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STABILIZED NONCONFORMING MIXED FINITE ELEMENT METHOD FOR LINEAR ELASTICITY ON RECTANGULAR OR CUBIC MESHES*

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Abstract

Based on the primal mixed variational formulation, a stabilized nonconforming mixed finite element method is proposed for the linear elasticity on rectangular and cubic meshes. Two kinds of penalty terms are introduced in the stabilized mixed formulation, which are the jump penalty term for the displacement and the divergence penalty term for the stress. We use the classical nonconforming rectangular and cubic elements for the displacement and the discontinuous piecewise polynomial space for the stress, where the discrete space for stress are carefully chosen to guarantee the well-posedness of discrete formulation. The stabilized mixed method is locking-free. The optimal convergence order is derived in the L^2-norm for stress and in the broken H^1-norm and L^2-norm for displacement. A numerical test is carried out to verify the optimal convergence of the stabilized method.

Mathematics subject classification: 65N15, 65N30.
Key words: Mixed finite element method, Nonconforming rectangular or cubic elements, Elasticity, Locking-free, Stabilization.

1. Introduction

For the linear elasticity problem, the pure displacement formulation is a very common one. However, the so-called locking phenomenon may appear when this formulation is numerically solved in the nearly incompressible or incompressible case. Therefore, some locking-free finite element methods (FEMs) for this pure displacement formulation have been developed, see e.g. [7,12,19,25,28,40]. For example, Brenner and Sung developed a locking-free nonconforming FEM in [7] by using the well-known Crouzeix-Raviart (CR) element [15]. Therein, the elasticity operator was split into the gradient part and the divergence part with appropriate coefficients. However, the CR element is not stable for the elasticity operator, since it does not fulfill the discrete Korn's inequality. To overcome this problem, Hansbo and Larson proposed a stabilized method for the CR element in [19]. In other words, they added a jump penalty term for the displacement to get a locking-free stabilized nonconforming FEM. The optimal convergence was

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proved in a mesh-dependent norm. Similar stabilization technique was also done in [18] for the nonconforming quadrilateral element [33].

By contrast, the mixed formulation, where both the stress and displacement are simultaneously solved, is preferable to the pure displacement one, since the stress is usually a physical quantity of primary interest. For the Hellinger-Reissner mixed formulation, there exist two possibilities to apply the derivatives to the displacement or stress, that lead to the primal mixed formulation and the dual mixed formulation. For the second one, it requires that the stress tensor element is symmetric with continuous normal components and satisfies the discrete inf-sup condition. it is very difficult to construct such elements, so most of these elements are quite complicated, especially in three dimensions, see e.g. [1, 3–5, 24]. We also mention the further development of stable conforming elements from the references [20–23], where a new class of stable conforming elements called the Hu-Zhang element is proposed. In order to use common elements, many stabilized methods are proposed for the dual mixed method, see [9–11, 27, 35, 38] and therein the references. For more discussions on other methods, such as nonconforming mixed FEMs, mixed FEMs with weaker symmetry and discontinuous Galerkin methods, we refer the readers to the related references mentioned above.

On the other hand, the primal mixed formulation is easy to discretize, because it does not need the continuity of stress and the discrete inf-sup condition can be easily satisfied in this case. However, this primal mixed formulation usually leads to the standard FEMs suffering locking as the pure displacement formulation, unless special techniques are applied. Based on the primal mixed formulation, Franca and Hughes proposed two classes of stabilized conforming mixed FEMs for elasticity, called circumventing Babuška-Brezzi condition method (CBB) and satisfying Babuška-Brezzi condition method (SBB), see [16]. For the CBB method, the discontinuous or continuous piecewise polynomial space can be used for the stress approximation and the continuous piecewise polynomial space for the displacement approximation. For the SBB method, only the discontinuous piecewise polynomial space is used for the stress approximation. The CBB method is convergent, provided the method is employed in the compressible case. The SBB method is uniformly convergent for the nearly incompressible or incompressible case. Recently, a stabilized nonconforming mixed FEM was shown to be locking-free in [37] where the displacement is approximated by the CR element and the stress by piecewise constants. Therein, the jump penalty term is added for the displacement to get the stability of the formulation. The uniform convergence was proved based on the fact that the discrete space for the CR element contains the subspace of continuous piecewise linear functions. We mention that for finite Lamé constant λ, the stabilized nonconforming mixed method in [37] is reduced to the one in [19] for the pure displacement formulation.

We mention that the assumed stress hybrid FEM on quadrilateral or hexahedral meshes is closely related to the stabilized methods proposed in [16, 37]. The pioneering work of assumed stress hybrid FEM was presented in [30], where the assumed stress field is assumed to satisfy the homogeneous equilibrium equations pointwisely. Then a hybrid stress quadrilateral element was constructed in [32], where the isoparametric bilinear element is used for the displacement approximation and a discontinuous piecewise polynomial space for the stress approximation. By eliminating the stress parameters at the element level, the hybrid stress method [32] finally leads to a symmetric and positive definite discrete system involving only displacement unknowns. The uniform convergence analysis for the hybrid stress method can be found in [26, 36]. For more works on hybrid FEMs for elasticity, please see [31, 39] and the references therein.

In this paper, we propose a stabilized nonconforming mixed FEM to discretize the primal mixed formulation on rectangular and cubic meshes. We use the nonconforming rectangular and cubic elements from [2] for the displacement and the discontinuous piecewise polynomial space for the stress, where the discrete space for stress needs to be carefully chosen to guarantee the well-posedness of discrete formulation. We give the weaker Z_h-ellipticity and weaker inf-sup condition (see (3.3)–(3.4)) for an unstable formulation without stabilization terms obtained by a direct discretization. Based on the weaker Z_h-ellipticity and weaker inf-sup condition, we introduce two kinds of penalty terms in the stabilized mixed formulation, which are the jump penalty term for the displacement and the divergence penalty term for the stress. Further we show the existence and uniqueness of the numerical solution to the stabilized method.

However, for the nonconforming rectangular and cubic elements from [2], the corresponding discrete space does not contain the subspace of continuous piecewise bilinear functions, such that the common technique used in [10, 37] is not applicable here. This brings a difficulty for analyzing the convergence of the nonconforming mixed FEM presented in this paper. To overcome this difficulty, by a direct way we first estimate the truncation error between the interpolation of the exact solution and the numerical solution, measured by some mesh-dependent norms. Then we obtain the optimal convergence in the L^2-norm for stress and in the broken H^1-norm and L^2-norm for displacement. We remark that the proposed method is uniformly convergent in the nearly incompressible or incompressible case, i.e., it is locking-free. We also remark that, due to the discontinuous stress approximation, the proposed method here has the local elimination property, i.e., the stress unknowns can be eliminated at the element level, as well as the hybrid stress method and other stabilized FEMs for the primal mixed formulation.

This paper is organized as follows. The elasticity model is presented in Section 2. In Section 3, the stabilized mixed method is proposed by using the nonconforming rectangular and cubic elements. The optimal convergence is shown to be uniform with respect to the Lamé constant λ in Section 4. Finally, a numerical test is carried out to verify the uniform convergence of the stabilized method in Section 5.

2. The Model Problem

Let Ω be a bounded domain in $\mathbb{R}^d$ with the dimension $d = 2, 3$. For any given open subset S of Ω, $(\cdot,\cdot)_S$ and $\|\cdot\|_S$ denote the usual integral inner product and the corresponding norm of $L^2(S)^d$, respectively. For a positive integer m, we shall use the common notation for the Sobolev spaces $H^m(S)$ and $H_0^m(S)$ with the corresponding norms $\|\cdot\|_{m,S}$ and $|\cdot|_{m,S}$. If $S = \Omega$, the subscript will be omitted. We use $H^{-1}(\Omega)$ to denote the dual space of $H_0^1(\Omega)$ with respect to the duality product $\langle\cdot,\cdot\rangle$. The dual norm in $H^{-1}(\Omega)$ is defined by

$$\|v\|_{-1} = \sup_{\phi\in H_0^1(\Omega)\backslash\{0\}} \frac{\langle v, \phi\rangle}{\|\phi\|_1}, \qquad v \in H^{-1}(\Omega).$$

Let σ and u be the symmetric stress tensor and the displacement field. As in [37], the corresponding spaces are defined as

$$\Sigma = \left\{\tau \in L^2(\Omega, \mathbb{S}); \int_\Omega \mathrm{tr}\tau \mathrm{d}s = 0\right\}, \quad V = H_0^1(\Omega)^d,$$

where $L^2(\Omega, \mathbb{S})$ stands for the space of symmetric tensors in $L^2(\Omega)^{d\times d}$.

Then the primal mixed variational formulation for the linear elasticity problem reads: find

$(\sigma, u) \in \Sigma \times V$ satisfying

$$\begin{cases} (\mathcal{A}\sigma, \tau) - (\tau, \varepsilon(u)) = 0, & \forall \tau \in \Sigma, \\ (\sigma, \varepsilon(v)) = (f, v), & \forall v \in V, \end{cases} \tag{2.1}$$

which is equivalent to the standard H^1-based variational formulation (pure displacement formulation). Here $(\mathcal{A}\sigma, \tau) = \int_\Omega \mathcal{A}\sigma : \tau \mathrm{d}x$ and $\sigma : \tau = \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij}\tau_{ij}$ denotes the Frobenius inner product of matrices σ and τ, f is the body force, $\mathcal{A}$ is the compliance tensor, and $\varepsilon(u) = \big(\varepsilon_{ij}(u)\big)_{d\times d}$ is the linearized strain tensor defined by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For an isotropic elastic material, the compliance tensor has the following expression

$$\mathcal{A}\sigma = \frac{1}{2\mu}\Big(\sigma - \frac{\lambda}{d\lambda + 2\mu}\mathrm{tr}\sigma\delta\Big).$$

Here $\delta = (\delta_{ij})_{d\times d}$ is the identity tensor, λ and μ are the Lamé constants such that $\mu \in [\mu_1, \mu_2]$ with $0 < \mu_1 < \mu_2$ and $\lambda \in (0, \infty]$. When λ is very large or infinite, materials are said to be nearly incompressible or incompressible. If the domain Ω is convex or its boundary is enough smooth, the regularity estimate

$$\|\sigma\|_1 + \|u\|_2 \le C\|f\| \tag{2.2}$$

holds, cf. [4, 7, 14, 17, 29].

For convenience, the symbol C with or without subscripts is used to denote a generic positive constant, possibly different at different occurrences, which is independent of the Lamé constant λ and the mesh size. For clearness, we also use other symbols (e.g. α, β, γ) to denote such a generic positive constant.

We conclude this section with showing the following stability conditions (see [37]) in order to ensure the existence and uniqueness of the solution to (2.1).

Z-ellipticity There exists a positive constant α such that

$$\alpha\|\tau\|^2 \le (\mathcal{A}\tau, \tau), \quad \forall \tau \in Z, \tag{2.3}$$

where $Z = \{\tau \in \Sigma;\ (\tau, \varepsilon(v)) = 0, \forall v \in V\}$.

Inf-sup condition There exists a positive constant β such that, for any $v \in V$,

$$\sup_{\tau\in\Sigma\setminus\{0\}} \frac{(\tau, \varepsilon(v))}{\|\tau\|} \ge \beta|v|_1. \tag{2.4}$$

3. The Stabilized Method

First we introduce a family of shape-regular rectangular (or cubic) meshes $\{\mathcal{T}_h\}$ of Ω. For a given mesh $\mathcal{T}_h$, we denote the set of all the edges (faces) in $\mathcal{T}_h$ by $\mathcal{E}_h$ and the set of interior edges (faces) by $\mathcal{E}_h^{\mathrm{int}}$, respectively. For an element $K \in \mathcal{T}_h$ and an edge (or face) $E \in \mathcal{E}_h$, let h_K be the diameter of K and h_E the diameter of E. Especially, we set $h = \max_{K\in\mathcal{T}_h}\{h_K\}$. n_K always denotes the exterior unit normal vector along the boundary of K. For each edge (face), we define its unit normal vector denoted by n, whose orientation is chosen arbitrarily but fixed for interior edges (faces) and coinciding with the exterior normal of Ω for boundary edges (faces). For a function v and an interior edge (face) E shared by the elements K and L in $\mathcal{T}_h$ such that $n|_E$ points from K to L, we define the jump operator $[\![\cdot]\!]$ through E by

$$[\![v]\!]|_E = (v|_K)|_E - (v|_L)|_E.$$

For any boundary edge (face) E, set $[\![v]\!]|_E = v|_E$. Similar notation is used for the jump of vector-valued functions, where the jump is taken componentwisely.

For any given element K and nonnegative integer m, $\mathbb{P}_m(K)$ denotes the polynomial space of order m or less. On $\mathcal{T}_h$, we introduce a pair of finite element spaces for $d = 2$

$$\Sigma_h = \left\{ \tau_h \in \Sigma;\ \tau_h|_K \in \begin{pmatrix} \mathbb{P}_1(x_1) & \mathbb{P}_0(K) \\ \mathbb{P}_0(K) & \mathbb{P}_1(x_2) \end{pmatrix},\ \forall K \in \mathcal{T}_h \right\},$$

$$V_h = \left\{ v_h \in L^2(\Omega)^d;\ v_h|_K \in \begin{pmatrix} \widetilde{\mathbb{P}}_1(x_1) \\ \widetilde{\mathbb{P}}_1(x_2) \end{pmatrix},\ \forall K \in \mathcal{T}_h,\ \text{and}\ \int_E [\![v_h]\!] \mathrm{d}s = 0,\ \forall E \in \mathcal{E}_h \right\},$$

where $\mathbb{P}_1(x_i) = \operatorname{span}\{1, x_i\}$, $\widetilde{\mathbb{P}}_1(x_i) = \mathbb{P}_1(K) \oplus \operatorname{span}\{x_i^2\}$, and for $d = 3$

$$\Sigma_h = \left\{ \tau_h \in \Sigma;\ \tau_h|_K \in \begin{pmatrix} \mathbb{P}_2(x_1) & \mathbb{P}_0(K) & \mathbb{P}_0(K) \\ \mathbb{P}_0(K) & \mathbb{P}_2(x_2) & \mathbb{P}_0(K) \\ \mathbb{P}_0(K) & \mathbb{P}_0(K) & \mathbb{P}_2(x_3) \end{pmatrix},\ \forall K \in \mathcal{T}_h \right\},$$

$$V_h = \left\{ v_h \in L^2(\Omega)^d;\ v_h|_K \in \begin{pmatrix} \widetilde{\mathbb{P}}_2(x_1) \\ \widetilde{\mathbb{P}}_2(x_2) \\ \widetilde{\mathbb{P}}_2(x_3) \end{pmatrix},\ \forall K \in \mathcal{T}_h,\ \text{and}\ \int_E [\![v_h]\!] \mathrm{d}s = 0,\ \forall E \in \mathcal{E}_h \right\},$$

where $\mathbb{P}_2(x_i) = \operatorname{span}\{1, x_i, x_i^2\}$, $\widetilde{\mathbb{P}}_2(x_i) = \mathbb{P}_1(K) \oplus \operatorname{span}\{x_i^2, x_i^3\}$. Here we remark that, for any $v_h \in V_h$, it yeilds $\varepsilon_h(v_h) \in \Sigma_h$ where $\varepsilon_h(v_h)$ is defined locally on each $K \in \mathcal{T}_h$, i.e. $\varepsilon_h(v_h)|_K = \varepsilon(v_h|_K)$.

We define the interpolation operator Π_h for the space Σ_h by setting, for $d = 2$

$$\int_K \Pi_h \tau : \phi \mathrm{d}x = \int_K \tau : \phi \mathrm{d}x, \qquad \forall \phi \in \begin{pmatrix} \mathbb{P}_1(x_1) & \mathbb{P}_0(K) \\ \mathbb{P}_0(K) & \mathbb{P}_1(x_2) \end{pmatrix}, \quad \forall K \in \mathcal{T}_h,$$

and for $d = 3$

$$\int_K \Pi_h \tau : \phi \mathrm{d}x = \int_K \tau : \phi \mathrm{d}x, \qquad \forall \phi \in \begin{pmatrix} \mathbb{P}_2(x_1) & \mathbb{P}_0(K) & \mathbb{P}_0(K) \\ \mathbb{P}_0(K) & \mathbb{P}_2(x_2) & \mathbb{P}_0(K) \\ \mathbb{P}_0(K) & \mathbb{P}_0(K) & \mathbb{P}_2(x_3) \end{pmatrix}, \quad \forall K \in \mathcal{T}_h,$$

where $\tau \in L^2(\Omega, \mathbb{S})$.

The interpolation operator I_h for the space V_h is defined by

$$\int_E I_h v \mathrm{d}s = \int_E v \mathrm{d}s, \qquad \forall E \in \mathcal{E}_h,$$

where $v \in H^1(\Omega)^d$. Here we remark that the interpolation operator I_h for V_h has been shown to be well-posed in [2], and similar arguments can also show the well-posedness of I_h for the three dimensional case.

For $\tau \in \Sigma$ and $v \in V$, it obviously holds that $\Pi_h \tau \in \Sigma_h$ and $I_h v \in V_h$. Furthermore, by the standard scaling arguments we have the following local interpolation error estimates:

$$|\tau - \Pi_h \tau|_{m,K} \le C h_K^{l-m} |\tau|_{l,K}, \quad m \le l,\ m = 0, 1,\ l = 0, 1, \tag{3.1}$$

$$|v - I_h v|_{m,K} \le C h_K^{l-m} |v|_{l,K}, \quad m = 0, 1,\ l = 1, 2. \tag{3.2}$$

If we directly discretize the primal mixed variational problem (2.1) with the finite dimensional spaces defined above, it leads to an unstable formulation:

$$\begin{cases} (\mathcal{A}\sigma_h, \tau_h) - (\tau_h, \varepsilon_h(u_h)) = 0, & \forall \tau_h \in \Sigma_h, \\ (\sigma_h, \varepsilon_h(v_h)) = (f, v_h), & \forall v_h \in V_h. \end{cases}$$

In fact, we can only verify the following two conditions:

Weaker Z_h-ellipticity There exist two positive constants α (that is the same as the one in (2.3)) and α_1 such that

$$\alpha \|\tau_h\|^2 \leq (\mathcal{A}\tau_h, \tau_h) + \alpha_1 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\mathrm{div}\tau_h\|_K^2 \right)^{\frac{1}{2}}, \quad \forall \tau_h \in Z_h, \tag{3.3}$$

where $Z_h = \{\tau_h \in \Sigma_h;\ (\tau_h, \varepsilon_h(v_h)) = 0, \forall v_h \in V_h\}$.

Weaker inf-sup condition There exist two positive constants β_1 and β_2 such that, for any $v_h \in V_h$,

$$\sup_{\tau_h \in \Sigma_h \backslash \{0\}} \frac{(\tau_h, \varepsilon_h(v_h))}{\|\tau_h\|} \geq \beta_1 \|v_h\|_{1,h} - \beta_2 \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \|[\![P_E^1 v_h]\!]\|_E^2 \right)^{\frac{1}{2}}, \tag{3.4}$$

where the operator P_E^1 is the orthogonal projection from $L^2(E)$ onto $\mathbb{P}_1(E)$ and $\|\cdot\|_{1,h}$ is the piecewise H^1-norm on the discrete space V_h defined by

$$\|v_h\|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |v_h|_{1,K}^2 \right)^{\frac{1}{2}}.$$

The proof of the weaker inf-sup condition is identical with that in [37], where it needs to apply the Korn's inequality on piecewise H^1 vector space (cf. [6]) to the space V_h. Next we verify the weaker Z_h-ellipticity. To this end, we introduce a fundamental inequality

$$\alpha \|\tau\|^2 \leq (\mathcal{A}\tau, \tau) + \|\mathrm{div}\tau\|_{-1}^2, \quad \forall \tau \in \Sigma, \tag{3.5}$$

see the references [4, 8, 37] for the details. For any given $\tau_h \in Z_h$ and $v \in H_0^1(\Omega)^d$, we have

$$\begin{aligned} (\tau_h, \varepsilon(v)) &= (\tau_h, \varepsilon_h(v - I_h v)) \\ &= \sum_{K \in \mathcal{T}_h} \left(\int_{\partial K} \tau_h n_K \cdot (v - I_h v) \mathrm{d}s - (\mathrm{div}\tau_h, v - I_h v)_K \right) \\ &= - \sum_{K \in \mathcal{T}_h} (\mathrm{div}\tau_h, v - I_h v)_K, \end{aligned}$$

where we have used the fact that $\tau_h n_K$ is constant on each edge (face) of element K and the property of interpolation operator I_h. Then the interpolation error estimates (3.2) imply

$$(\tau_h, \varepsilon(v)) \leq \alpha_1 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\mathrm{div}\tau_h\|_K^2 \right)^{\frac{1}{2}} \|v\|_1,$$

which, together with the inequality (3.5) and definition of the dual norm $\|\cdot\|_{-1}$, leads to the weaker Z_h-ellipticity.

Based on the weaker Z_h-ellipticity (3.3) and weaker inf-sup condition (3.4), we propose a stabilized mixed finite element method: find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ satisfying

$$\mathcal{Q}((\sigma_h, u_h), (\tau_h, v_h)) = F(\tau_h, v_h), \quad \forall (\tau_h, v_h) \in \Sigma_h \times V_h, \tag{3.6}$$

where the bilinear form is defined by

$$\begin{aligned}\mathcal{Q}((\sigma_h, u_h), (\tau_h, v_h)) &= (\mathcal{A}\sigma_h, \tau_h) + \gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 (\mathrm{div}\sigma_h, \mathrm{div}\tau_h)_K - (\tau_h, \varepsilon_h(u_h)) \\ &\quad + (\sigma_h, \varepsilon_h(v_h)) + \gamma_2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E [\![u_h]\!] \cdot [\![v_h]\!] \mathrm{d}s, \\ F(\tau_h, v_h) &= -\gamma_1 \sum_K h_K^2 (f, \mathrm{div}\tau_h)_K + (f, v_h),\end{aligned}$$

and γ_1, γ_2 are positive constants to be properly chosen. Since $V_h \not\subseteq H_0^1(\Omega)^d$, the mixed method (3.6) is nonconforming.

We conclude this section by showing the existence and uniqueness of the solution to (3.6). To this end, we assume $f = 0$, then it is sufficient to show that the system of homogeneous equations has only zero solution. By setting $\tau_h = \sigma_h$ and $v_h = u_h$, we have

$$\mathcal{Q}((\sigma_h, u_h), (\sigma_h, u_h)) = (\mathcal{A}\sigma_h, \sigma_h) + \gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 \|\mathrm{div}\sigma_h\|_K^2 + \gamma_2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \|[\![u_h]\!]\|_E^2 = 0,$$

which leads to

$$(\mathcal{A}\sigma_h, \sigma_h) = 0, \quad \|\mathrm{div}\sigma_h\|_K = 0,\ \forall K \in \mathcal{T}_h \quad \text{and } [\![u_h]\!]|_E = 0,\ \forall E \in \mathcal{E}_h.$$

Due to the fact that $(\sigma_h, \varepsilon_h(v_h)) = 0$ for all $v_h \in V_h$, i.e., $\sigma_h \in Z_h$, the weaker Z_h-ellipticity (3.3) implies $\sigma_h = 0$. Thus, setting $v_h = 0$ in (3.6) leads to

$$(\tau_h, \varepsilon_h(u_h)) = 0, \quad \forall \tau_h \in \Sigma_h,$$

which, together with the weaker inf-sup condition (3.4) and the fact that the jump of $P_E^1 u_h$ on each edge E vanishes, yields $u_h = 0$.

4. Error Analysis

Let (σ, u) be the exact solution to problem (2.1) and (σ_h, u_h) the approximate solution satisfying (3.6). We first present the estimates on the truncation error $\Pi_h\sigma - \sigma_h$ and $I_h u - u_h$, where $\Pi_h\sigma$ and $I_h u$ are the interpolation functions of σ and u. In the proofs of the following lemmas, we will frequently use the trace inequality [13]

$$\|v\|_E^2 \le C(h_K^{-1}\|v\|_K^2 + h_K|v|_{1,K}^2), \qquad \forall E \subset \partial K,\ \forall K \in \mathcal{T}_h. \tag{4.1}$$

Lemma 4.1. *Under the condition of the regularity estimate* (2.2)*, it holds*

$$\begin{aligned}&(\mathcal{A}(\Pi_h\sigma - \sigma_h), \Pi_h\sigma - \sigma_h) + \gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 \|\mathrm{div}(\Pi_h\sigma - \sigma_h)\|_K^2 + \gamma_2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \|[\![I_h u - u_h]\!]\|_E^2 \\ &\le C h^2 \|f\|^2.\end{aligned}$$

Proof. Let $(\tau_h, v_h) \in \Sigma_h \times V_h$ be arbitrary. The equation (2.1) implies

$$\mathcal{A}\sigma = \varepsilon(u), \quad -\mathrm{div}\sigma = f.$$

Thus by integration by parts, we get

$$(\mathcal{A}\sigma,\tau_h)-(\tau_h,\varepsilon(u))+(\sigma,\varepsilon_h(v_h))=(f,v_h)+\sum_{E\in\mathcal{E}_h}\int_E \sigma n\cdot[\![v_h]\!]\mathrm{d}s.$$

Then by using the definition of Π_h and the fact that $\varepsilon_h(v_h)\in\Sigma_h$, we get

$$(\mathcal{A}(\Pi_h\sigma),\tau_h)-(\tau_h,\varepsilon(u))+(\Pi_h\sigma,\varepsilon_h(v_h))=(f,v_h)+\sum_{E\in\mathcal{E}_h}\int_E \sigma n\cdot[\![v_h]\!]\mathrm{d}s. \tag{4.2}$$

Subtracting (3.6) from (4.2) leads to the error equation

$$\begin{aligned}(\mathcal{A}(\Pi_h\sigma-\sigma_h),\tau_h)&-\gamma_1\sum_{K\in\mathcal{T}_h}h_K^2(\mathrm{div}\sigma_h,\mathrm{div}\tau_h)_K-(\tau_h,\varepsilon_h(u-u_h))+(\Pi_h\sigma-\sigma_h,\varepsilon_h(v_h))\\ &-\gamma_2\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E[\![u_h]\!]\cdot[\![v_h]\!]\mathrm{d}s=\gamma_1\sum_{K\in\mathcal{T}_h}h_K^2(f,\mathrm{div}\tau_h)_K+\sum_{E\in\mathcal{E}_h}\int_E\sigma n\cdot[\![v_h]\!]\mathrm{d}s. \end{aligned}\tag{4.3}$$

Then we obtain

$$\begin{aligned}&\mathcal{Q}((\Pi_h\sigma-\sigma_h,I_hu-u_h),(\tau_h,v_h))\\ =&(\tau_h,\varepsilon_h(u-I_hu))+\gamma_1\sum_{K\in\mathcal{T}_h}h_K^2(\mathrm{div}(\Pi_h\sigma-\sigma),\mathrm{div}\tau_h)_K\\ &+\gamma_2\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E[\![I_hu]\!]\cdot[\![v_h]\!]\mathrm{d}s+\sum_{E\in\mathcal{E}_h}\int_E\sigma n\cdot[\![v_h]\!]\mathrm{d}s. \end{aligned}\tag{4.4}$$

Next we start to estimate each term on the right-hand side of (4.4). For the first one, we recall the fact that $\tau_h n_K$ is constant on each edge (or face) of element K and the property of interpolation operator I_h and obtain

$$(\tau_h,\varepsilon_h(u-I_hu))=\sum_{K\in\mathcal{T}_h}(\mathrm{div}\tau_h,u-I_hu)_K,$$

which, together with the interpolation estimate (3.2) and the regularity estimate (2.2), leads to

$$|(\tau_h,\varepsilon_h(u-I_hu))|\le Ch\|f\|\Big(\sum_{K\in\mathcal{T}_h}h_K^2\|\mathrm{div}\tau_h\|_K^2\Big)^{1/2}. \tag{4.5}$$

By using the interpolation estimates (3.1)-(3.2), regularity estimate (2.2) and trace inequality (4.1), the second and third terms on the right-hand side of (4.4) can be estimated as follows

$$\gamma_1\Big|\sum_{K\in\mathcal{T}_h}h_K^2(\mathrm{div}(\Pi_h\sigma-\sigma),\mathrm{div}\tau_h)_K\Big|\le Ch\|f\|\Big(\sum_{K\in\mathcal{T}_h}h_K^2\|\mathrm{div}\tau_h\|_K^2\Big)^{1/2}, \tag{4.6}$$

$$\begin{aligned}\gamma_2\Big|\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E[\![I_hu]\!]\cdot[\![v_h]\!]\mathrm{d}s\Big|&=\gamma_2\Big|\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E[\![u-I_hu]\!]\cdot[\![v_h]\!]\mathrm{d}s\Big|\\ &\le\gamma_2\sum_{E\in\mathcal{E}_h}h_E^{-1}\|[\![u-I_hu]\!]\|_E\|[\![v_h]\!]\|_E\le Ch\|f\|\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\|[\![v_h]\!]\|_E^2\Big)^{1/2}. \end{aligned}\tag{4.7}$$

For the last term, we observe that the jump average of v_h on each edge (or face) vanishes and obtain

$$\sum_{E\in\mathcal{E}_h}\int_E\sigma n\cdot[\![v_h]\!]\mathrm{d}s=\sum_{E\in\mathcal{E}_h}\int_E(\sigma-P_E^0\sigma)n\cdot[\![v_h]\!]\mathrm{d}s\le\sum_{E\in\mathcal{E}_h}\|\sigma-P_E^0\sigma\|_E\|[\![v_h]\!]\|_E,$$

where the operator P_E^0 is the orthogonal projection from $L^2(E)$ onto $\mathbb{P}_0(E)$. Then we use the trace inequality (4.1), Poincaré/Friedrichs inequality [34] and regularity estimate (2.2) to obtain

$$\Big|\sum_{E\in\mathcal{E}_h}\int_E \sigma n\cdot [\![v_h]\!]\mathrm{d}s\Big| \le Ch\|f\|\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![v_h]\!]\|_E^2\Big)^{1/2}. \tag{4.8}$$

Since (τ_h, v_h) is arbitrary, we set $(\tau_h, v_h) = (\Pi_h\sigma - \sigma_h, I_h u - u_h)$ and substitute (4.5)-(4.8) into (4.4), which yields the desired result. $\square$

Lemma 4.2. *Under the condition of the regularity estimate* (2.2)*, it holds*

$$\|\Pi_h\sigma - \sigma_h\| \le Ch\|f\|. \tag{4.9}$$

Proof. The fundamental inequality (3.5) implies

$$\alpha\|\Pi_h\sigma - \sigma_h\|^2 \le (\mathcal{A}(\Pi_h\sigma - \sigma_h), \Pi_h\sigma - \sigma_h) + \|\mathrm{div}(\Pi_h\sigma - \sigma_h)\|_{-1}^2. \tag{4.10}$$

For any $v \in H_0^1(\Omega)$, we have

$$\begin{aligned}\langle \mathrm{div}(\Pi_h\sigma - \sigma_h), v\rangle &= -(\Pi_h\sigma - \sigma_h, \varepsilon(v))\\ &= -(\Pi_h\sigma - \sigma_h, \varepsilon_h(v - I_h v)) - (\Pi_h\sigma - \sigma_h, \varepsilon_h(I_h v))\\ &= \sum_{K\in\mathcal{T}_h}(\mathrm{div}(\Pi_h\sigma - \sigma_h), v - I_h v)_K - (\Pi_h\sigma - \sigma_h, \varepsilon_h(I_h v))\\ &\le \sum_{K\in\mathcal{T}_h}\|\mathrm{div}(\Pi_h\sigma - \sigma_h)\|_K\|v - I_h v\|_K - (\Pi_h\sigma - \sigma_h, \varepsilon_h(I_h v)),\end{aligned} \tag{4.11}$$

where we have used the fact that $(\Pi_h\sigma - \sigma_h)n_K$ is constant on each edge (or face) of element K and the property of interpolation operator I_h. By using the interpolation estimate (3.2) and Lemma 4.1, we get

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\|\mathrm{div}(\Pi_h\sigma - \sigma_h)\|_K\|v - I_h v\|_K\\ \le& C\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}(\Pi_h\sigma - \sigma_h)\|_K^2\Big)^{1/2}|v|_1 \le Ch\|f\||v|_1.\end{aligned} \tag{4.12}$$

In order to estimate the second term in (4.11), we let $\tau_h = 0$ and $v_h = I_h v$ in equation (4.4) and obtain

$$\begin{aligned}&(\Pi_h\sigma - \sigma_h, \varepsilon_h(I_h v))\\ =&\gamma_2\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\![I_h u]\!]\cdot[\![I_h v]\!]\mathrm{d}s - \gamma_2\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\![I_h u - u_h]\!]\cdot[\![I_h v]\!]\mathrm{d}s + \sum_{E\in\mathcal{E}_h}\int_E \sigma n\cdot[\![I_h v]\!]\mathrm{d}s.\end{aligned} \tag{4.13}$$

We use the trace inequality (4.1), interpolation estimate (3.2) and regularity estimate (2.2) to obtain

$$\begin{aligned}&\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\![I_h u]\!]\cdot[\![I_h v]\!]\mathrm{d}s\\ =&\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\![u - I_h u]\!]\cdot[\![v - I_h v]\!]\mathrm{d}s \le Ch|u|_2|v|_1 \le Ch\|f\||v|_1.\end{aligned} \tag{4.14}$$

By Lemma 4.1, the trace inequality (4.1) and interpolation estimate (3.2), we estimate the second term in (4.13)

$$\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\![I_h u-u_h]\!]\cdot[\![I_h v]\!]\mathrm{d}s=\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\![I_h u-u_h]\!]\cdot[\![I_h v-v]\!]\mathrm{d}s$$

$$\leq\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![I_h u-u_h]\!]\|_E^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![v-I_h v]\!]\|_E^2\Big)^{1/2}\leq Ch\|f\||v|_1. \tag{4.15}$$

For the last term in (4.13), we use the estimate (4.8), trace inequality (4.1) and interpolation estimate (3.2) to obtain

$$\sum_{E\in\mathcal{E}_h}\int_E \sigma n\cdot[\![I_h v]\!]\mathrm{d}s\leq Ch\|f\|\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![I_h v]\!]\|_E^2\Big)^{1/2}$$

$$=Ch\|f\|\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![v-I_h v]\!]\|_E^2\Big)^{1/2}\leq Ch\|f\||v|_1. \tag{4.16}$$

Combining (4.13)-(4.16), we obtain

$$|(\Pi_h\sigma-\sigma_h,\varepsilon_h(I_h v))|\leq Ch\|f\||v|_1. \tag{4.17}$$

Substituting estimates (4.12) and (4.17) into (4.11), we get

$$\langle\mathrm{div}(\Pi_h\sigma-\sigma_h),v\rangle\leq Ch\|f\||v|_1.$$

Due to the arbitrariness of v, we get

$$\|\mathrm{div}(\Pi_h\sigma-\sigma_h)\|_{-1}\leq Ch\|f\|,$$

which, together with inequality (4.10) and Lemma 4.1, concludes the proof. □

Lemma 4.3. *Under the condition of the regularity estimate* (2.2)*, it holds*

$$\|I_h u-u_h\|_{1,h}\leq Ch\|f\|. \tag{4.18}$$

Proof. The weaker inf-sup condition (3.4) implies

$$\beta_1\|I_h u-u_h\|_{1,h}$$
$$\leq\sup_{\tau_h\in\Sigma_h\setminus\{0\}}\frac{(\tau_h,\varepsilon_h(I_h u-u_h))}{\|\tau_h\|}+\beta_2\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![I_h u-u_h]\!]\|_E^2\Big)^{1/2}. \tag{4.19}$$

Next we estimate the first term in the above inequality. To this end, we let $v_h=0$ in equation (4.4) and obtain

$$(\tau_h,\varepsilon_h(I_h u-u_h))=(\mathcal{A}(\Pi_h\sigma-\sigma_h),\tau_h)-(\tau_h,\varepsilon_h(u-I_h u)) \tag{4.20}$$
$$+\gamma_1\sum_{K\in\mathcal{T}_h} h_K^2\Big((\mathrm{div}(\Pi_h\sigma-\sigma_h),\mathrm{div}\tau_h)_K+(\mathrm{div}(\sigma-\Pi_h\sigma),\mathrm{div}\tau_h)_K\Big).$$

Recalling the interpolation estimates (3.1)-(3.2), regularity estimate (2.2), Lemma 4.1 and the inverse inequality, we obtain

$$|(\mathcal{A}(\Pi_h\sigma-\sigma_h),\tau_h)|\leq\|\mathcal{A}^{\frac12}(\Pi_h\sigma-\sigma_h)\|\|\mathcal{A}^{\frac12}\tau_h\|\leq Ch\|f\|\|\tau_h\|, \tag{4.21}$$

$$|(\tau_h,\varepsilon_h(u-I_h u))|\leq\|u-I_h u\|_{1,h}\|\tau_h\|\leq Ch\|f\|\|\tau_h\|, \tag{4.22}$$

and

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} h_K^2\Big((\mathrm{div}(\Pi_h\sigma-\sigma_h),\mathrm{div}\tau_h)_K+(\mathrm{div}(\sigma-\Pi_h\sigma),\mathrm{div}\tau_h)_K\Big)\\&\le C_1\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}(\Pi_h\sigma-\sigma_h)\|_K^2\Big)^{1/2}\|\tau_h\|+C_2\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}(\sigma-\Pi_h\sigma)\|_K^2\Big)^{1/2}\|\tau_h\|\\&\le Ch\|f\|\|\tau_h\|. \end{aligned}\tag{4.23}$$

Combining (4.20)-(4.23), we obtain

$$\sup_{\tau_h\in\Sigma_h\backslash\{0\}}\frac{(\tau_h,\varepsilon_h(I_hu-u_h))}{\|\tau_h\|}\le Ch\|f\|,$$

which, together with inequality (4.19) and Lemma 4.1, concludes the proof. □

With the above preparations, we start to estimate the approximation errors $\|\sigma-\sigma_h\|$ and $\|u-u_h\|_{1,h}$. By Lemmas 4.2-4.3 and interpolation estimates (3.1)-(3.2), it is easy to see the following convergence result.

Theorem 4.1. *Under the condition of the regularity estimate* (2.2)*, it holds*

$$\|\sigma-\sigma_h\|+\|u-u_h\|_{1,h}\le Ch\|f\|.$$

With the help of Theorem 4.1, we further derive the error estimate for $u-u_h$ in L^2-norm.

Theorem 4.2. *Under the condition of the regularity estimate* (2.2)*, it holds*

$$\|u-u_h\|\le Ch^2\|f\|.$$

Proof. The proof relies on the usual duality argument. Let (σ^*,u^*) be the solution to the auxiliary problem

$$\left\{\begin{array}{ll}\mathcal{A}\sigma^*-\varepsilon(u^*)=0, & \text{in }\Omega,\\ -\mathrm{div}\sigma^*=u-u_h, & \text{in }\Omega,\\ u^*=0, & \text{on }\partial\Omega.\end{array}\right.\tag{4.24}$$

From the regularity estimate (2.2), it immediately follows that

$$\|\sigma^*\|_1+\|u^*\|_2\le C\|u-u_h\|.\tag{4.25}$$

On both sides of the second equation of (4.24), we take an L^2-inner product with respect to $u-u_h$ and obtain

$$\begin{aligned}\|u-u_h\|^2&=-(\mathrm{div}\sigma^*,u-u_h)\\&=(\sigma^*,\varepsilon_h(u-u_h))+\sum_{E\in\mathcal{E}_h}\int_E\sigma^*n\cdot[\![u_h]\!]\mathrm{d}s\\&=(\sigma^*-\Pi_h\sigma^*,\varepsilon_h(u-u_h))+(\Pi_h\sigma^*,\varepsilon_h(u-u_h))\\&\quad+\sum_{E\in\mathcal{E}_h}\int_E(\sigma^*-P_E^0\sigma^*)n\cdot[\![(u_h-u)-P_E^0(u_h-u)]\!]\mathrm{d}s.\end{aligned}\tag{4.26}$$

For the second term above, the equations (2.1), (3.6) and (4.24) imply that

$$
\begin{aligned}
&(\Pi_h\sigma^*, \varepsilon_h(u-u_h))\\
&= (\mathcal{A}(\sigma-\sigma_h), \Pi_h\sigma^*-\sigma^*) + (\mathcal{A}(\sigma-\sigma_h), \sigma^*) - \gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 (f+\mathrm{div}\sigma_h, \mathrm{div}\Pi_h\sigma^*)_K\\
&= (\mathcal{A}(\sigma-\sigma_h), \Pi_h\sigma^*-\sigma^*) + (\sigma-\sigma_h, \varepsilon(u^*)) - \gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 (f+\mathrm{div}\sigma_h, \mathrm{div}\Pi_h\sigma^*)_K\\
&= (\mathcal{A}(\sigma-\sigma_h), \Pi_h\sigma^*-\sigma^*) + (\sigma-\sigma_h, \varepsilon_h(u^*-I_hu^*)) + (\sigma-\sigma_h, \varepsilon_h(I_hu^*))\\
&\quad -\gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 (f+\mathrm{div}\sigma_h, \mathrm{div}\Pi_h\sigma^*)_K. \qquad (4.27)
\end{aligned}
$$

Combining (4.26) and (4.27), it yields

$$
\begin{aligned}
&\|u-u_h\|^2\\
&= (\sigma^*-\Pi_h\sigma^*, \varepsilon_h(u-u_h)) + (\mathcal{A}(\sigma-\sigma_h), \Pi_h\sigma^*-\sigma^*) + (\sigma-\sigma_h, \varepsilon_h(u^*-I_hu^*))\\
&\quad +(\sigma-\sigma_h, \varepsilon_h(I_hu^*)) - \gamma_1 \sum_{K\in\mathcal{T}_h} h_K^2 (f+\mathrm{div}\sigma_h, \mathrm{div}\Pi_h\sigma^*)_K\\
&\quad + \sum_{E\in\mathcal{E}_h} \int_E (\sigma^*-P_E^0\sigma^*)n \cdot [\![(u_h-u)-P_E^0(u_h-u)]\!] \mathrm{d}s.
\end{aligned}
$$

Next, we start to bound each term on the right side of the above equation. For the first three terms, we use the interpolation error estimates (3.1) and (3.2) to obtain

$$
\begin{aligned}
&(\sigma^*-\Pi_h\sigma^*, \varepsilon_h(u-u_h)) \le Ch\|u-u_h\|_{1,h}|\sigma^*|_1,\\
&(\mathcal{A}(\sigma-\sigma_h), \Pi_h\sigma^*-\sigma^*) \le Ch\|\sigma-\sigma_h\||\sigma^*|_1,\\
&(\sigma-\sigma_h, \varepsilon_h(u^*-I_hu^*)) \le Ch\|\sigma-\sigma_h\||u^*|_2.
\end{aligned}
$$

In order to estimate the fourth term, we use the trace inequality (4.1), Poincaré/Friedrichs inequality [34] to obtain

$$
\|(u-u_h)-P_E^0(u-u_h)\|_E \le Ch_K^{\frac{1}{2}}|u-u_h|_{1,K}, \quad E\subset\partial K.
$$

In error equation (4.3), we set $\tau_h = 0$ and $v_h = I_hu^*$ to obtain

$$
\begin{aligned}
&(\sigma-\sigma_h, \varepsilon_h(I_hu^*))\\
&= \sum_{E\in\mathcal{E}_h} \int_E \sigma n \cdot [\![I_hu^*]\!] \mathrm{d}s + \gamma_2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E [\![u_h]\!] \cdot [\![I_hu^*]\!] \mathrm{d}s\\
&= \sum_{E\in\mathcal{E}_h} \int_E (\sigma-P_E^0\sigma) n \cdot [\![I_hu^*]\!] \mathrm{d}s + \gamma_2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E [\![(u_h-u)-P_E^0(u_h-u)]\!] \cdot [\![I_hu^*]\!] \mathrm{d}s\\
&= \sum_{E\in\mathcal{E}_h} \int_E (\sigma-P_E^0\sigma) n \cdot [\![I_hu^*-u^*]\!] \mathrm{d}s + \gamma_2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E [\![(u_h-u)-P_E^0(u_h-u)]\!] \cdot [\![I_hu^*-u^*]\!] \mathrm{d}s.
\end{aligned}
$$

Thus by using the trace inequality (4.1), Poincaré/Friedrichs inequality [34] and interpolation estimate (3.2), the fourth term can be estimated as follows

$$
(\sigma-\sigma_h, \varepsilon_h(I_hu^*)) \le C(h^2|\sigma|_1 + h\|u-u_h\|_{1,h})|u^*|_2.
$$

For the fifth term, we use the interpolation error estimate (3.1) and inverse inequality to obtain

$$\begin{aligned}\|f+\mathrm{div}\sigma_h\|_K &\le \|\mathrm{div}(\sigma-\Pi_h\sigma)\|_K+\|\mathrm{div}(\Pi_h\sigma-\sigma_h)\|_K\\ &\le \|\mathrm{div}(\sigma-\Pi_h\sigma)\|_K+Ch_K^{-1}\|\Pi_h\sigma-\sigma_h\|_K\\ &\le \|\mathrm{div}(\sigma-\Pi_h\sigma)\|_K+Ch_K^{-1}(\|\Pi_h\sigma-\sigma\|_K+\|\sigma-\sigma_h\|_K)\\ &\le C(|\sigma|_{1,K}+h_K^{-1}\|\sigma-\sigma_h\|_K).\end{aligned}$$

Thus the fifth term can be bounded by

$$\gamma_1\sum_{K\in\mathcal{T}_h}h_K^2(f+\mathrm{div}\sigma_h,\mathrm{div}\Pi_h\sigma^*)_K\le C(h^2|\sigma|_1+h\|\sigma-\sigma_h\|)|\sigma^*|_1.$$

For the last term, we use the trace inequality (4.1) and Poincaré/Friedrichs inequality [34] to obtain

$$\sum_{E\in\mathcal{E}_h}\int_E(\sigma^*-P_E^0\sigma^*)n\cdot[\![(u_h-u)-P_E^0(u_h-u)]\!]\mathrm{d}s\le Ch\|u-u_h\|_{1,h}|\sigma^*|_1.$$

Collecting all the above results, we obtain

$$\|u-u_h\|^2\le Ch(h|\sigma|_1+\|\sigma-\sigma_h\|+\|u-u_h\|_{1,h})(|\sigma^*|_1+|u^*|_2).$$

Combining Theorem 4.1 and the regularity estimates (2.2) and (4.25) concludes the proof. $\square$

5. Numerical Results

In this section, we verify the uniform convergence of the stabilized nonconforming mixed FEM by an example. In this example, we set $\Omega=(-1,1)\times(-1,1)$ and $\mu=1$, and take the right-hand side as

$$f(x_1,x_2)=\begin{pmatrix}-8(x_1+x_2)\big((3x_1x_2-2)(x_1^2+x_2^2)+5(x_1x_2-1)^2-2x_1^2x_2^2\big)\\ -8(x_1-x_2)\big((3x_1x_2+2)(x_1^2+x_2^2)-5(x_1x_2+1)^2+2x_1^2x_2^2\big)\end{pmatrix}.$$

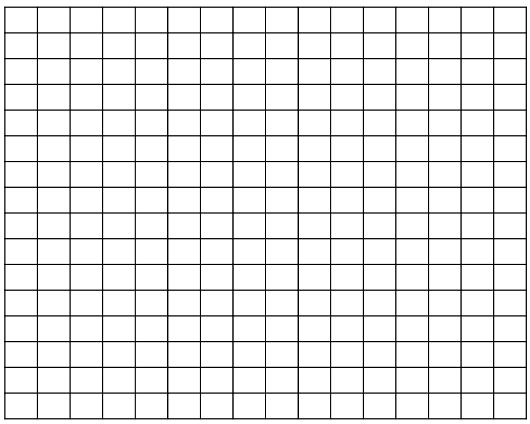

Fig. 5.1. The uniform mesh with $h=2^{-3}$.

Table 5.1: The errors for $\lambda = 1$ and different h.

h	$\|u-u_h\|$	$\|u-u_h\|_{1,h}$	$\|\sigma-\sigma_h\|$
2^0	6.722403E-01	2.171532E+00	2.395017E+00
2^{-1}	2.952428E-01	1.777865E+00	8.198931E-01
2^{-2}	6.870318E-02	8.111081E-01	2.772765E-01
2^{-3}	1.685703E-02	4.138433E-01	8.622177E-02
2^{-4}	4.194904E-03	2.116063E-01	3.188727E-02
2^{-5}	1.046885E-03	1.071030E-01	1.444655E-02
Rate	2.00	0.98	1.14

Table 5.2: The errors for $\lambda = 10$ and different h.

h	$\|u-u_h\|$	$\|u-u_h\|_{1,h}$	$\|\sigma-\sigma_h\|$
2^0	1.001529E+00	3.235462E+00	3.464014E+00
2^{-1}	2.994347E-01	1.845460E+00	9.852803E-01
2^{-2}	6.482320E-02	7.914514E-01	2.936679E-01
2^{-3}	1.539517E-02	3.884958E-01	8.764870E-02
2^{-4}	3.791173E-03	1.960038E-01	3.140186E-02
2^{-5}	9.431509E-04	9.877749E-02	1.394900E-02
Rate	2.01	0.99	1.17

Table 5.3: The errors for $\lambda = 10^9$ and different h.

h	$\|u-u_h\|$	$\|u-u_h\|_{1,h}$	$\|\sigma-\sigma_h\|$
2^0	1.122275E+00	3.625587E+00	3.848302E+00
2^{-1}	3.173446E-01	1.991295E+00	1.185754E+00
2^{-2}	6.980646E-02	8.787252E-01	3.769842E-01
2^{-3}	1.677049E-02	4.342520E-01	1.126229E-01
2^{-4}	4.151051E-03	2.190600E-01	3.684028E-02
2^{-5}	1.034788E-03	1.102995E-01	1.484875E-02
Rate	2.00	0.99	1.31

It can be checked that the corresponding exact solution of (2.1) is

$$u(x_1,x_2) = \begin{pmatrix} -4x_2(1-x_2^2)(1-x_1^2)^2 \\ 4x_1(1-x_1^2)(1-x_2^2)^2 \end{pmatrix} + \frac{1}{2+\lambda}\begin{pmatrix} -4x_1(1-x_1^2)(1-x_2^2)^2 \\ -4x_2(1-x_2^2)(1-x_1^2)^2 \end{pmatrix}.$$

The stress σ can be obtained by $\sigma = \mathcal{C}\varepsilon(u)$.

We use the stabilized mixed formulation (3.6) with $\gamma_1 = 0.05$, $\gamma_2 = 1$ to simultaneously approximate the stress and the displacement on a sequence of uniform $n \times n$ meshes with $h = 1/n$, as shown in Fig. 5.1.

We first let the Lamé constant λ be taken to be $1, 10$ and 10^9, respectively, in order to present the optimal convergence of the stabilized mixed method. The corresponding numerical results for different values of λ are given in Tables 5.1-5.3, where the convergence rate is computed by using the numerical results over the last two meshes. From Tables 5.1-5.3, we see that the convergence rate of $\|u-u_h\|$, $\|u-u_h\|_{1,h}$ and $\|\sigma-\sigma_h\|$ is $\mathcal{O}(h^2)$, $\mathcal{O}(h)$ and $\mathcal{O}(h)$, respectively. We notice that the optimal convergence rate is still maintained even in the nearly incompressible case ($\lambda = 10^9$). These numerical results confirm the theoretical results in Theorems 4.1-4.2.

Table 5.4: The errors for $h = 2^{-5}$ and different λ.

λ	$\|u-u_h\|$	$\|u-u_h\|_{1,h}$	$\|\sigma-\sigma_h\|$
10^0	1.046885E-03	1.071030E-01	1.444655E-02
10^1	9.431509E-04	9.877749E-02	1.394900E-02
10^2	1.020211E-03	1.086649E-01	1.466796E-02
10^3	1.033254E-03	1.101301E-01	1.482908E-02
10^4	1.034634E-03	1.102825E-01	1.484676E-02
10^5	1.034773E-03	1.102978E-01	1.484854E-02
10^6	1.034787E-03	1.102993E-01	1.484872E-02
10^7	1.034788E-03	1.102995E-01	1.484874E-02
10^8	1.034788E-03	1.102995E-01	1.484874E-02
10^9	1.034788E-03	1.102995E-01	1.484875E-02

Next we test the robustness of the stabilized mixed method with respect to λ whose value is changed from 1 to 10^9 on a fixed mesh with $h = 2^{-5}$. The corresponding numerical results for different values of λ are given in Table 5.4. From that, we observe that the errors are hardly affected by the choice of λ, which confirms that the stabilized mixed method is robust with respect to λ, i.e. it is locking-free.

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