Big(\frac{(3\ell+1)h_r(1)+(\ell-1)(\ell+1)h(1)}{2\ell+1}\Big)W_{\ell m}, \\ 2e\Big(h(r)X_{\ell m}\Big)\mathbf{n}|_{\mathbb{S}^2} &= \Big(h_r(1)-h(1)\Big)X_{\ell m}, \end{aligned} \tag{5.9}$$

where $\mathbf{n}$ *is the outward pointing unit normal vector of the unit sphere* $\mathbb{S}^2$.

Proof. We consider $h(r)V_{\ell m}$ first. Following (4.9) and (4.12), we have,

$$\begin{aligned} &\Big(\nabla(h(r)V_{\ell m}) + \nabla(h(r)V_{\ell m})^\top\Big)\mathbf{n}|_{\mathbb{S}^2} \\ =&h_r(1)\big(V_{\ell m}\hat{r}^\top + \hat{r}V_{\ell m}^\top\Big)\mathbf{n} + h(r)(\nabla_{\rm s}V_{\ell m} + \nabla_{\rm s}V_{\ell m}^\top)\mathbf{n}|_{\mathbb{S}^2} \\ =&h_r(1)\big(V_{\ell m} + \hat{r}V_{\ell m}^\top)\hat{r} + h(1)(\nabla_{\rm s}V_{\ell m} + \nabla_{\rm s}V_{\ell m}^\top)\hat{r}, \end{aligned} \tag{5.10}$$

where we also use the fact that $\mathbf{n}$ is equal to the radial basis $\hat{r}$ on the unit sphere $\mathbb{S}^2$. Consider now the first term $h_r(1)(V_{\ell m}+\hat{r}V_{\ell m}^\top)\hat{r}$. To compute $\hat{r}V_{\ell m}^\top\hat{r}$, we need first the relation $\nabla_{\rm s}Y_{\ell m}^\top\hat{r} = 0$ following from (4.3). Then according to the definition of the vector spherical harmonics $V_{\ell m}$ given by (4.4), we have

$$(\hat{r}V_{\ell m}^\top)\hat{r} = (\hat{r}\nabla_{\rm s}Y_{\ell m}^\top)\hat{r} - (\ell+1)Y_{\ell m}\hat{r}\hat{r}^\top\hat{r} = -(\ell+1)Y_{\ell m}\hat{r} = \frac{(\ell+1)}{2\ell+1}(-W_{\ell m}+V_{\ell m}). \tag{5.11}$$

Therefore, the first term in (5.10) yields

$$V_{\ell m} + \hat{r} V_{\ell m}^\top \hat{r} = \frac{3\ell+2}{2\ell+1} V_{\ell m} - \frac{\ell+1}{2\ell+1} W_{\ell m}.$$

Now consider the second term $(\nabla_{\mathrm{s}} V_{\ell m} + \nabla_{\mathrm{s}} V_{\ell m}^\top)\hat{r}$. According to (4.3), we have $\nabla_{\mathrm{s}} V_{\ell m}\hat{r} = 0$. The definition of $V_{\ell m}$ (4.4) gives

$$\nabla_{\mathrm{s}} V_{\ell m}^\top \hat{r} = \nabla_{\mathrm{s}}(\nabla_{\mathrm{s}} Y_{\ell m})^\top \hat{r} - (\ell+1)\nabla_{\mathrm{s}}(Y_{\ell m}\hat{r})^\top \hat{r}.$$

We compute the two terms separately. Using the product rule (4.9), we have

$$\nabla_{\mathrm{s}}(\nabla_{\mathrm{s}} Y_{\ell m})^\top \hat{r} = \nabla_{\mathrm{s}}(\nabla_{\mathrm{s}} Y_{\ell m}^\top \hat{r}) - \nabla_{\mathrm{s}} \hat{r}^\top \nabla_{\mathrm{s}} Y_{\ell m} = -(\mathrm{Id} - \hat{r}\hat{r}^\top)^\top \nabla_{\mathrm{s}} Y_{\ell m} = -\nabla_{\mathrm{s}} Y_{\ell m}.$$

For $\nabla_{\mathrm{s}}(Y_{\ell m}\hat{r})^\top \hat{r}$, we use (4.9) and have

$$\nabla_{\mathrm{s}}(Y_{\ell m}\hat{r})^\top \hat{r} = \nabla_{\mathrm{s}} Y_{\ell m} \hat{r}^\top \hat{r} + (Y_{\ell m}\nabla_{\mathrm{s}}\hat{r})^\top \hat{r} = \nabla_{\mathrm{s}} Y_{\ell m} + Y_{\ell m}\nabla_{\mathrm{s}}\hat{r}^\top \hat{r} = \nabla_{\mathrm{s}} Y_{\ell m},$$

where we use the fact that $\nabla_{\mathrm{s}}\hat{r}$ is a symmetric matrix and $\nabla_{\mathrm{s}}\hat{r}\hat{r} = 0$. Therefore, there holds

$$\nabla_{\mathrm{s}} V_{\ell m}^\top \hat{r} = -(\ell+2)\nabla_{\mathrm{s}} Y_{\ell m} = -\frac{\ell+2}{2\ell+1}\big(\ell V_{\ell m} + (\ell+1) W_{\ell m}\big). \tag{5.12}$$

The computation of $e\big(h(r)V_{\ell m}\big)\mathbf{n}$ is thus completed in view of (5.11) and (5.12). A similar computation gives $h(r)W_{\ell m}$. Consider now $h(r)X_{\ell m}$. Similar to (5.10), we have to compute the sum:

$$e\big(h(r)X_{\ell m}\big)\mathbf{n}|_{\mathbb{S}^2} = h_r(1)\big(X_{\ell m} + \hat{r} X_{\ell m}^\top\big)\hat{r} + h(1)(\nabla_{\mathrm{s}} X_{\ell m} + \nabla_{\mathrm{s}} X_{\ell m}^\top)\hat{r}.$$

Notice that $X_{\ell m} = \hat{r} \times \nabla_{\mathrm{s}} Y_{\ell m}$ is orthogonal to $\hat{r}$ and it follows immediately that $X_{\ell m}^\top \hat{r} = 0$. Further, by (4.3), there holds $\nabla_{\mathrm{s}} X_{\ell m}\hat{r} = 0$. Then, it remains to consider $X_{\ell m}\hat{r}$ and $\nabla_{\mathrm{s}} X_{\ell m}^\top \hat{r}$. For the term $\nabla_{\mathrm{s}} X_{\ell m}^\top \hat{r}$, we use the relation (4.10) and have

$$\nabla_{\mathrm{s}} X_{\ell m}^\top \hat{r} = \big(\nabla_{\mathrm{s}}(\hat{r} \times \nabla_{\mathrm{s}} Y_{\ell m})\big)^\top \hat{r} = (\nabla_{\mathrm{s}}\hat{r} \times \nabla_{\mathrm{s}} Y_{\ell m})^\top \hat{r} + \big(\hat{r} \times \nabla_{\mathrm{s}}(\nabla_{\mathrm{s}} Y_{\ell m})\big)^\top \hat{r}.$$

Both terms can be computed by (4.11):

$$\big(\hat{r} \times \nabla_{\mathrm{s}}(\nabla_{\mathrm{s}} Y_{\ell m})\big)^\top \hat{r} = \nabla_{\mathrm{s}}(\nabla_{\mathrm{s}} Y_{\ell m})^\top (\hat{r} \times \hat{r}) = 0,$$

and

$$\begin{aligned}(\nabla_{\mathrm{s}}\hat{r} \times \nabla_{\mathrm{s}} Y_{\ell m})^\top \hat{r} =& (\hat{r} \times \nabla_{\mathrm{s}}\hat{r})^\top \nabla_{\mathrm{s}} Y_{\ell m} = \big(\hat{r} \times (\mathrm{Id} - \hat{r}\hat{r}^\top)\big)^\top \nabla_{\mathrm{s}} Y_{\ell m} \\ =& (\hat{r} \times \mathrm{Id})^\top \nabla_{\mathrm{s}} Y_{\ell m} = \nabla_{\mathrm{s}} Y_{\ell m} \times \hat{r}.\end{aligned}$$

This gives

$$\nabla_{\mathrm{s}} X_{\ell m}^\top \hat{r} = -X_{\ell m}.$$

Then we get the result for $h(r)X_{\ell m}$.

By (4.13), (5.9), we get, for a displacement $h(r)V_{\ell m} + g(r)W_{\ell m} + h(r)X_{\ell m}$, it holds that

$$\begin{aligned}&\mathcal{T}_{\mathbf{n}}^-\big(f(r)V_{\ell m} + g(r)W_{\ell m} + h(r)X_{\ell m}\big) = \mathcal{T}_{\mathbf{n}}^+\big(f(r)V_{\ell m} + g(r)W_{\ell m} + h(r)X_{\ell m}\big) \\ =& \Big(\frac{\mu}{2\ell+1}\big((3\ell+2)f_r(1) - \ell(\ell+2)f(1) - \ell g_r(1) + \ell(\ell-1)g(1)\big) \\ &+ \frac{\lambda}{2\ell+1}\big((\ell+1)f_r(1) + (\ell+1)(\ell+2)f(1) - \ell g_r(1) + \ell(\ell-1)g(1)\big)\Big)V_{\ell m}\end{aligned}$$

$$+\Big(\frac{\mu}{2\ell+1}\big(-(\ell+1)f_r(1)-(\ell+1)(\ell+2)f(1)+(3\ell+1)g_r(1)+(\ell+1)(\ell-1)g(1)\big)$$
$$+\frac{\lambda}{2\ell+1}\big(-(\ell+1)f_r(1)-(\ell+1)(\ell+2)f(1)+\ell g_r(1)-\ell(\ell-1)g(1)\big)\Big)W_{\ell m}$$
$$+\mu\big(h_r(1)-h(1)\big)X_{\ell m}. \tag{5.13}$$

The flowing lemma concerns the double layer boundary operator and its adjoint. In particular, on a sphere, we have the following lemma.

Lemma 5.2. *Let $\mathcal{K}$ be the double layer boundary operators defined by* (3.4) *on a sphere and $\mathcal{K}^*$ its adjoint operators. Then for $v \in L^2(\mathbb{S}^2)^3$, we have $\mathcal{K}v = \mathcal{K}^*v$.*

Proof. Indeed, we have

$$\partial_{x_k}G_{ji}(x-y) =\frac{1}{8\pi\mu|x-y|^3}\Bigg(-\frac{\lambda+3\mu}{\lambda+2\mu}\delta_{ij}(x_k-y_k) + \frac{\lambda+\mu}{\lambda+2\mu}\Big((x_i-y_i)\delta_{jk}+(x_j-y_j)\delta_{ik}-\frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^2}\Big)\Bigg),$$

and

$$\operatorname{Tr} e_x(G_j)\operatorname{Id} = \operatorname{div}_x G_j(x-y) = -\frac{(x_j-y_j)}{4\pi|x-y|^3}\frac{1}{\lambda+2\mu}.$$

Hence, we have

$$\Big(2\mu e_x(G_j)+\lambda\operatorname{Tr} e_x(G_j)\operatorname{Id}\Big)_{ik}\frac{x_k}{|x|}$$
$$= -\frac{1}{4\pi(\lambda+2\mu)|x-y|^3}\Big(\frac{\mu}{\lambda+3\mu}(\delta_{ij}(x_k-y_k)+\delta_{jk}(x_i-y_i)+\delta_{ik}(x_j-y_j)) + \frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^2}\Big)\Big)\frac{x_k}{|x|}$$

The same result holds for $\Big(2\mu e_y(G_i)+\lambda\operatorname{Tr} e_y(G_i)\operatorname{Id}\Big)_{jk}\frac{y_k}{|y|}$ by replacing x by y. Further, for $x, y \in \mathbb{S}^2$, the following relation holds:

$$(x-y)\cdot\frac{x}{|x|} = 1-\frac{x\cdot y}{r} = (y-x)\cdot\frac{y}{|y|}.$$

Therefore, we have

$$(\mathcal{K}^*v)_i(x) =\sum_j\int_\Gamma\frac{1}{4\pi(\lambda+3\mu)|x-y|^3}\Bigg(\Big(-\frac{\mu}{\lambda+3\mu}\delta_{ij}-\frac{3(x_i-y_i)(x_j-y_j)}{|x-y|^2}\Big)(x-y)\cdot\frac{x}{|x|}v_j(y) + \frac{\mu}{\lambda+3\mu}(x_jy_i-x_iy_j)v_j(y)\Bigg)dy$$
$$=\sum_j\int_\Gamma\frac{1}{4\pi(\lambda+2\mu)|x-y|^3}\Bigg(\Big(-\frac{\mu}{\lambda+3\mu}\delta_{ij}-\frac{3(y_i-x_i)(y_j-x_j)}{|x-y|^2}\Big)(y-x)\cdot\frac{y}{|y|}v_j(y) + \frac{\mu}{\lambda+3\mu}(x_jy_i-x_iy_j)v_j(y)\Bigg)dy = (\mathcal{K}v)_i(x).$$

This completes the proof of the lemma. □

5.3. Proof of the principal results

We are now ready to prove Theorem 5.1.

Proof. [Proof of Theorem 5.1] Consider the single layer potential $\mathcal{S}$ defined by (3.1). Note that for any $\phi \in H^{-\frac{1}{2}}(\mathbb{S}^2)^3$, as announced in Lemma 3.1, $u = \mathcal{S}\phi$ satisfies the following linear isotropic elasticity system

$$\mathbf{L}u = -\operatorname{div}\Big(2\mu e(u) + \lambda \operatorname{Tr} e(u)\operatorname{Id}\Big) = 0, \quad \text{in } \mathbb{R}^3 \backslash \mathbb{S}^2 . \tag{5.14}$$

Now we determine $u = \mathcal{S}V_{\ell m}$ by means of separation of variables in spherical coordinates. That is, we propose the Ansatz displacement field $\mathcal{S}V_{\ell m}$ as a function of the spherical coordinates of form $u = \mathcal{S}V_{\ell m} = h(r)V_{\ell m} + g(r)W_{\ell m} + h(r)X_{\ell m}$ where f, g, h are three scalar functions of r to be determined. Using the relation

$$\operatorname{div}(\operatorname{Tr} e(u)\operatorname{Id}) = \nabla(\operatorname{div} u),$$

and plugging the Ansatz into (5.14), together with (4.13)-(4.15), we have the following equation:

$$\begin{aligned}
&\Bigg[\Big(\mu + \frac{\ell+1}{2\ell+1}(\mu+\lambda)\Big)\Big(f_{rr} + \frac{2}{r}f_r - \frac{(\ell+1)(\ell+2)}{r^2}f\Big) \\
&\qquad\qquad - \frac{\ell}{2\ell+1}(\mu+\lambda)\Big(g_{rr} - \frac{2\ell-1}{r}g_r + \frac{(\ell-1)(\ell+1)}{r^2}g\Big)\Bigg]V_{\ell m} \\
&+ \Bigg[\Big(\mu + \frac{\ell}{2\ell+1}\Big(\mu+\lambda\Big)\Big)\Big(g_{rr} + \frac{2}{r}g_r - \frac{(\ell-1)\ell}{r^2}g\Big) - \frac{\ell+1}{2\ell+1}(\mu+\lambda)\Big(f_{rr} + \frac{2\ell+3}{r}f_r + \frac{\ell(\ell+2)}{r^2}f\Big)\Bigg]W_{\ell m} \\
&+ \mu\Big[h_{rr} + \frac{2}{r}h - \frac{\ell(\ell+1)}{r^2}h\Big]X_{\ell m} = 0.
\end{aligned} \tag{5.15}$$

Since $V_{\ell m}, W_{\ell m}, X_{\ell m}$ is an orthogonal basis, all the coefficients of $V_{\ell m}, W_{\ell m}, X_{\ell m}$ must be zero in (5.15). Let first $\ell \geq 1$ and we have six sets of analytical solutions to (5.15) reading

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
f	$r^{-\ell-2}$	$-\frac{\ell}{2\ell+1}(\mu+\lambda)r^{-\ell}$	0	$\frac{2}{2l+3}\Big(\mu + \frac{\ell}{2\ell+1}(\mu+\lambda)\Big)r^{\ell+1}$	0	0
g	0	$\frac{2}{2\ell-1}\Big(\mu + \frac{\ell+1}{2\ell+1}(\mu+\lambda)\Big)r^{-\ell}$	0	$\frac{\ell+1}{2\ell+1}(\mu+\lambda)r^{\ell+1}$	$r^{\ell-1}$	0
h	0	0	$r^{-\ell-1}$	0	0	r^{ℓ}

in which $(i), (ii), (iii)$ are admissible only for $|r| > 0$ while $(iv), (v), (vi)$ are unbounded when $|r| \to \infty$. Now, we consider the exterior and the interior of the unit sphere separately in which we aim to get $\mathcal{S}V_{\ell m}$. For the sake of simplicity, write

$$\begin{aligned}
p_{12} &= -\frac{\ell}{2\ell+1}(\mu+\lambda), \qquad p_{22} = \frac{2}{2\ell-1}\Big(\mu + \frac{\ell+1}{2\ell+1}(\mu+\lambda)\Big), \\
q_{11} &= \frac{2}{2l+3}\Big(\mu + \frac{\ell}{2\ell+1}(\mu+\lambda)\Big)r^{\ell+1}, \qquad q_{21} = \frac{\ell+1}{2\ell+1}(\mu+\lambda).
\end{aligned}$$

Write $\mathcal{S}V_{\ell m}$ as a linear combination of the three solutions in the exterior and three in the interior of the unit sphere:

$$\mathcal{S}V_{\ell m} = \begin{cases} a^{\text{in}}_{\ell m} q_{11} r^{\ell+1} V_{\ell m} + \Big(a^{\text{in}}_{\ell m} q_{21} r^{\ell+1} + b^{\text{in}}_{\ell m} r^{\ell-1}\Big)W_{\ell m} + c^{\text{in}}_{\ell m} r^{\ell} X_{\ell m} & |r| < 1, \\ \Big(a^{\text{out}}_{\ell m} r^{-\ell-2} + b^{\text{out}}_{\ell m} p_{12} r^{-\ell}\Big)V_{\ell m} + b^{\text{out}}_{\ell m} p_{22} r^{-\ell} W_{\ell m} + c^{\text{out}}_{\ell m} r^{-\ell-1} X_{\ell m} & |r| > 1, \end{cases} \tag{5.16}$$

where $a_{\ell m}^{\text{in/out}}, b_{\ell m}^{\text{in/out}}, c_{\ell m}^{\text{in/out}} \in \mathbb{R}$ are constants to be determined. In order to determine these six unknown constants, we use the jump relation given by Theorems 3.2:

$$[\![\mathcal{S}V_{\ell m}]\!] = 0, \quad [\![\mathcal{T}\mathcal{S}V_{\ell m}]\!] = V_{\ell m}. \tag{5.17}$$

Since $V_{\ell m}, W_{\ell m}, X_{\ell m}$ is an orthogonal basis, it follows immediately from the first equality in (5.17) that

$$a_{\ell m}^{\text{in}} q_{11} = a_{\ell m}^{\text{out}} + b_{\ell m}^{\text{out}} p_{12} \quad a_{\ell m}^{\text{in}} q_{21} + b_{\ell m}^{\text{in}} = b_{\ell m}^{\text{out}} p_{22}, \quad c_{\ell m}^{\text{in}} = c_{\ell m}^{\text{out}}.$$

Now take the inner product of the second equation (5.17) with $V_{\ell m}, W_{\ell m}, X_{\ell m}$. With the computations in (5.13), we have

$$\begin{aligned}
2\ell - 1 =& \mu a_{\ell m}^{\text{in}} q_{11}(3\ell+2)(\ell+1) + \lambda_{\ell m}^{\text{in}} q_{11}(\ell+1)^2 - (\mu+\lambda)\Big(a_{\ell m}^{\text{in}} q_{21}\ell(\ell+1) + b_{\ell m}^{\text{in}}\ell(\ell-1)\Big) \\
&- \mu\Big(a_{\ell m}^{\text{out}}(3\ell+2)(-\ell-2) - b_{\ell m}^{\text{out}} p_{12}(3\ell+2)\ell\Big) \\
&- \lambda\Big((\ell+1)(\ell-2)a_{\ell m}^{\text{out}} - b_{\ell m}^{\text{out}} p_{12}(\ell+1)l\Big) - (\mu+\lambda) b_{\ell m}^{\text{out}} p_{22}\ell^2, \\
0 =& \mu\Big(a_{\ell m}^{\text{in}} q_{21}(3\ell+1)(\ell+1) + b_{\ell m}^{\text{in}}(3\ell+1)(\ell-1)\Big) + \lambda\Big(a_{\ell m}^{\text{in}} q_{21}\ell(\ell+1) + b_{\ell m}^{\text{in}}\ell(\ell-1)\Big) \\
&- (\mu+\lambda)a_{\ell m}^{\text{in}} q_{11}(\ell+1)^2 + \mu b_{\ell m}^{\text{out}} p_{22}\ell(3\ell+1) + \lambda b_{\ell m}^{\text{out}} p_{22}\ell^2 \\
&- (\mu+\lambda)\Big(a_{\ell m}^{\text{out}}(\ell+1)(\ell+2) + b_{\ell m}^{\text{out}} p_{12}(\ell+\ell)\ell\Big), \\
0 =& \mu(\ell c_{\ell m}^{\text{in}} + (\ell+1)c_{\ell m}^{\text{out}}).
\end{aligned}$$

Hence, we conclude

$$\begin{aligned}
a_{\ell m}^{\text{in}} &= \frac{1}{2\mu(2\mu+\lambda)}, \qquad a_{\ell m}^{\text{out}} = \frac{1}{2\ell+3}\Big(\mu + \frac{\ell}{2\ell+1}(\mu+\lambda)\Big)\big(\mu(2\mu+\lambda)\big)^{-1}, \\
b_{\ell m}^{\text{in}} &= -\frac{\ell+1}{2\ell+1}(\mu+\lambda)\big(2\mu(2\mu+\lambda)\big)^{-1}, \qquad b_{\ell m}^{\text{out}} = c_{\ell m}^{\text{in}} = c_{\ell m}^{\text{out}} = 0.
\end{aligned}$$

Similar computations following the same logic give the results for $W_{\ell m}, X_{\ell m}$ for $\ell \geq 1$.

In the case where $\ell = 0$, we only have to treat V_{00} since $W_{00} = X_{00} = 0$. Using (5.16)–(5.17), we have

$$\mathcal{S}V_{00} = \begin{cases} \dfrac{1}{3(2\mu+\lambda)} r^2 V_{00} & r \leq 1, \\ \dfrac{1}{3(2\mu+\lambda)} r^{-1} V_{00} & r > 1. \end{cases} \tag{5.18}$$

Notice that the result (5.18) is indeed consistent with the cases where $\ell \geq 1$. We have proved therefore the theorem. □

Corollary 5.1 can now be deduced.

Proof. [Proof of Corollary 5.1] As is shown in (3.9), we have

$$2\mathcal{K}^* V_{\ell m} = \mathcal{T}_{\mathbf{n}}^+ \mathcal{S}V_{\ell m} + \mathcal{T}_{\mathbf{n}}^- \mathcal{S}V_{\ell m}.$$

Again, apply the computation in (5.13), we have

$$\mathcal{T}_{\mathbf{n}}^- \mathcal{S}V_{\ell m} + \mathcal{T}_{\mathbf{n}}^+ \mathcal{S}V_{\ell m} = \frac{-2(2\ell^2+6\ell+1)\mu + 3\lambda}{(2\ell+1)(2\ell+3)(2\mu+\lambda)} V_{\ell m}.$$

Hence, we have

$$\mathcal{K}^* V_{\ell m} = \frac{2(2\ell^2 + 6\ell + 1)\mu + 3\lambda}{2(2\ell + 1)(2\ell + 3)(2\mu + \lambda)}.$$

Similar computations give the results for the other two components $W_{\ell m}, X_{\ell m}$. Further, Lemma 5.2 provides the result for the double layer operator $\mathcal{K}$. □

Finally, the proof of Theorem 5.2 follows the same structure to that of Theorem 5.1 in employing the jump relations

$$[\![\mathcal{D}\phi]\!] = -\phi, \quad [\![\mathcal{T}\mathcal{D}\phi]\!] = 0$$

for the choices $\phi = V_{\ell m}, W_{\ell m}, X_{\ell m}$ respectively.

6. Application

We study here a case of an elasticity problem involving several spherical inclusions as an application of the results in the above sections, derive an integral equation formulation and propose a Galerkin formulation thereof based on the vectorial spherical harmonics.

6.1. Problem setting

Set the sets of indices J_1, J_2, J such that $M + 1 \in J_2$, $J_1 \cap J_2 = \emptyset$ and

$$J_1 \cup J_2 = \{1, 2, \ldots, M, M + 1\}, \qquad J = J_1 \cup J_2 \backslash \{M + 1\}, \tag{6.1}$$

and let $\Omega_i \subset \mathbb{R}^3$, $i \in J$ be non-overlapping balls, centered at $x_i \in \mathbb{R}^3$ with radius r_i, all contained in an additional ball B_R centred at the origin with the radius R. Moreover, define the domains

$$\Omega_{M+1} = \mathbb{R}^3 \backslash B_R, \quad \Omega_0 := B_R \backslash \bigcup_{i \in J_1 \cup J_2} \overline{\Omega}_i, \quad \Omega := B_R \backslash \bigcup_{i \in J_2} \overline{\Omega}_i. \tag{6.2}$$

Denote the boundaries $\Gamma_i = \partial\Omega_i$, $i \in \{0\} \cup J_1 \cup J_2$. Then, it holds that $\Gamma_0 = \bigcup\limits_{i \in J_1 \cup J_2} \Gamma_i$. Set further $\mathbf{n}_0$ as the outward pointing unit normal vector with respect to the domain Ω_0 and $\mathbf{n}_i$ the outward pointing normal vector with respect to each domain Ω_i, $i \in J_1 \cup J_2$. Then it holds that $\mathbf{n}_i = -\mathbf{n}_0$. We refer to Fig. 6.1 for an illustration of geometry configuration.

In the numerical example presented below, we assume that each inclusion Ω_i, $i \in J_1$ is filled with an isotropic elastic medium associated with Lamé parameters μ_i, λ_i. The remaining background domain Ω_0 is filled with medium of Lamé constants μ_0, λ_0. Further, we denote ${\mathcal{T}^i_{\mathbf{n}_j}}^{\pm}$ the normal derivative operator acting on the boundary Γ_j with Lamé constants μ_j, λ_j and the normal vector $\mathbf{n}_j$:

$${\mathcal{T}^i_{\mathbf{n}_j}}^{\pm} u = \gamma_j^{\pm}\Big((2\mu_i e(u) + \lambda_i \operatorname{Tr} e(u) \operatorname{Id})\mathbf{n}_j\Big)$$

where $\gamma_j^{\pm}$ are the exterior and interior the trace operators on Γ_j, following the notations given by (2.4) by taking $\Omega^- = \Omega_j$. Define the parameter s_i by

$$s_i = \begin{cases} -1 & i = M + 1, \\ 1 & \text{else}. \end{cases} \tag{6.3}$$

In particular, write

$$[\![\mathcal{T}u]\!] = {\mathcal{T}^0_{s_i \mathbf{n}_i}}^{-} u - {\mathcal{T}^0_{s_i \mathbf{n}_i}}^{+} u, \qquad \text{on} \quad \Gamma_i, \ i \in J_1 \cup J_2 \tag{6.4}$$

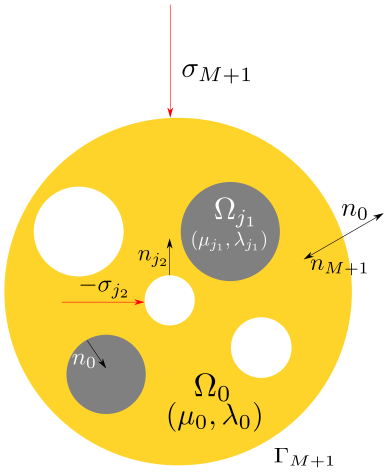

Fig. 6.1. Geometry setting of the model where $j_1 \in J_1$ and $j_2 \in J_2$.

and it is obvious that

$$\mathcal{T}^{0\ \pm}_{s_i\mathbf{n}_i} = s_i \mathcal{T}^{0\ \pm}_{\mathbf{n}_i}.$$

For given $f_i \in H^{-\frac{1}{2}}(\Gamma_i)^3$, we impose the transmission condition:

$$[\![u]\!] = 0, \qquad [\![\mathcal{T}u]\!] = f_i, \qquad \text{on} \quad \Gamma_i, i \in J_1.$$

Further, we let the domain Ω be subjected of a given stress tensor on its boundary. Indeed, we let each part of the boundary Γ_i, $i \in J_2$ be subjected to a given stress tensor $\sigma_i \in H^{1/2}(\Gamma_i)^3$:

$$\mathcal{T}^{0\ +}_{\mathbf{n}_i} u = -s_i \sigma_i. \tag{6.5}$$

Then, we consider the solution $u \in V_0(\Omega)^3$ to the following interface problem:

$$\begin{array}{ll} -\operatorname{div}\big(2\mu_i e(u) + \lambda_i \operatorname{Tr} e(u) \operatorname{Id}\big) = 0, & \text{in } \Omega_i,\ i \in \{0\} \cup J_1, \\ [\![u]\!] = 0, & \text{on } \Gamma_i\ i \in J_1, \\ [\![\mathcal{T}u]\!] = f_i, & \text{on } \Gamma_i\ i \in J_1, \\ \mathcal{T}^{0\ +}_{\mathbf{n}_i} u = -s_i\sigma_i, & \text{on } \Gamma_i, i \in J_2. \end{array} \tag{6.6}$$

Standard arguments involving the Lax-Milgram theorem yields the well-posedness of the problem.

Remark 6.1. In our case, we pre-defined the index set J_2 with the condition $M+1 \in J_2$. However, by equipping the domain Ω_{M+1} (which is indeed the exterior domain of the sphere B_R) with Lamé parameters μ_{M+1}, λ_{M+1}, one can also relax the setting and consider the case where $M+1 \in J_1$.

6.2. Integral equation

Let $u \in V_0(\Omega)^3$ denote the solution to (6.6) satisfying $[\![u]\!] = 0$ on Γ_i for $i \in J_1$ and define ν on Γ_0 by

$$\nu = \gamma_0^- u \qquad \text{as well as} \qquad \nu_i = \nu|_{\Gamma_i}, \quad \forall i \in J_1 \cup J_2,$$

where γ_0^- is the trace operator $\gamma_0^-: H^1(\Omega_0)^3 \to H^{\frac{1}{2}}(\Gamma_0)^3$ defined by (2.4) for $\Omega^- = \Omega_0$. To deduce an integral equation for ν, we first introduce an auxiliary problem: find a solution $v \in H^1(\Omega)^3$ to

$$\begin{aligned} -\operatorname{div}\big(2\mu_0 e(v) + \lambda_0 \operatorname{Tr} e(v) \operatorname{Id}\big) &= 0, && \text{in } \Omega\backslash\Gamma_0, \\ \gamma_0^- v = \gamma_0^- u, & && \text{on } \Gamma_0. \end{aligned} \tag{6.7}$$

The auxiliary problem (6.7) admits a unique solution v in Ω and observe that

$$v = u, \quad \text{in } \Omega_0,$$

but is different in all Ω_i, $i \in J_1$. Further, there exists a global density ϕ supported on Γ_0 such that

$$\begin{aligned} v = \mathcal{S}_G^0 \phi &= \sum_{i \in J_1 \cup J_2} \mathcal{S}_i^0 \phi_i && \text{in } \Omega_0 \\ v = \mathcal{V}_G^0 \phi &= \sum_{i \in J_1 \cup J_2} \mathcal{V}_i^0 \phi_i && \text{on } \Gamma_0, \end{aligned} \tag{6.8}$$

where $\mathcal{S}_G^0$ is the global layer potential (3.1) with Lamé parameters μ_0, λ_0 defined on the whole boundary Γ_0 while $\mathcal{S}_i^0$ is the local single layer potential with Lamé parameters μ_0, λ_0 defined locally on the sphere Γ_i and $\mathcal{V}_G^0, \mathcal{V}_i^0$ their corresponding single layer boundary operators (3.2). Further, according to Theorem 3.2, the density ϕ_i is given by the jump relation

$$\phi_i = [\![\mathcal{T} v]\!] = \Big(\mathcal{T}_{s_i \mathbf{n}_i}^{0\;-} v - \mathcal{T}_{s_i \mathbf{n}_i}^{0\;+} v\Big) = s_i \Big(\mathcal{T}_{\mathbf{n}_i}^{0\;-} u - \mathcal{T}_{\mathbf{n}_i}^{0\;+} v\Big), \quad \text{on } \Gamma_i, i \in J_1 \cup J_2. \tag{6.9}$$

The last equivalence in (6.9) is obtained because $u = v$ on Ω_0. Further, both solutions u, v can be represented by some local densities in each domain Ω_i:

$$\begin{aligned} u|_{\Omega_i} &= \mathcal{S}_i \varphi_i, && i \in J_1 \\ v|_{\Omega_i} &= \mathcal{S}_i^0 \psi_i && i \in J_1 \cup J_2, \end{aligned}$$

where $\mathcal{S}_i$ is the local layer potential with Lamé parameters μ_i, λ_i while $\mathcal{S}_i^0$ with μ_0, λ_0, both of which are defined locally on the sphere Γ_i. For the corresponding single layer boundary operators $\mathcal{V}_i^0, \mathcal{V}_i$ defined by (3.2), we have

$$\mathcal{V}_i^0 \psi_i = \mathcal{V}_i \varphi_i = \nu_i, \quad i \in J_1.$$

According to Corollary 3.1, the single layer boundary operators $\mathcal{V}_i^0, \mathcal{V}_i$ are invertible, so we have

$$\begin{aligned} \psi_i &= \mathcal{V}_i^{0^{-1}}(\gamma_i v) = \mathcal{V}_i^{0^{-1}} \nu_i, && \text{on } \Gamma_i, i \in J_1 \cup J_2, \\ \varphi_i &= \mathcal{V}_i^{-1}(\gamma_i u) = \mathcal{V}_i^{-1} \nu_i, && \text{on } \Gamma_i, i \in J_1. \end{aligned} \tag{6.10}$$

Now, according to Theorem 3.1, we have

$$\begin{aligned} \mathcal{T}_{\mathbf{n}_i}^{0\;-} \mathcal{S}_i^0 \psi_i &= s_i \frac{1}{2} \psi_i + \mathcal{K}_i^{0^*} \psi_i, && \text{on } \Gamma_i, i \in J_1 \cup J_2, \\ \mathcal{T}_{\mathbf{n}_i}^{i\;-} \mathcal{S}_i \varphi_i &= s_i \frac{1}{2} \varphi_i + \mathcal{K}_i^* \varphi_i, && \text{on } \Gamma_i, i \in J_1, \end{aligned} \tag{6.11}$$

where $\mathcal{K}_i^{0^*}$, $\mathcal{K}_i^*$ are adjoint double layer boundary operators (3.5) defined locally on Γ_i with Lamé constants μ_0, λ_0 and μ_i, λ_i respectively. In problem (6.6), we have

$$\mathcal{T}_{\mathbf{n}_i}^{0\;+} u = \begin{cases} \mathcal{T}_{\mathbf{n}_i}^{i\;-} \mathcal{S}_i \varphi_i - f_i, & \text{on } \Gamma_i, i \in J_1, \\ -s_i \sigma_i, & \text{on } \Gamma_i, i \in J_2, \end{cases}$$

where $f_i \in H^{-\frac{1}{2}}(\Gamma_i)^3, \sigma_i \in H^{\frac{1}{2}}(\Gamma_i)^3$ are given transmission conditions and Neuman boundary conditions respectively. Providing (6.9)–(6.11), we have

$$\phi_i = \begin{cases} \frac{1}{2}(\mathcal{V}_i^{0^{-1}} - \mathcal{V}_i^{-1})\nu_i + (\mathcal{K}_i^{0^*}\mathcal{V}_i^{0^{-1}} - \mathcal{K}_i{}^*\mathcal{V}_i^{-1})\nu_i + f_i, & i \in J_1, \\ \frac{1}{2}\mathcal{V}_i^{0^{-1}}\nu_i + s_i\mathcal{K}_i^{0^*}\mathcal{V}_i^{0^{-1}}\nu_i + s_i\sigma_i, & i \in J_2. \end{cases}$$

According to (6.8), we have

$$(\mathrm{Id} - \mathcal{V}_G^0\mathcal{L})\nu = \mathcal{V}_G^0\Sigma \tag{6.12}$$

where $\mathcal{L}$ is an operator defined on each Γ_i:

$$\mathcal{L}|_{\Gamma_i} = \mathcal{L}_i = \frac{1}{2}(\mathcal{V}_i^{0^{-1}} - \mathcal{V}_i^{-1}) + (\mathcal{K}_i^{0^*}\mathcal{V}_i^{0^{-1}} - \mathcal{K}_i{}^*\mathcal{V}_i^{-1}), \qquad i \in J_1,$$
$$\mathcal{L}|_{\Gamma_i} = \mathcal{L}_i = \frac{1}{2}\mathcal{V}_i^{0^{-1}}\nu_i + s_i\mathcal{K}_i^{0^*}\mathcal{V}_i^{0^{-1}}\nu_i, \qquad i \in J_2,$$

and the vector Σ satisfies

$$\Sigma|_{\Gamma_i} = \begin{cases} f_i, & i \in J_1, \\ \sigma_i, & i \in J_2. \end{cases}$$

6.3. Galerkin approximation

Introduce $V_{N,i}, i \in J_1 \cup J_2$ the set spanned by vector spherical harmonics (4.4) on the sphere Γ_i with a maximum degree N:

$$V_{N,i} = \left\{ \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \sum_{k=1}^{3} [y_i]_{\ell m}^k Y_{\ell m}^{ki}(x) \middle| [y_i]_{\ell m}^k \in \mathbb{R} \right\}, \tag{6.13}$$

where we write $Y_{\ell m}^1 = V_{\ell m}, Y_{\ell m}^2 = W_{\ell m}, Y_{\ell m}^3 = X_{\ell m}$ and $Y_{\ell m}^{ki}(x) = Y_{\ell m}^k\left(\frac{x-x_i}{r_i}\right)$. Define also the global set

$$V_N = \bigotimes_{i \in J_1 \cup J_2} V_{N,i}. \tag{6.14}$$

We look for the approximation of $\nu_N \in V_N$ to (6.6) with

$$\forall i \in J_1 \cup J_2, \forall v_{N,i} \in V_{N,i}: \quad \langle \nu_{N,i} - \mathcal{V}_G^0\mathcal{L}\nu_N, v_{N,i}\rangle_{\Gamma_i} = \langle \mathcal{V}_G^0\Sigma, v_{N,i}\rangle_{\Gamma_i}. \tag{6.15}$$

In practice, we use the quadrature (4.7) to approximate the inner product and the approximate solution ν_N on each Γ_i thus satisfies

$$\forall i \in J_1 \cup J_2, \forall v_{N,i} \in V_{N,i}: \quad \langle \nu_{N,i} - \mathcal{V}_G^0\mathcal{L}\nu_N, v_{N,i}\rangle_{\Gamma_i,t} = \langle \mathcal{V}_G^0\Sigma, v_{N,i}\rangle_{\Gamma_i,t}.$$

Denote by $\mathcal{M} = 3(N+1)^2(|J_1| + |J_2|)$ the number of degrees of freedom. The $\mathbb{R}^{\mathcal{M}}$-vector collecting all the coefficients $[y]_{\ell m}^k$ denoted by $\mathbf{\Lambda}$ yields the linear system

$$(\mathbf{D} - \mathbf{N})\mathbf{\Lambda} = \mathbf{F}, \tag{6.16}$$

where by (4.5), (4.6), the $\mathcal{M} \times \mathcal{M}$ diagonal matrix $\mathbf{D}$ is given by

$$[D_{ii}]_{\ell m,\ell m}^{11} = (2\ell+1)(\ell+1), \qquad [D_{ii}]_{\ell m,\ell m}^{22} = (2\ell+1)\ell, \qquad [D_{ii}]_{\ell m,\ell m}^{33} = (\ell+1)\ell,$$

and where $\mathbf{N}$ is a $\mathcal{M} \times \mathcal{M}$ matrix with coefficients

$$[N_{ij}]^{kk'}_{\ell m,l'm'} = \left\langle \mathcal{V}^0_G \mathcal{L}_j Y^{k'}_{\ell' m'}, Y^{ki}_{\ell m} \right\rangle_{\Gamma_i,t}. \tag{6.17}$$

The right hand side $\mathbf{F} \in \mathbb{R}^{\mathcal{M}}$ is given by its coefficients:

$$[F_i]^k_{\ell m} = \langle \mathcal{V}^0_G \Sigma, Y^{ki}_{\ell m} \rangle_{\Gamma_i,t}. \tag{6.18}$$

To derive the entries of the matrix (6.17), recall the spectral results in Lemma 5.1 so that we have on Γ_j

$$\begin{aligned} \mathcal{V}_j^{-1} Y^{k'j}_{\ell' m'} &= \frac{1}{r_j \tau^{k'j}_{\mathcal{V},\ell'}} Y^{k'j}_{\ell' m'}, & \mathcal{V}_j^{0^{-1}} Y^{k'j}_{\ell' m'} &= \frac{1}{r_j \tau^{k'0}_{\mathcal{V},\ell'}} Y^{k'j}_{\ell' m'}, \\ \mathcal{K}^*_j Y^{k'j}_{\ell' m'} &= \tau^{k'j}_{\mathcal{K}^*,\ell'} Y^{k'j}_{\ell' m'}, & \mathcal{K}^{*0}_j Y^{k'j}_{\ell' m'} &= \tau^{k'0}_{\mathcal{K}^*,\ell'} Y^{k'j}_{\ell' m'}, \end{aligned}$$

where $\tau^{k'j}_{\mathcal{V},\ell'}, \tau^{k'j}_{\mathcal{K}^*,\ell'}$ are eigenvalues of the single and double layer boundary operator with Lamé constants μ_j, λ_j given by (5.2), (5.5) respectively. Hence, we have

$$\mathcal{L}_j Y^{k'j}_{\ell' m'} = \frac{1}{r_j} C_{j\ell' k'} Y^{k'j}_{\ell' m'},$$

where the constant $C_{j\ell'k'}$ reads

$$C_{j\ell'k'} = \begin{cases} \dfrac{1}{2}\Big(\dfrac{1}{\tau^{k'0}_{\mathcal{K}^*,\ell'}} - \dfrac{1}{\tau^{k'j}_{\mathcal{V},l'}}\Big) + \Big(\dfrac{\tau^{k'0}_{\mathcal{K}^*,\ell'}}{\tau^{k'0}_{\mathcal{V},\ell'}} - \dfrac{\tau^{k'j}_{\mathcal{K}^*,\ell'}}{\tau^{k'j}_{\mathcal{V},\ell'}}\Big) & j \in J_1, \\ \dfrac{1/2 + s_i \tau^{k'0}_{\mathcal{K}^*,\ell'}}{\tau^{k'0}_{\mathcal{V},\ell'}} & j \in J_2. \end{cases}$$

Recall that s_i denotes the parameter defined by (6.3). Further, by (5.3), (5.4),

$$(\mathcal{S}^0_j Y^{k'}_{\ell' m'})(x) = \begin{cases} r_j \left[\underline{Y_{\ell' m'}}\Big(\dfrac{x - x_j}{|x - x_j|}\Big) A^{\text{in}}_{\mathcal{V},\ell'}\Big(\dfrac{x - x_j}{r_j}\Big)\right]_{k'} & |x - x_j| \le r_j, \\ r_j \left[\underline{Y_{\ell' m'}}\Big(\dfrac{x - x_j}{|x - x_j|}\Big) A^{\text{out}}_{\mathcal{V},\ell'}\Big(\dfrac{x - x_j}{r_j}\Big)\right]_{k'} & |x - x_j| > r_j, \end{cases}$$

where $[\cdot]_{k'}$ denotes the k'-th column of the obtained matrix. Hence, the coefficient $[N_{ij}]^{kk'}_{\ell m,\ell' m'}$ of the matrix $\mathbf{N}$ (6.17) reads

$$\begin{aligned} [N_{ij}]^{kk'}_{\ell m,\ell' m'} &= \langle \mathcal{V}^0_G \mathcal{L}_j Y^{k'}_{\ell' m'}, Y^{ki}_{\ell m} \rangle_{\Gamma_i,t} \\ &= C_{j\ell'k'} \sum_{t=1}^{T_g} w_t Y^{ki}_{\ell m}(s_t) \mathcal{S}^0_j Y^{k'}_{\ell' m'}(x_i + r_i s_t) \\ &= C_{j\ell'k'} \sum_{t=1}^{T_g} w_t Y^{ki}_{\ell m}(s_t) \left[\underline{Y_{\ell' m'}}\Big(\frac{y^t_{ij}}{|y^t_{ij}|}\Big) A^{f(j)}_{\mathcal{V},\ell'}\Big(\frac{y^t_{ij}}{r_j}\Big)\right]_{k'}, \end{aligned}$$

where $y^t_{ij} = x_i + r_i s_t - x_j$ and $f(j)$ takes the value

$$f(j) = \begin{cases} in & j = M+1, \\ out & else. \end{cases}$$

In particular, when $i = j$, we use (4.5) for the exact value of the inner product and obtain:

$$[N_{ii}]^{kk'}_{\ell m,\ell' m'} = \sum_{l'=0}^{N} \sum_{m'=-\ell'}^{\ell'} C_{i\ell' k'} \tau^{k',0}_{\mathcal{V},\ell} \langle Y^{ki}_{\ell m}, Y^{k'i}_{\ell' m'} \rangle_{\Gamma_i}.$$

Finally, the right-hand side vector $\mathbf{F}$ with coefficient $[F_i]^k_{\ell m}$ is given by:

$$[F_i]^k_{\ell m} = \sum_{\ell'=0}^{N} \sum_{m'=-\ell'}^{\ell'} \sum_{k'=1}^{3} \Bigg(r_i [\Sigma_i]^{k'}_{\ell' m'} \tau^{k'0}_{\mathcal{V},\ell'} \langle Y^{k'i}_{\ell' m'}, Y^{ki}_{\ell m} \rangle_{\Gamma_i} + \sum_{\substack{j \in J_1 \cup J_2 \\ j \neq i}} r_j \sum_{t=1}^{T_g} [\Sigma_j]^{k'}_{\ell' m'} w_t Y^{ki}_{\ell m}(s_t) \Big[\underline{Y_{\ell' m'}} \Big(\frac{y^t_{ij}}{|y^t_{ij}|} \Big) A^{f(j)}_{\mathcal{V},\ell'} \Big(\frac{y^t_{ij}}{r_j} \Big) \Big]_{k'} \Bigg), \quad i \in J_1 \cup J_2$$

where $[\Sigma_i]^k_{\ell m} \in \mathbb{R}$ is determined by the right-hand side vector Σ in (6.12) :

$$\Sigma|_{\Gamma_i} = \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \sum_{k=1}^{3} [\Sigma_i]^k_{\ell m} Y^{ki}_{\ell m}(x).$$

Remark 6.2. A similar physical model called "Finite Cluster Model" was considered in [22] in where an algebraic formulation is derived through the use of M2L-operators (using the fast multipole method terminology). However, with the jump relations given in (3.10), the algebraic formulation of the "Finite Cluster Model" can be proven to be equivalent to the discrete integral formulation (6.15) presented above.

It shall be noted that the mathematical framework introduced here through the use of layer potentials and boundary operators in order to derive an integral equation (6.12) defining the exact solution and and the subsequent introduction of the Galerkin discretization (6.15) allows a mathematical analysis which will be subject of an upcoming work.

7. Numerical Tests

For all following computations, we chose the number of Lebedev integration points T_g such that, for given N, products of two scalar spherical harmonics of maximal degree N, thus spherical harmonics of degree $2N$, are integrated exactly. The number of points can then be extracted from Table 4.1.

7.1. One sphere model

We start with a simple model involving only one single sphere whose solution can be computed analytically in order to assess the convergence of the method in this simple setting. For simplicity, let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere on which a stress tensor $\sigma \in H^{\frac{1}{2}}(\mathbb{S}^2)^3$ is imposed and let the Lamé constants be $\mu_0 = \lambda_0 = 1$. Let $\mathbf{n}$ be the outward pointing normal vector with respect to the unit sphere $\mathbb{S}^2$. The solution $u \in V_0(B)^3$ to the problem

$$\begin{aligned} &-\operatorname{div}\big(2e(u) + \operatorname{Tr} e(u) \operatorname{Id}\big) = 0, && \text{in } B, \\ &\mathcal{T}_{\mathbf{n}}^{-} u = \sigma, && \text{on } \mathbb{S}^2, \end{aligned}$$

reads

$$u(x) = \big(\mathcal{S}(1 + \mathcal{K}^*)^{-1}\sigma\big)(x), \quad \forall x \in \mathbb{R}^3, \tag{7.1}$$

with $\mathcal{S}$ being the single layer potential (3.1) and $\mathcal{K}^*$ the adjoint of the double layer boundary operator (3.5). For the given tensor σ, if there exists an integer $\ell_{\rm ex}$ such that we can expand σ by means of vector spherical harmonics up to order $\ell_{\rm ex}$:

$$\sigma(x) = \sum_{\ell=1}^{\ell_{\rm ex}} \sum_{m=-\ell}^{\ell} \sum_{k=1}^{3} [\Sigma]_{\ell m}^{k} Y_{\ell m}^{k}(x),$$

then the exact solution restricted to the sphere $\Lambda_{\rm ex} = u|_{\mathbb{S}^2}$ is given explicitly by

$$\Lambda_{\rm ex} = \sum_{\ell=1}^{\ell_{\rm ex}} \sum_{m=-\ell}^{\ell} \sum_{k=1}^{3} \frac{\tau_{\mathcal{V},\ell}^{k}}{\frac{1}{2} + \tau_{\mathcal{K}^*,\ell}^{k}} [\Sigma]_{\ell m}^{k} Y_{\ell m}^{k}(x), \tag{7.2}$$

where $\tau_{\mathcal{V},\ell}^{k}, \tau_{\mathcal{K}^*,\ell}^{k}$ are the eigenvalues of the single layer boundary operator and the adjoint double layer boundary operator given by (5.2) and (5.5) resp. They only concern in computing (7.2) is that the denominator tends to zero if $\tau_{\mathcal{K}^*,\ell}^{k}$ approaches $-1/2$. Recall that according to Remark 5.1, the only possible eigenvectors of $\mathcal{K}^*$ with the eigenvalue $-1/2$ are $W_{1,-1}, W_{1,0}, W_{1,1}$. According to Appendix A, we see that they are constant and parallel to the cartesian basis $\mathbf{e}_i, i = 1, 2, 3$. To ensure that (7.2) is well defined and these modes avoided, we simply impose that

$$\int_{\mathbb{S}^2} \sigma = 0.$$

We consider the following four cases:

- Case 1. $\sigma = -(x, y, z)^\top$.
- Case 2. $\sigma = -(x, 0, 0)^\top$.
- Case 3. $\sigma = -(x^7, y^7, z^7)^\top$.
- Case 4. $\sigma = -\big(\sin(2\pi x), \sin(2\pi y), \sin(2\pi z)\big)^\top$.

Table 7.1 lists the L^2 norm of the numerical solution on the unit sphere $||\Lambda_s||_{L^2}$ in each case with different degrees of vector spherical harmonics and the relative error is defined by

$$\texttt{Re} = \frac{||\Lambda_s - \Lambda_{\rm ex}||_{L^2}}{||\Lambda_{\rm ex}||_{L^2}}. \tag{7.3}$$

The exact solutions $\Lambda_{\rm ex}$ in the first three cases are exactly computed by the (7.1), (7.2) while the in the last cases, the "exact" solution is obtained by taking a large enough $\ell_{\rm ex}$ (in this case $\ell_{\rm ex} = 50$).

Table 7.1: Relative error Re of the approximation to the one-sphere model.

N	Case 1	Case 2	Case 3	Case 4
2	0	0	1.985e-01	5.375e-01
5	0	0	4.020e-03	6.797e-02
8	0	0	4.796e-09	1.370e-04
11	0	0	0	7.892e-13
14	0	0	0	7.097e-13

7.2. Convergence with respect to the degree N

We study now the convergence of the error measured in the L^2 norm with respect to the degree N of the vector spherical harmonics. Using the notation introduced in Section 6, we test a model with $M = 2$ and

$$J_1 = \{1\}, \quad J_2 = \{2, 3\}.$$

Let Γ_1 be the sphere centered at $(1, 0, 0)$ respectively with radius 0.1 and Γ_2 centered at $(-1, 0, 0)$ with radius 0.1 while Γ_3 is centered at $(0, 0, 0)$ with radius 2. The inclusion Ω_1 is filled with a medium represented by the Lamé constants $\mu_1 = 10, \lambda_1 = 10$ while the background domain Ω_0 uses $\mu_0 = 1, \lambda_0 = 1$ as Lamé parameters.

The interface condition $[\![\mathcal{T}u]\!] = 0$ is imposed on Γ_1 while the spheres Γ_2, Γ_3 are subjected to a stress tensor $\mathcal{T}^+_{\mathbf{n}_2} u = -\sigma_2$ respectively $\mathcal{T}^+_{\mathbf{n}_3} u = \sigma_3$. Table 7.2 illustrates the parameters of the above geometry configuration.

Table 7.2: Geometric configuration of the case study for convergence with respect to the degree of vector spherical harmonics N involving three spheres.

Set	Sphere	Center	Radius	Lamé constants	Stress tensor	Transmission
J_1	Γ_1	$(1,0,0)$	0.1	$\mu_1 = 10, \lambda_1 = 10$	———	$[\![\mathcal{T}u]\!] = 0$
J_2	Γ_2	$(-1,0,0)$	0.1	———	$\mathcal{T}^+_{\mathbf{n}_2} u = -\sigma_2$	———
J_2	Γ_3	$(0,0,0)$	2	$\mu_0 = 1, \lambda_0 = 1$	$\mathcal{T}^+_{\mathbf{n}_3} u = \sigma_3$	———

We now test two cases to see the relation between the relative error and the degree of the spherical harmonics for different kinds of imposed stress tensors σ_2, σ_3:

1. The two stress tensors are set to be smooth functions such that

$$\begin{aligned} \mathcal{T}^+_{\mathbf{n}_2} u &= -\sigma_2 = 10\big(\sin(2\pi(x+1)), \sin(2\pi(y+1)), \sin(2\pi(z+1))\big)^\top, \\ \mathcal{T}^+_{\mathbf{n}_3} u &= \sigma_3 = -2\big(\sin(2\pi x), \sin(2\pi y), \sin(2\pi z)\big)^\top. \end{aligned} \tag{7.4}$$

2. The stress tensors are set to be piecewise smooth such that

$$\mathcal{T}^+_{\mathbf{n}_2} u = -\sigma_2 = \begin{cases} (0.2, 0, 0)^\top & x \geq 0, \\ -(0.2, 0, 0)^\top & x < 0, \end{cases} \tag{7.5}$$

and

$$\mathcal{T}^+_{\mathbf{n}_3} u = \sigma_3 = \begin{cases} (1, 0, 0)^\top & x \geq 0, \\ -(1, 0, 0)^\top & x < 0. \end{cases} \tag{7.6}$$

We compute the "exact" solution to the problem with a large degree of vector spherical harmonics ($N_{\text{ex}} = 50$) for both cases. In the Fig. 7.1, we illustrate the log of the relative error (7.3) with respect to the degree N of spherical harmonics of the two tests above. We observe exponential convergence in the first case where the given stress tensor is regular. In the second case, the situation is less clear as an initial pre-asymptotic is followed by a very fast convergence and the asymptotic regime is not yet reached, but the absolute error is already very small.

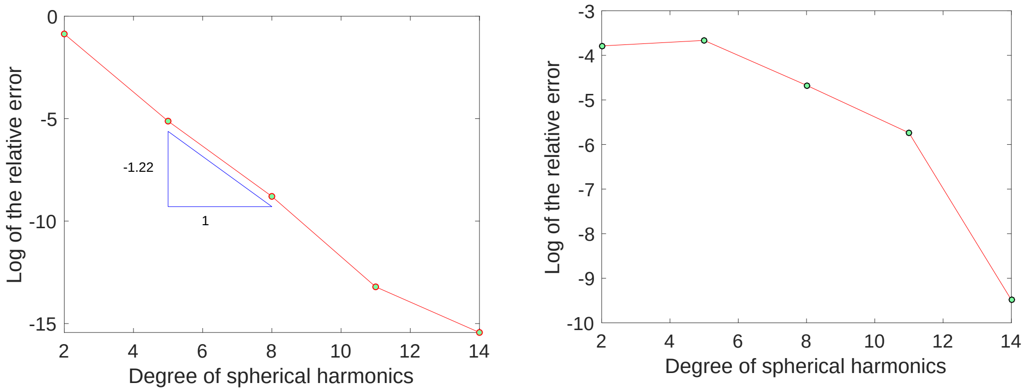

Fig. 7.1. The L^2 error of the approximation with respect to the degree of spherical harmonics for the test cases (7.4) (left) and (7.5)–(7.6) (right).

7.3. Computational cost

Next, we study the computational cost of our numerical method by considering an "embedded model" with M inclusions by increasing the value of M. We do the following test with Matlab on an iMac with a 2,7 GHz Intel Core i5 processor.

We consider a case where a stress tensor $\mathcal{T}^0 \mathbf{n}_{M+1}{}^+ u = -\frac{1}{R}(x,y,z)^\top$ is imposed on a origin-centered sphere with radius R, denoted by $\mathbb{S}^2_R$. Inclusions are taken to be all the spheres with radii 0.1, centered on a cubic lattice $\mathbb{Z}^3$ and which are contained in $\mathbb{S}^2_R$. We increase the number of inclusions M by increasing the value of the radius R of the big sphere. Table 7.3 lists the number of spheres with respect to the radius R that grows of course cubically.

Table 7.3: Number of spheres w.r.t the radius R.

Radius of the big sphere	1	2	3	4	5
Number of total spheres	2	28	94	252	486

We fill each small inclusion with a medium associated with the Lamé constants $\mu_i = 10, \lambda_i = 10$, $i = 1, ..., M$ and take the transmission condition $[\![\mathcal{T} u]\!] = 0$ on each embedded sphere. Further, the Lamé constants of the background domain are fixed to be $\mu_0 = 1, \lambda_0 = 1$. The degree of the vector spherical harmonics is chosen to be $N = 3$. Further, we stop the iterative solver of the linear system when the residual is smaller than 10^{-6}. Fig. 7.2 illustrates the computed elastic deformation of the model computed when $R = 3$. The colorcode represents the modulus of the displacement.

We report the result of the computational time in Fig. 7.3 which illustrates that the computational cost with respect to the number of spheres grows as $O(M^2)$. This is the normal scaling for an integral equation involving M spheres, whose iterative solver requires a number of iterations that is independent of M which we observe.

7.4. The effect of an inclusion

We now consider the unit ball B_1 which contains an additional inclusion Ω_1 in form of a sphere centered at the origin with radius 0.5. We study how the displacement on the unit sphere

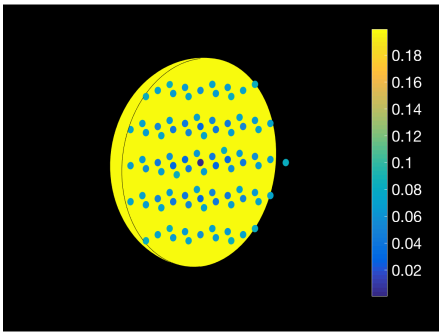

Fig. 7.2. The elastic deformation of the embedded model when $R = 3$. The colorcode represents the modulus of the displacement.

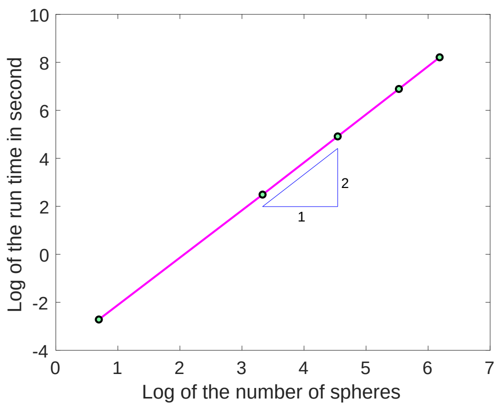

Fig. 7.3. The run time in second with respect to the number M of spheres (log-log scale).

$\mathbb{S}^2$ is influenced by the compressibility of the small inclusions Ω_1. We will use the Poisson's ratio as the parameter defined by (1.2) describing the compressibility of a substance.

Let the stress tensor $-(x, y, z)^\top$ be imposed on the unit sphere $\mathbb{S}^2$ and fix the shear modulus of the exterior shell $\Omega_0 = B_1 \backslash \overline{\Omega}_1$ to be $\mu_0 = 1$ and the shear modulus $\mu_1 = 1$ for the inclusion Ω_1. We test several cases where the the exterior shell and the inclusion are associated to different Poisson's ratio ν_0, ν_1. Recall that $\nu_0, \nu_1 \in (-1, 1/2)$ according to the definition, we have the limit values of the first Lamé parameter λ_1:

$$\lambda_1 \xrightarrow[\nu_1 \to -1]{} -\frac{2}{3}, \quad \lambda_1 \xrightarrow[\nu_1 \to \frac{1}{2}]{} \infty.$$

In Fig. 7.4, we plot the L^2 norm of the displacement on the unit sphere by letting the Poisson's ratio ν_1 vary in $[-1, 0.4998]$ with different given values of Poisson's ratio ν_0 of the background domain Ω_0. In Fig. 7.5, we give two solutions with different Poisson's ratios: the left solution is obtained by setting $\nu_1 = 0.4995$ while the other is obtained by setting $\nu = -1$, both embedded into a background domain with $\nu_0 = 1/6$.

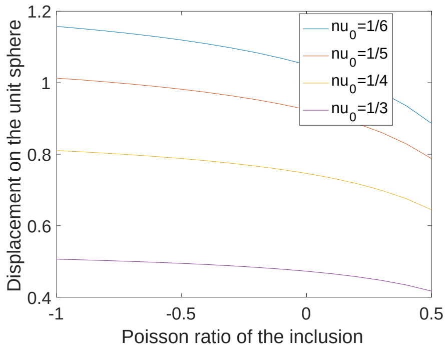

Fig. 7.4. The L^2 norm of the solution on the unit sphere with respect to the Poisson's ratio ν_1 of the inclusion. Each curve is obtained by a given background Poisson's ratio ν_0 specified by the legend.

Fig. 7.5. Two solutions with the Poisson's ratio $\nu_0 = \frac{1}{6}$. The left solution is obtained for $\nu_1 = 0.4998$ while the left for $\nu_1 = -1$. The colorcode represents the modulus of the displacement.

8. Conclusion

In this article, we have discussed the layer potentials and their corresponding integral operators on arbitrary bounded domains with Lipschitz boundary in the context of isotropic elasticity. We proved jump relations of layer potentials and the invertibility of the single layer boundary operator. In the particular case where the body is a unit ball, we present spectral properties of the boundary operators on the base of the vector spherical harmonics. We then derived a second-kind integral equation for isotropic elastic materials with spherical inclusions that was then discretized by employing the vector spherical harmonics as basis functions and exploiting the spectral properties to enhance efficiency of the discretization. In the last part, we effect some numerical tests to asses the properties of the method: the accuracy with respect to the degree of the vector spherical harmonics and the complexity of the computational cost with respect to the number of spherical inclusions. We also used the method to explore how the deformation of the elastic material is effected by the value of the Poisson's ratio.

Appendix A: Computation of the First Few Vector Spherical Harmonics

We first start considering the table of vector spherical harmonics up to the second order as listed below:

$$\ell = 0: \qquad Y_{0,0} = \frac{1}{2}\sqrt{\frac{1}{\pi}},$$

$$\ell = 1: \qquad Y_{1,-1} = \sqrt{\frac{3}{4\pi}}y, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}z, \qquad Y_{1,1} = \sqrt{\frac{3}{4\pi}}x,$$

$$\ell = 2: \qquad Y_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}xy, \qquad Y_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}yz, \qquad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(-x^2 - y^2 + 2z^2),$$

$$Y_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}xz, \qquad Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{\pi}}(x^2 - y^2).$$

This gives first the obvious result that

$$\ell = 0: \qquad V_{0,0} = -\frac{1}{2}\sqrt{\frac{1}{\pi}}(x, y, z)^\top \qquad \text{and} \qquad W_{00} = X_{00} = 0.$$

Using the definition of the surface gradient (4.3), we obtain

$$\begin{aligned}
\ell = 1: \qquad & \nabla_{\mathrm{s}} Y_{1,-1} = \sqrt{\frac{3}{4\pi}}\big((0,1,0) - y(x,y,z)\big)^\top, \\
& \nabla_{\mathrm{s}} Y_{1,0} = \sqrt{\frac{3}{4\pi}}\big((0,0,1) - z(x,y,z)\big)^\top, \\
& \nabla_{\mathrm{s}} Y_{1,1} = \sqrt{\frac{3}{4\pi}}\big((1,0,0) - x(x,y,z)\big)^\top, \\
\ell = 2: \qquad & \nabla_{\mathrm{s}} Y_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((y,x,0) - 2xy(x,y,z)\big)^\top, \\
& \nabla_{\mathrm{s}} Y_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((0,z,y) - 2yz(x,y,z)\big)^\top, \\
& \nabla_{\mathrm{s}} Y_{2,0} = \frac{1}{2}\sqrt{\frac{5}{\pi}}\big((-x,-y,2z) - (-x^2 - y^2 + 2z^2)(x,y,z)\big)^\top, \\
& \nabla_{\mathrm{s}} Y_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((z,0,x) - 2xz(x,y,z)\big)^\top, \\
& \nabla_{\mathrm{s}} Y_{2,2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((x,-y,0) - (x^2 - y^2)(x,y,z)\big)^\top.
\end{aligned}$$

The spherical harmonics $V_{\ell m}$ up to order 2 are then given as follows

$$\begin{aligned}
\ell = 0: \qquad & V_{0,0} = -\frac{1}{2}\sqrt{\frac{1}{\pi}}(x,y,z)^\top, \\
\ell = 1: \qquad & V_{1,-1} = \sqrt{\frac{3}{4\pi}}\big((0,1,0) - 2y(x,y,z)\big)^\top,
\end{aligned}$$

$$V_{1,0} = \sqrt{\frac{3}{4\pi}}\big((0,0,1) - 2z(x,y,z)\big)^\top,$$

$$V_{1,1} = \sqrt{\frac{3}{4\pi}}\big((1,0,0) - 2x(x,y,z)\big)^\top,$$

$$\ell = 2: \qquad V_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((y,x,0) - 5xy(x,y,z)\big)^\top,$$

$$V_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((0,z,y) - 5yz(x,y,z)\big)^\top,$$

$$V_{2,0} = \frac{1}{2}\sqrt{\frac{5}{\pi}}\big((-x,-y,2z) - \frac{5}{2}(-x^2 - y^2 + 2z^2)(x,y,z)\big)^\top,$$

$$V_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((z,0,x) - 5xz(x,y,z)\big)^\top,$$

$$V_{2,2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\big((x,-y,0) - \frac{5}{2}(x^2 - y^2)(x,y,z)\big)^\top.$$

The spherical harmonics $W_{\ell m}$ up to order 2 are given by

$$\ell = 0: \qquad W_{0,0} = 0,$$

$$\ell = 1: \qquad W_{1,-1} = \sqrt{\frac{3}{4\pi}}(0,1,0)^\top, \qquad W_{1,0} = \sqrt{\frac{3}{4\pi}}(0,0,1)^\top, \qquad W_{1,1} = \sqrt{\frac{3}{4\pi}}(1,0,0)^\top,$$

$$\ell = 2: \qquad W_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(y,x,0)^\top, \quad W_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(0,z,y)^\top, \quad W_{2,0} = \frac{1}{2}\sqrt{\frac{5}{\pi}}(-x,-y,2z)^\top,$$

$$W_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(z,0,x)^\top, \qquad W_{2,2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(x,-y,0)^\top.$$

And finally, the spherical harmonics $X_{\ell m}$ up to order 2 are given by

$$\ell = 0: \qquad X_{0,0} = 0,$$

$$\ell = 1: \qquad X_{1,-1} = \sqrt{\frac{3}{4\pi}}(-z,0,x)^\top, \qquad X_{1,0} = \sqrt{\frac{3}{4\pi}}(y,-x,0)^\top,$$

$$X_{1,1} = \sqrt{\frac{3}{4\pi}}(0,z,-y)^\top,$$

$$\ell = 2: \qquad X_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(-xz, yz, x^2 - y^2)^\top, \qquad X_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(y^2 - z^2, -xy, xz)^\top,$$

$$X_{2,0} = \frac{1}{2}\sqrt{\frac{5}{\pi}}(3yz, -3xz, 0)^\top, \qquad X_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(xy, z^2 - x^2, -yz)^\top,$$

$$X_{2,2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}(yz, xz, -2xy)^\top.$$

Appendix B: Entries of Matrices $A^{\mathrm{in}}_{\mathcal{D},\ell}$ and $A^{\mathrm{out}}_{\mathcal{D},\ell}$

The coefficients in $A^{\mathrm{in}}_{\mathcal{D},\ell}$ and $A^{\mathrm{out}}_{\mathcal{D},\ell}$ are given as follows:

$$a_{11}^{\mathrm{in},\mathcal{D},\ell} = -\frac{(\ell+2)\big((3\ell+2)\mu + (\ell+1)\lambda\big)\big((3\ell+1)\mu + \ell\lambda\big)}{(2\ell+3)(2\ell+1)^2\mu(2\mu+\lambda)},$$

$$a_{21,1}^{\text{in},\mathcal{D},\ell} = -\frac{(\ell+1)(\ell+2)\big((3\ell+2)\mu+(\ell+1)\lambda\big)(\mu+\lambda)}{2(2\ell+1)^2(2\mu+\lambda)},$$

$$a_{21,2}^{\text{in},\mathcal{D},\ell} = \frac{(\ell+1)(\ell+2)\big((3\ell+2)\mu+(\ell+1)\lambda\big)(\mu+\lambda)}{2(2\ell-1)(2\ell+1)\mu(2\mu+\lambda)},$$

$$a_{12}^{\text{in},\mathcal{D},\ell} = -\frac{\ell(\ell-1)(\mu+\lambda)\big((3\ell+1)\mu+\ell\lambda\big)}{(2\ell+3)(2\ell+1)^2\mu(2\mu+\lambda)},$$

$$a_{22,1}^{\text{in},\mathcal{D},\ell} = -\frac{\ell(\ell-1)(\ell+1)(\mu+\lambda)^2}{2(2\ell+1)^2\mu(2\mu+\lambda)},$$

$$a_{22,2}^{\text{in},\mathcal{D},\ell} = \frac{(\ell^3+24\ell^2-5\ell-8)\mu^2+2(\ell^3+6\ell^2-2\ell-2)\mu\lambda+(\ell^3-\ell)\lambda^2}{(2\ell-1)(2\ell+1)\mu(2\mu+\lambda)},$$

and

$$a_{11,1}^{\text{out},\mathcal{D},\text{l}} = \frac{(\ell+1)\big((\ell^2+10\ell+4)\mu^2+(2\ell^2+8\ell+2)\mu\lambda+(\ell^2+\ell)\lambda\big)}{2(2\ell+1)(2\ell+3)\mu(2\mu+\lambda)},$$

$$a_{11,2}^{\text{out},\mathcal{D},\ell} = -\frac{\ell(\ell+1)(\ell+2)(\mu+\lambda)^2}{2(2\ell+1)^2\mu(2\mu+\lambda)},$$

$$a_{21}^{\text{out},\mathcal{D},\ell} = \frac{(\ell+1)(\ell+2)(\mu+\lambda)\big((3\ell+2)\mu+(\ell+1)\lambda\big)}{(2\ell-1)(2\ell+1)^2\mu(2\mu+\lambda)},$$

$$a_{12,1}^{\text{out},\mathcal{D},\ell} = -\frac{\ell(\ell-1)(\mu+\lambda)\big((3\ell+1)\mu+l\lambda\big)}{2(2\ell+3)(2\ell+1)\mu(2\mu+\lambda^2)},$$

$$a_{12,2}^{\text{out},\mathcal{D},\ell} = \frac{\ell(\ell-1)\big((3\ell+1)\mu+\ell\lambda\big)(\mu+\lambda)}{2(2\ell+1)(2\ell+3)\mu(2\mu+\lambda)},$$

$$a_{22}^{\text{out},\mathcal{D},\ell} = \frac{(\ell-1)\big((3\ell+1)\mu+\ell\lambda\big)\big((3\ell+2)\mu+(\ell+1)\lambda\big)}{(2\ell-1)(2\ell+1)^2\mu(2\mu+\lambda)}.$$

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