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A TWO-GRID FINITE ELEMENT APPROXIMATION FOR NONLINEAR TIME FRACTIONAL TWO-TERM MIXED SUB-DIFFUSION AND DIFFUSION WAVE EQUATIONS*

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Abstract

In this paper, we develop a two-grid method (TGM) based on the FEM for 2D nonlinear time fractional two-term mixed sub-diffusion and diffusion wave equations. A two-grid algorithm is proposed for solving the nonlinear system, which consists of two steps: a nonlinear FE system is solved on a coarse grid, then the linearized FE system is solved on the fine grid by Newton iteration based on the coarse solution. The fully discrete numerical approximation is analyzed, where the Galerkin finite element method for the space derivatives and the finite difference scheme for the time Caputo derivative with order $\alpha \in (1,2)$ and $\alpha_1 \in (0,1)$. Numerical stability and optimal error estimate $O(h^{r+1} + H^{2r+2} + \tau^{\min\{3-\alpha,2-\alpha_1\}})$ in L^2-norm are presented for two-grid scheme, where t, H and h are the time step size, coarse grid mesh size and fine grid mesh size, respectively. Finally, numerical experiments are provided to confirm our theoretical results and effectiveness of the proposed algorithm.

Mathematics subject classification: 65N30, 65M60, 26A33.
Key words: Two-grid method, Finite element method, Nonlinear time fractional mixed sub-diffusion and diffusion-wave equations, L1-CN scheme, Stability and convergence.

1. Introduction

Fractional partial differential equations (FPDEs) have been the focus of many studies due to their frequent appearance in various fields such as physics, chemistry, biology and engineering [4, 8, 14, 28]. Compared with integer-order PDEs, they are better choices for describing some phenomena or processes with diffusion, relaxation vibrations, memory, hereditary and long-range interaction in viscoelasticity, electrochemistry and fluid mechanics.

In this paper, we consider the numerical solution of the following nonlinear time-fractional two-term mixed sub-diffusion and diffusion wave equations:

$$\begin{cases} {}_0^C D_t^{\alpha_1} u(\boldsymbol{x},t) + {}_0^C D_t^{\alpha} u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = g(\boldsymbol{u}), & (\boldsymbol{x},t) \in \Omega \times (0,T], \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad u_t(\boldsymbol{x},0) = \tilde{u}_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal region with boundary $\partial\Omega$, $\boldsymbol{x}$=[x,y], $g(\cdot)$ is twice continuously differentiable.

The Caputo fractional derivative ${}_0^C D_t^{\alpha_1}$, ${}_0^C D_t^{\alpha}$ are defined by ([14])

$$ {}_0^C D_t^{\alpha_1} u(\boldsymbol{x},t) = \frac{1}{\Gamma(1-\alpha_1)} \int_0^t \frac{\partial u(\boldsymbol{x},s)}{\partial s} \frac{ds}{(t-s)^{\alpha_1}}, \qquad 0<\alpha_1<1. \tag{1.2} $$

$$ {}_0^C D_t^{\alpha} u(\boldsymbol{x},t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\boldsymbol{x},s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \qquad 1<\alpha<2. \tag{1.3} $$

The nonlinear time-fractional two-term mixed sub-diffusion and diffusion wave equations have been widely applied in depicting the anomalous diffusion process, modeling viscoelastic damping, capturing power-law frequency dependence [6,7,10,25,27]. There have existed some jobs in the area of numerical analysis for linear fractional mixed sub-diffusion and diffusion wave equations. Sun [20] proposed a new analytical technique of the L-type difference schemes for time fractional mixed sub-diffusion and diffusion wave equations with the min$\{2-\alpha_1, 3-\alpha\}$ order accuracy in a discrete H^1-norm in time and the second order accuracy in space, respectively. A Galerkin finite element method combined with L1-CN time discrete scheme for finding the numerical solution of two-term time-fractional mixed diffusion and diffusion wave equations in [22]. By use of anisotropic linear triangle finite element method, Zhao [27] presented a fully-discrete scheme for multi-term time-fractional mixed diffusion and diffusion wave equations with variable coefficient on 2D bounded domain. To the best of our knowledge, no article is available in the literature concerning a numerical analysis for fully discrete finite element approximations for the nonlinear time-fractional two-term mixed sub-diffusion and diffusion wave equations.

As we all know, the TGM is usually regarded as an efficient discretization technique for solving the nonsymmetric indefinite and nonlinear equations based on a coarse mesh with size H and a finer mesh with size h ($h \ll H$). More precisely, a nonlinear or nonsymmetric problem is solved on the coarse mesh. Then the solution obtained from coarse grid is used as a initial guess to solve a linearized problem on the finer mesh [23,24]. Later on, Chen [1,2] proposed two grid mixed finite element methods to solve nonlinear reaction-diffusion equations. Liu [15,16] considered a two-grid finite element approximation for a nonlinear time-fractional Cable equation and nonlinear fourth-order fractional differential equations with Caputo fractional derivative. Recently, Chen [11,12] also have done some work on the two grid method of fractional differential equations.

In this article, our main task is to take the Galerkin finite element to construct a fully discrete TGM scheme for 2D nonlinear time fractional two-term mixed sub-diffusion and diffusion wave equations, and derive the analysis of the corresponding stability and the error estimate in L^2-norm.

The remainder of the paper is organized as follows. In Section 2, we propose a fully-discrete scheme for (1.1) based on Galerkin FEM and L1-CN approximation. The unconditional stability analysis and the corresponding error estimate are deduced. In Section 3, we set up the TGM, and the stability and a priori error estimate of TGM are proved. In Section 4, the numerical example is presented to verify our theoretical analysis and some comparisons of computing time are done. The paper is concluded with some remarks in the last section.

Throughout this paper, let $L^p(\Omega)$ be the Lebesgue space with norm $\|\cdot\|_{0,p}$ for $0 \le p \le \infty$, then denote as $W^{m,p}(\Omega)$ the Sobolev with norm $\|\cdot\|_{m,p}$ given by

$$\|u\|_{m,p}=\begin{cases}\left(\sum_{|\alpha|\le m}\|D^\alpha u\|_{0,p}^p\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \max_{|\alpha|\le m}\|D^\alpha u\|_{0,\infty}, & p=\infty.\end{cases}\tag{1.4}$$

When p=2, denote $H^m(\Omega)=W^{m,2}(\Omega)$, $\|\cdot\|_m=\|\cdot\|_{m,2}$ and $\|\cdot\|=\|\cdot\|_{0,2}$. The generic constant $C>0$ (with or without subscript) is independent of n (time level), h (spatial size), τ (time step size) and may be different in different places.

2. Stability and Convergence Analysis for FEM Scheme

Let $\mathcal{T}_h$ be a regular rectangular partition of Ω with mesh size h, and let V_h be the two-dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree $r(r\ge 1)$ on $\mathcal{T}_h$ and $V_h^0=\{v\in V_h, v|_{\partial\Omega}=0\}$. Moreover, we also let $R_h: H_0^1(\Omega)\to V_h^0$ be the Rize-projection operator satisfying

$$\left(\nabla\left(u-R_hu\right),\nabla\chi\right)=0,\quad \forall\chi\in V_h^0.\tag{2.1}$$

Lemma 2.1 ([3]). *For $u\in H_0^1(\Omega)\cap H^{r+1}(\Omega)$, it has been proved that*

$$\|u-R_hu\|+h\|u-R_hu\|_1\le Ch^{r+1}\|u\|_{r+1}.\tag{2.2}$$

Note that the corresponding weak formulation of (1.1) is to find $u(x,t):(0,T]\to H_0^1(\Omega)$ such that

$$\begin{cases}\left({}_0^CD_t^{\alpha_1}u\left(\boldsymbol{x},t\right),v\right)+\left({}_0^CD_t^{\alpha}u\left(\boldsymbol{x},t\right),v\right)+\left(\nabla u\left(\boldsymbol{x},t\right),\nabla v\right)=\left(g(\boldsymbol{u}),v\right), & \forall v\in H_0^1(\Omega),\\ u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\quad u_t(\boldsymbol{x},0)=\tilde{u}_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega.\end{cases}\tag{2.3}$$

Let $\{t_n|=t_n=n\tau; 0\le n\le N\}$ be a uniform partition in time with time step τ. For a sequence of smooth functions $\{\phi(t)\}_{n=0}^N$ on $[0,T]$, we denote

$$\begin{aligned}&\phi^n=\phi(t_n),\quad \tilde{\phi}(\mathbf{x},t)=u_t(\mathbf{x},t),\quad \tilde{\phi}^0(\mathbf{x})=u_t(\mathbf{x},0)=\tilde{u}_0,\\ &\phi^{n-\frac{1}{2}}=\frac{\phi^n+\phi^{n-1}}{2},\quad \partial_t\phi^{k-\frac{1}{2}}=\frac{\phi^k-\phi^{k-1}}{\tau}\ (1\le k\le N),\quad \partial_t\phi^0=0.\end{aligned}\tag{2.4}$$

Now we give the L1-CN approximation of the Caputo derivative ${}_0^CD_t^{\alpha_1}u(\boldsymbol{x},t)$, ${}_0^CD_t^{\alpha}u(\boldsymbol{x},t)$ as follows:

$$\tilde{D}_t^{\alpha_1}\phi^{n-\frac{1}{2}}=\frac{\lambda_1}{2}\left(\sum_{k=1}^n\tilde{b}_{\alpha_1,n-k}\partial_t\phi^{k-\frac{1}{2}}+\sum_{k=1}^{n-1}\tilde{b}_{\alpha_1,n-k-1}\partial_t\phi^{k-\frac{1}{2}}\right),\tag{2.5a}$$

$$\tilde{D}_t^{\alpha}\phi^{n-\frac{1}{2}}=\lambda\left(\tilde{b}_{\alpha,0}\partial_t\phi^{n-1/2}+\sum_{k=1}^{n-1}(\tilde{b}_{\alpha,n-k}-\tilde{b}_{\alpha,n-k-1})\partial_t\phi^{k-\frac{1}{2}}-\tilde{b}_{\alpha,n-1}\tilde{\phi}_0\right),\tag{2.5b}$$

where $\lambda=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}$, $\lambda_1=\frac{\tau^{1-\alpha_1}}{\Gamma(2-\alpha_1)}$, and

$$\tilde{b}_{\alpha_1,k}=(k+1)^{1-\alpha_1}-k^{1-\alpha_1},\quad \tilde{b}_{\alpha,k}=(k+1)^{2-\alpha}-k^{2-\alpha},\quad 0\le k\le N-1.$$

The following useful lemmas are given, which are important for proving the stability and error estimate of the fully discrete scheme and TGM.

Lemma 2.2 ([21]). (1) *For each* $t \in (0,T]$, *assume that* $u_{tt}(x,t) \in L^2(\Omega)$. *Let* $R_{\alpha_1}^{n-\frac{1}{2}} = {}_0^C D_t^{\alpha_1} u^{n-\frac{1}{2}} - \widetilde{D}_t^{\alpha_1} u^{n-\frac{1}{2}}$, *which satisfies*

$$\|R_{\alpha_1}^{n-\frac{1}{2}}\| \le C \max_{0\le t\le T} \|u_{tt}(x,t)\| \tau^{2-\alpha_1}. \tag{2.6}$$

(2) *Assume that* $u_{ttt}(x,t) \in L^2(\Omega)$, *and let* $R_{\alpha}^{n-\frac{1}{2}} = {}_0^C D_t^{\alpha} u^{n-\frac{1}{2}} - \widetilde{D}_t^{\alpha} u^{n-\frac{1}{2}}$. *Then we have*

$$\|R_{\alpha}^{n-\frac{1}{2}}\| \le C \max_{0\le t\le T} \|u_{ttt}(x,t)\| \tau^{3-\alpha}. \tag{2.7}$$

Lemma 2.3 ([18, 26]). *For* $\tilde{b}_{\alpha,k}, k = 0,1,2,\cdots,n$, *we have*

$$1 = \tilde{b}_{\alpha,0} > \tilde{b}_{\alpha,1} > \tilde{b}_{\alpha,2} > \cdots > \tilde{b}_{\alpha,k} > \cdots > 0, \tag{2.8}$$

$$(2-\alpha)(k+1)^{1-\alpha} < \tilde{b}_{\alpha,k} < (2-\alpha)k^{1-\alpha}. \tag{2.9}$$

Lemma 2.4 ([5]). *For* $\tilde{b}_{\alpha_1,k} = (k+1)^{1-\alpha_1} - k^{1-\alpha_1}, k = 0,1,\cdots,n$, *any positive integer* N *and vector* $P = \left[w^1, w^2, \cdots, w^N\right] \in \mathbb{R}^N$, *we have*

$$\sum_{n=1}^{N}\sum_{k=1}^{n} \tilde{b}_{\alpha_1,n-k} w^k w^n \ge 0, \qquad \sum_{n=1}^{N}\sum_{k=1}^{n} \tilde{b}_{\alpha_1,n-k} w^k w^n + \sum_{n=1}^{N}\sum_{k=1}^{n-1} \tilde{b}_{\alpha_1,n-k-1} w^k w^n \ge 0. \tag{2.10}$$

Lemma 2.5 ([17]). *Assume that* $\{c_n\}$ *and* $\{q_n\}$ *are non-negative sequences, and the sequence* $\{\phi_n\}$ *satisfies*

$$\phi_0 \le g_0,\ \phi_n \le g_0 + \sum_{l=0}^{n-1} q_l + \sum_{l=0}^{n-1} c_l \phi_l, \quad n \ge 1,$$

where $g_0 \ge 0$. *Then the sequence* $\{\phi_n\}$ *satisfies*

$$\phi_n \le \left(g_0 + \sum_{l=0}^{n-1} q_l\right) \exp\left(\sum_{l=0}^{n-1} c_l\right).$$

Then the L1-CN scheme of (2.3) is to find $U^n \in V_h^0$ for n=1, $\cdots$, N, such that

$$\begin{cases} \left(\tilde{D}_t^{\alpha_1} U^{n-\frac{1}{2}}, v^h\right) + \left(\tilde{D}_t^{\alpha} U^{n-\frac{1}{2}}, v^h\right) + \left(\nabla U^{n-\frac{1}{2}}, \nabla v^h\right) = \left(g(U^{n-\frac{1}{2}}), v^h\right), & \forall v^h \in V_h^0, \\ U^0 = R_h u_0(\mathbf{x}), \qquad \tilde{U}^0 = R_h \tilde{u}_0, & \mathbf{x} \in \Omega. \end{cases} \tag{2.11}$$

Next, we present the stability and error analysis for fully-discrete scheme, which are then used in the next section to derive the stability and error analysis for TGM.

Theorem 2.1 (Stability). *The fully-discrete scheme* (2.11) *is unconditional stable.*

Proof. Let $\{U^n\}$ be the solution of (2.11). Setting $v^h = \partial_t U^{n-\frac{1}{2}} \in V_h^0$ in (2.11), we have

$$\begin{aligned} &\left(\tilde{D}_t^{\alpha_1} U^{n-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right) + \left(\tilde{D}_t^{\alpha} U^{n-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right) + (\nabla U^{n-\frac{1}{2}}, \nabla \partial_t U^{n-\frac{1}{2}}) \\ =&(g(U^{n-\frac{1}{2}}), \partial_t U^{n-\frac{1}{2}}). \end{aligned} \tag{2.12}$$

It follows from the definitions of $\tilde{D}_t^{\alpha_1}U^{n-\frac{1}{2}}$, $\tilde{D}_t^{\alpha}U^{n-\frac{1}{2}}$, $\partial_t U^{n-\frac{1}{2}}$ defined by (2.4) that

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}U^{n-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\\ =&\frac{\lambda_1}{2}\left[\left(\sum_{k=1}^{n}\tilde{b}_{\alpha_1,n-k}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right) + \left(\sum_{k=1}^{n-1}\tilde{b}_{\alpha_1,n-k-1}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\right], \end{aligned}\tag{2.13}$$

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha}U^{n-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\\ =&\lambda\left(\tilde{b}_{\alpha,0}\partial_t U^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k} - \tilde{b}_{\alpha,n-k-1}\right)\partial_t U^{k-\frac{1}{2}} - \tilde{b}_{\alpha,n-1}\tilde{U}^0, \partial_t U^{n-\frac{1}{2}}\right) \\ =&\lambda\left[\|\partial_t U^{n-\frac{1}{2}}\|^2 - \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k-1} - \tilde{b}_{\alpha,n-k}\right)\left(\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right) - \left(\tilde{b}_{\alpha,n-1}\tilde{U}^0, \partial_t U^{n-\frac{1}{2}}\right)\right],\end{aligned}\tag{2.14}$$

and

$$\left(\nabla U^{n-\frac{1}{2}}, \nabla\partial_t U^{n-\frac{1}{2}}\right) = \frac{1}{2\tau}\left(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\right). \tag{2.15}$$

Substituting (2.13)–(2.15) into (2.12), and multiplying 2τ on both sides of this equality, we have

$$\begin{aligned}&2\lambda\tau\|\partial_t U^{n-\frac{1}{2}}\|^2 + \|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\\ =&2\lambda\tau\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k-1} - \tilde{b}_{\alpha,n-k}\right)\left(\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\\ &+ 2\lambda\tau\left(\tilde{b}_{\alpha,n-1}\tilde{U}^0, \partial_t U^{n-\frac{1}{2}}\right) + 2\tau\left(g(U^{n-\frac{1}{2}}), \partial_t U^{n-\frac{1}{2}}\right)\\ &- \lambda_1\tau\left[\sum_{k=1}^{n}\left(\tilde{b}_{\alpha_1,n-k}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right) + \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha_1,n-k-1}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\right]\\ \le&\lambda\tau\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k-1} - \tilde{b}_{\alpha,n-k}\right)\left(\|\partial_t U^{k-\frac{1}{2}}\|^2 + \|\partial_t U^{n-\frac{1}{2}}\|^2\right)\\ &+ \lambda\tau\tilde{b}_{\alpha,n-1}\left(\|\tilde{U}^0\|^2 + \|\partial_t U^{n-\frac{1}{2}}\|^2\right) + 2\tau\left(g(U^{n-\frac{1}{2}}), \partial_t U^{n-\frac{1}{2}}\right)\\ &- \lambda_1\tau\left[\sum_{k=1}^{n}\left(\tilde{b}_{\alpha_1,n-k}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right) + \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha_1,n-k-1}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\right].\end{aligned}\tag{2.16}$$

Denoting $\beta^0 = \|\nabla U^0\|^2$ and $\beta^n = \|\nabla U^n\|^2 + \lambda\tau\sum_{k=1}^{n}\tilde{b}_{\alpha,n-k}\|\partial_t U^{k-\frac{1}{2}}\|^2$, we then have

$$\begin{aligned}\beta^n \le&\beta^{n-1} + \lambda\tau\tilde{b}_{\alpha,n-1}\|\tilde{U}^0\|^2 + 2\tau\left|\left(g(U^{n-\frac{1}{2}}), \partial_t U^{n-\frac{1}{2}}\right)\right|\\ &- \lambda_1\tau\left[\sum_{k=1}^{n}(\tilde{b}_{\alpha_1,n-k}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}) + \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha_1,n-k-1}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{n-\frac{1}{2}}\right)\right],\end{aligned}$$

which, by induction, gives

$$\beta^n \le \beta^0 + \lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,l-1}\|\tilde{U}^0\|^2 + 2\tau\sum_{l=1}^{n}\left|\left(g(U^{l-\frac{1}{2}}), \partial_t U^{l-\frac{1}{2}}\right)\right|$$
$$-\lambda_1\tau\sum_{l=1}^{n}\left[\sum_{k=1}^{l}\left(\tilde{b}_{\alpha_1,l-k}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{l-\frac{1}{2}}\right) + \sum_{k=1}^{l-1}\left(\tilde{b}_{\alpha_1,l-k-1}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{l-\frac{1}{2}}\right)\right]. \quad (2.17)$$

Using Cauchy-Schwarz inequality, Young′s inequality and the nonlinear property of $g(\cdot)$, it is easy to show that

$$2\tau\sum_{l=1}^{n}\left|\left(g(U^{l-\frac{1}{2}}), \partial_t U^{l-\frac{1}{2}}\right)\right| \le 2\tau\sum_{l=1}^{n}\left\{\frac{2}{\lambda\tilde{b}_{\alpha,n-l}}\|g(U^{l-\frac{1}{2}})\|^2 + \frac{\lambda\tilde{b}_{\alpha,n-l}}{2}\|\partial_t U^{l-\frac{1}{2}}\|^2\right\}$$
$$= \tau\sum_{l=1}^{n}\frac{4}{\lambda\tilde{b}_{\alpha,n-l}}\|g(U^{l-\frac{1}{2}})\|^2 + \lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,n-l}\|\partial_t U^{l-\frac{1}{2}}\|^2$$
$$\le \tau\sum_{l=1}^{n}\frac{4\Gamma(2-\alpha)}{T^{1-\alpha}}C^2\|U^{l-\frac{1}{2}}\|^2 + \lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,n-l}\|\partial_t U^{l-\frac{1}{2}}\|^2, \quad (2.18)$$

where above analysis uses

$$\tilde{b}_{\alpha,n-l} = (n-l+1)^{2-\alpha} - (n-l)^{2-\alpha}$$
$$=(2-\alpha)\int_{n-k}^{n-k+1} x^{1-\alpha}dx \ge (2-\alpha)(n-k+1)^{1-\alpha}. \quad (2.19)$$

By Lemma 2.3, we obtain

$$\lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,l-1}\|\tilde{U}^0\|^2 \le \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|\tilde{U}^0\|^2. \quad (2.20)$$

It follows from Lemma 2.4 that

$$-\lambda_1\tau\sum_{l=1}^{n}\left[\sum_{k=1}^{l}\left(\tilde{b}_{\alpha_1,l-k}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{l-\frac{1}{2}}\right) + \sum_{k=1}^{l-1}\left(\tilde{b}_{\alpha_1,l-k-1}\partial_t U^{k-\frac{1}{2}}, \partial_t U^{l-\frac{1}{2}}\right)\right] \le 0. \quad (2.21)$$

Substituting the estimates (2.18), (2.20) and (2.21) into (2.17), and using the Poincaré inequality, we have

$$\|U^n\|^2 \le \|\nabla U^0\|^2 + \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|\tilde{U}^0\|^2 + \sum_{l=1}^{n}C\tau\|U^{l-\frac{1}{2}}\|^2. \quad (2.22)$$

By Lemma 2.5, when $\tau < \frac{C}{2}$, we have

$$\|U^n\|^2 \le C\left(\|U^0\|_1^2 + \|\tilde{U}^0\|^2\right), \quad (2.23)$$

which is the desired result. □

We now obtain the L^2 error estimate for the fully-discrete scheme (2.11), which will be useful to prove the error estimate for TGM.

Theorem 2.2 (Error Analysis). *Let u^n and U^n be the solution of* (2.3) *at $t = t_n$ and* (2.11), *respectively. Assume that $u \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$, $u_t \in H^2(\Omega)$, $u_{tt} \in L^2(\Omega)$, $u_{ttt} \in L^2(\Omega)$, then we have*

$$\|u^n - U^n\| = O\left(h^{r+1} + \tau^{min\{2-\alpha_1, 3-\alpha\}}\right).$$

Proof. Combining (2.3) with (2.11) to get

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}\left(u^{n-\frac{1}{2}} - U^{n-\frac{1}{2}}\right), v^h\right) + \left(\tilde{D}_t^{\alpha}\left(u^{n-\frac{1}{2}} - U^{n-\frac{1}{2}}\right), v^h\right) + \left(\nabla\left(u^{n-\frac{1}{2}} - U^{n-\frac{1}{2}}\right), \nabla v^h\right)\\ =&\left(g(u^{n-\frac{1}{2}}) - g(U^{n-\frac{1}{2}}), v^h\right) - \left(R_{\alpha_1}^{n-\frac{1}{2}}, v^h\right) - \left(R_{\alpha}^{n-\frac{1}{2}}, v^h\right). \end{aligned} \tag{2.24}$$

Denoting $\rho^n = u^n - R_h u^n, \sigma^n = R_h u^n - U^n$ and choosing $v^h = \partial_t \sigma^{n-\frac{1}{2}}$ in (2.24), we obtain

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}\sigma^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) + \left(\tilde{D}_t^{\alpha}\sigma^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) + \left(\nabla\sigma^{n-\frac{1}{2}}, \nabla\partial_t\sigma^{n-\frac{1}{2}}\right)\\ =&\left(g(u^{n-\frac{1}{2}}) - g(U^{n-\frac{1}{2}}), \partial_t\sigma^{n-\frac{1}{2}}\right) - \left(\tilde{D}_t^{\alpha_1}\rho^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) - \left(\tilde{D}_t^{\alpha}\rho^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right)\\ &- \left(\nabla\rho^{n-\frac{1}{2}}, \nabla\partial_t\sigma^{n-\frac{1}{2}}\right) - \left(R_{\alpha_1}^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) - \left(R_{\alpha}^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right). \end{aligned} \tag{2.25}$$

From the definitions of $\tilde{D}_t^{\alpha_1}U^{n-\frac{1}{2}}$, $\tilde{D}_t^{\alpha}U^{n-\frac{1}{2}}$ and $\partial_t U^{n-\frac{1}{2}}$, we have

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}\sigma^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right)\\ =&\frac{\lambda_1}{2}\left[\left(\sum_{k=1}^{n}\tilde{b}_{\alpha_1,n-k}\partial_t\sigma^{k-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) + \left(\sum_{k=1}^{n-1}\tilde{b}_{\alpha_1,n-k-1}\partial_t\sigma^{k-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right)\right], \end{aligned} \tag{2.26}$$

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha}\sigma^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right)\\ =&\lambda\left(\tilde{b}_{\alpha,0}\partial_t\sigma^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k} - \tilde{b}_{\alpha,n-k-1}\right)\partial_t\sigma^{k-\frac{1}{2}} - \tilde{b}_{\alpha,n-1}\tilde{\sigma}_0, \partial_t\sigma^{n-\frac{1}{2}}\right) \\ =&\lambda\left[\|\partial_t\sigma^{n-\frac{1}{2}}\|^2 - \sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k-1} - \tilde{b}_{\alpha,n-k}\right)\left(\partial_t\sigma^{k-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) - \left(\tilde{b}_{\alpha,n-1}\tilde{\sigma}^0, \partial_t\sigma^{n-\frac{1}{2}}\right)\right]. \end{aligned} \tag{2.27}$$

and

$$\left(\nabla\sigma^{n-\frac{1}{2}}, \nabla\partial_t\sigma^{n-\frac{1}{2}}\right) = \frac{1}{2\tau}\left(\|\nabla\sigma^n\|^2 - \|\nabla\sigma^{n-1}\|^2\right). \tag{2.28}$$

Since $g(\cdot)$ is twice continuously differentiable, we have

$$\begin{aligned}&\left(g(u^{n-\frac{1}{2}}) - g(U^{n-\frac{1}{2}}), \partial_t\sigma^{n-\frac{1}{2}}\right)\\ \le& C\|u^{n-\frac{1}{2}} - U^{n-\frac{1}{2}}\|\|\partial_t\sigma^{n-\frac{1}{2}}\| \le C\left(\|\rho^{n-\frac{1}{2}}\| + \|\sigma^{n-\frac{1}{2}}\|\right)\|\partial_t\sigma^{n-\frac{1}{2}}\|. \end{aligned} \tag{2.29}$$

Substituting above results into (2.25), and multiplying 2τ on both sides of the resulting identity, we have

$$\begin{aligned}
&2\lambda\tau\|\partial_t\sigma^{n-\frac{1}{2}}\|^2+\|\nabla\sigma^n\|^2-\|\nabla\sigma^{n-1}\|^2\\
\le&\lambda\tau\tilde{b}_{\alpha,n-1}\left(\|\tilde{\sigma}^0\|^2+\|\partial_t\sigma^{n-\frac{1}{2}}\|^2\right)\\
&+\lambda\tau\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k-1}-\tilde{b}_{\alpha,n-k}\right)\left(\|\partial_t\sigma^{k-\frac{1}{2}}\|^2+\|\partial_t\xi^{n-\frac{1}{2}}\|^2\right)\\
&-2\tau\left(\tilde{D}_t^{\alpha_1}\rho^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)-2\tau\left(\tilde{D}_t^{\alpha}\rho^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)-2\tau\left(R_{\alpha_1}^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)\\
&-2\tau\left(R_{\alpha}^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)+2\tau C\left(\|\rho^{n-\frac{1}{2}}\|+\|\sigma^{n-\frac{1}{2}}\|\right)\|\partial_t\sigma^{n-\frac{1}{2}}\|\\
&-\lambda_1\tau\left[\sum_{k=1}^{n}\left(\tilde{b}_{\alpha_1,n-k}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)+\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha_1,n-k-1}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)\right].
\end{aligned}\tag{2.30}$$

Denoting $\theta^0=\|\nabla\sigma^0\|^2$ and $\theta^n=\|\nabla\sigma^n\|^2+\sum\limits_{k=1}^{n}\lambda\tau\tilde{b}_{\alpha,n-k}\|\partial_t\sigma^{k-\frac{1}{2}}\|^2$, we have

$$\begin{aligned}
\theta^n\le&\theta^{n-1}+\lambda\tau\tilde{b}_{\alpha,n-1}\|\tilde{\sigma}^0\|^2-2\tau\left(\tilde{D}_t^{\alpha_1}\rho^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)\\
&-2\tau\left(\tilde{D}_t^{\alpha}\rho^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)-2\tau\left(R_{\alpha_1}^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)-2\tau\left(R_{\alpha}^{n-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)\\
&+2\tau C\|\rho^{n-\frac{1}{2}}\|\|\partial_t\sigma^{n-\frac{1}{2}}\|+2\tau C\|\sigma^{n-\frac{1}{2}}\|\|\partial_t\sigma^{n-\frac{1}{2}}\|\\
&-\lambda_1\tau\left[\sum_{k=1}^{n}\left(\tilde{b}_{\alpha_1,n-k}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)+\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha_1,n-k-1}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{n-\frac{1}{2}}\right)\right],
\end{aligned}$$

which, by induction, gives

$$\begin{aligned}
\theta^n\le&\theta^0+\lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,l-1}\|\tilde{\sigma}^0\|^2-2\tau\sum_{l=1}^{n}\left(\tilde{D}_t^{\alpha_1}\rho^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\\
&-2\tau\sum_{l=1}^{n}\left(\tilde{D}_t^{\alpha}\rho^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)-2\tau\sum_{l=1}^{n}\left(R_{\alpha_1}^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)-2\tau\sum_{l=1}^{n}\left(R_{\alpha}^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\\
&+2\tau C\sum_{l=1}^{n}\|\rho^{l-\frac{1}{2}}\|\|\partial_t\sigma^{l-\frac{1}{2}}\|+2\tau C\sum_{l=1}^{n}\|\sigma^{l-\frac{1}{2}}\|\|\partial_t\sigma^{l-\frac{1}{2}}\|\\
&-\lambda_1\tau\sum_{l=1}^{n}\left[\sum_{k=1}^{l}\left(\tilde{b}_{\alpha_1,l-k}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)+\sum_{k=1}^{l-1}\left(\tilde{b}_{\alpha_1,l-k-1}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\right].
\end{aligned}\tag{2.31}$$

By Cauchy-Schwarz inequality, Young′s inequality and Lemma 2.2, we obtain

$$2\tau\sum_{l=1}^{n}\left(\tilde{D}_t^{\alpha_1}\rho^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\le\sum_{l=1}^{n}\frac{6\tau\|\tilde{D}_t^{\alpha_1}\rho^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}+\sum_{l=1}^{n}\frac{\lambda\tau}{6}\tilde{b}_{\alpha,n-l}\|\partial_t\sigma^{l-\frac{1}{2}}\|^2, \tag{2.32}$$

$$2\tau\sum_{l=1}^{n}\left(\tilde{D}_t^{\alpha}\rho^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\le\sum_{l=1}^{n}\frac{6\tau\|\tilde{D}_t^{\alpha}\rho^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}+\sum_{l=1}^{n}\frac{\lambda\tau\tilde{b}_{\alpha,n-l}\|\partial_t\sigma^{l-\frac{1}{2}}\|^2}{6}, \tag{2.33}$$

$$2\tau\sum_{l=1}^{n}\left(R_{\alpha_1}^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\le\frac{C\tau}{\lambda\tilde{b}_{a,n-l}}\max_{0\le t\le T}\|u_{tt}(X,t)\|^2\tau^{4-2\alpha_1}+\sum_{l=1}^{n}\frac{\lambda\tau}{6}\tilde{b}_{\alpha,n-l}\|\partial_t\sigma^{l-\frac{1}{2}}\|^2, \tag{2.34}$$

$$2\tau\sum_{l=1}^{n}\left(R_{\alpha}^{l-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\le\frac{C\tau}{\lambda\tilde{b}_{a,n-l}}\max_{0\le t\le T}\|u_{ttt}(X,t)\|^2\tau^{6-2\alpha}+\sum_{l=1}^{n}\frac{\lambda\tau}{6}\tilde{b}_{\alpha,n-l}\|\partial_t\sigma^{l-\frac{1}{2}}\|^2, \tag{2.35}$$

$$2\tau C\sum_{l=1}^{n}\|\rho^{l-\frac{1}{2}}\|\|\partial_t\sigma^{l-\frac{1}{2}}\|\le\sum_{l=1}^{n}\frac{6\tau C^2\|\rho^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}+\sum_{l=1}^{n}\frac{\lambda\tau}{6}\tilde{b}_{\alpha,n-l}\|\partial_t\sigma^{l-\frac{1}{2}}\|^2, \tag{2.36}$$

$$2\tau C\sum_{l=1}^{n}\|\sigma^{l-\frac{1}{2}}\|\|\partial_t\sigma^{l-\frac{1}{2}}\|\le\sum_{l=1}^{n}\frac{6\tau C^2\|\sigma^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}+\sum_{l=1}^{n}\frac{\lambda\tau}{6}\tilde{b}_{\alpha,n-l}\|\partial_t\sigma^{l-\frac{1}{2}}\|^2, \tag{2.37}$$

By Lemma 2.3, we have

$$\sum_{l=1}^{n}\frac{1}{\tilde{b}_{\alpha,n-l}}\le\frac{n(n+1)^{\alpha-1}}{(2-\alpha)}\le\frac{N^\alpha}{2-\alpha}\le\frac{T^\alpha\tau^{-\alpha}}{(2-\alpha)}. \tag{2.38}$$

It follows from Lemma 2.4 that

$$-\lambda_1\tau\sum_{l=1}^{n}\left[\sum_{k=1}^{l}\left(\tilde{b}_{\alpha_1,l-k}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)+\sum_{k=1}^{l-1}\left(\tilde{b}_{\alpha_1,l-k-1}\partial_t\sigma^{k-\frac{1}{2}},\partial_t\sigma^{l-\frac{1}{2}}\right)\right]\le 0. \tag{2.39}$$

Based on the above estimates and Lemma 2.1, we have

$$\begin{aligned}\sum_{l=1}^{n}\frac{6\tau\|\tilde{D}_t^{\alpha_1}\rho^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}&\le\frac{6T^\alpha\Gamma(3-\alpha)}{2-\alpha}\max_{1\le l\le n}\|\tilde{D}_t^{\alpha_1}\rho^{l-\frac{1}{2}}\|^2\\&\le Ch^{2r+2}\max_{1\le l\le n}\|\tilde{D}_t^{\alpha_1}u^{l-\frac{1}{2}}\|_{r+1}^2+C\max_{1\le l\le n}\|u_{tt}(\mathbf{x},t)\|^2\tau^{4-2\alpha_1},\end{aligned} \tag{2.40}$$

$$\begin{aligned}\sum_{l=1}^{n}\frac{6\tau\|\tilde{D}_t^{\alpha}\rho^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}&\le\frac{6T^\alpha\Gamma(3-\alpha)}{2-\alpha}\max_{1\le l\le n}\|\tilde{D}_t^{\alpha}\rho^{l-\frac{1}{2}}\|^2\\&\le Ch^{2r+2}\max_{1\le l\le n}\|\tilde{D}_t^{\alpha}u^{l-\frac{1}{2}}\|_{r+1}^2+C\max_{1\le l\le n}\|u_{ttt}(X,t)\|^2\tau^{6-2\alpha},\end{aligned} \tag{2.41}$$

$$\sum_{l=1}^{n}\frac{C\tau\max\limits_{0\le t\le T}\|u_{tt}(X,t)\|^2\tau^{4-2\alpha_1}}{\lambda\tilde{b}_{\alpha,n-l}}\le\frac{CT^\alpha\Gamma(3-\alpha)}{2-\alpha}\max_{0\le t\le T}\|u_{tt}(X,t)\|^2\tau^{4-2\alpha_1}, \tag{2.42}$$

$$\sum_{l=1}^{n}C\tau\max_{0\le t\le T}\frac{\|u_{ttt}(X,t)\|^2\tau^{6-2\alpha}}{\lambda\tilde{b}_{\alpha,n-l}}\le\frac{CT^\alpha\Gamma(3-\alpha)}{2-\alpha}\max_{0\le t\le T}\|u_{ttt}(X,t)\|^2\tau^{6-2\alpha}, \tag{2.43}$$

$$\sum_{l=1}^{n}\frac{6\tau\|\rho^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}\le\frac{6T^\alpha\Gamma(3-\alpha)\|\rho^{l-\frac{1}{2}}\|^2}{2-\alpha}\le Ch^{2r+2}\|u^{l-\frac{1}{2}}\|_{r+1}^2. \tag{2.44}$$

Combining (2.31)–(2.44), we have

$$\|\nabla\sigma^n\|^2 \le C\left(\sum_{l=1}^{n}\frac{6L^2\tau\|\sigma^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}} + h^{2r+2} + \tau^{\min\{4-2\alpha_1,6-2\alpha\}}\right). \tag{2.45}$$

By Poincaré inequality, Lemma 2.1 and Lemma 2.5, it follows that

$$\|\sigma^n\|^2 \le C\left(h^{2r+2} + \tau^{\min\{4-2\alpha_1,6-2\alpha\}}\right), \tag{2.46}$$

which implies

$$\|u^n - U^n\| = O\left(h^{r+1} + \tau^{min\{2-\alpha_1,3-\alpha\}}\right). \tag{2.47}$$

The proof is complete. □

3. Unconditional Stability and Error Estimate for TGM

In this section, we present the following two-grid algorithm based on Newton iteration idea and derive the stability and error estimate results. TGM has two steps as follows:

Algorithm 3.1. Step 1: On the coarse grid $\mathcal{T}_H$, find $u_H^n \in V_H^0$ for the following nonlinear system, such that

$$\begin{cases} \left(\tilde{D}_t^{\alpha_1}u_H^{n-\frac{1}{2}}, w_H\right) + \left(\tilde{D}_t^{\alpha}u_H^{n-\frac{1}{2}}, w_H\right) + \left(\nabla u_H^{n-\frac{1}{2}}, \nabla w_H\right) \\ = \left(g(u_H^{n-\frac{1}{2}}), w_H\right), \forall w_H \in V_H^0, \\ U^0 = R_H u_0(\mathbf{x}), \quad \tilde{U}^0 = R_H\tilde{u}_0, \quad \mathbf{x}\in\Omega. \end{cases} \tag{3.1}$$

Step 2: On the fine grid $\mathcal{T}_h$, find $u_h^n \in V_h^0$ for the following linear system, such that

$$\begin{cases} \left(\tilde{D}_t^{\alpha_1}u_h^{n-\frac{1}{2}}, w_h\right) + \left(\tilde{D}_t^{\alpha}u_h^{n-\frac{1}{2}}, w_h\right) + \left(\nabla u_h^{n-\frac{1}{2}}, \nabla w_h\right) \\ = \left(g(u_H^{n-\frac{1}{2}}) + g'(u_H^{n-\frac{1}{2}})\left(u_h^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right), w_h\right), \forall w_h \in V_h^0, \\ U^0 = R_h u_0(\mathbf{x}), \quad \tilde{U}^0 = R_h\tilde{u}_0, \quad \mathbf{x}\in\Omega. \end{cases} \tag{3.2}$$

Remark 3.1. In the two-grid algorithm, we solve the nonlinear fractional equation on the coarse grid $\mathcal{T}_H$ to produce a rough approximation, and then use the rough approximation as the initial guess to solve the linearized equation on the fine grid $\mathcal{T}_h$.

First, we derive the stability analysis for TGM of step 2.

Theorem 3.1. *For the two-grid FE system* (3.1) *and* (3.2), *the following stable inequality for* $u_h^n \in V_h^0$ *holds*

$$\|u_h^n\|^2 \le C\Big(\|\nabla u_H^0\|^2 + \|\nabla u_h^0\|^2 + \|\tilde{u}^0\|^2 + \|u^0\|^2\Big). \tag{3.3}$$

Proof. Considering (3.2), and we have

$$\begin{aligned}&\left(\frac{\lambda_1}{2}\left(\sum_{k=1}^{n}\tilde{b}_{\alpha_1,n-k}\partial_t u_h^{k-\frac{1}{2}}+\sum_{k=1}^{n-1}\tilde{b}_{\alpha_1,n-k-1}\partial_t u_h^{k-\frac{1}{2}}\right),w_h\right)+\left(\lambda\left[\tilde{b}_{\alpha,0}\partial_t u_h^{n-\frac{1}{2}}\right.\right.\\&\qquad\left.\left.+\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha,n-k}-\tilde{b}_{\alpha,n-k-1}\right)\partial_t u_h^{k-\frac{1}{2}}-\tilde{b}_{\alpha,n-1}\tilde{u}^0\right],w_h\right)+\left(\nabla u_h^{n-\frac{1}{2}},\nabla w_h\right)\\&=\left(g(u_H^{n-\frac{1}{2}})+g'(u_H^{n-\frac{1}{2}})\left(u_h^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}}\right),w_h\right)\\&\le\left(C(u_H^{n-\frac{1}{2}})+C_1\left(u_h^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}}\right),w_h\right)\\&=\left(C_2(u_H^{n-\frac{1}{2}})+C_1(u_h^{n-\frac{1}{2}}),w_h\right).\end{aligned}\tag{3.4}$$

where C, C_1, C_2 are positive constants. Let $w_h=\partial_t u_h^{n-\frac{1}{2}}$, $\rho^n=\|\nabla u_h^n\|^2+\lambda\tau\sum_{k=1}^{n}\tilde{b}_{\alpha,n-k}\|\partial_t u_h^{k-\frac{1}{2}}\|^2$, and we have

$$\begin{aligned}\rho^n\le&\rho^{n-1}+\lambda\tau\tilde{b}_{\alpha,n-1}\|\tilde{u}^0\|^2\\&+2\tau\left(C_2\left(u_H^{n-\frac{1}{2}}\right)+C_1(u_h^{n-\frac{1}{2}}),\partial_t u_h^{n-\frac{1}{2}}\right)\\&-\lambda_1\tau\left[\sum_{k=1}^{n}\left(\tilde{b}_{\alpha_1,n-k}\partial_t u_h^{k-\frac{1}{2}},\partial_t u_h^{n-\frac{1}{2}}\right)+\sum_{k=1}^{n-1}\left(\tilde{b}_{\alpha_1,n-k-1}\partial_t u_h^{k-\frac{1}{2}},\partial_t u_h^{n-\frac{1}{2}}\right)\right],\end{aligned}$$

which, by induction, gives

$$\begin{aligned}\rho^n\le&\rho^0+\lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,l-1}\|\tilde{u}_0\|^2+\sum_{l=1}^{n}\frac{\tau C_1^2\|u_h^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}\\&+\sum_{l=1}^{n}\frac{\tau C_2^2\|u_H^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}+\sum_{l=1}^{n}\lambda\tau\tilde{b}_{\alpha,n-l}\|\partial_t u_h^{l-\frac{1}{2}}\|^2\\&-\lambda_1\tau\sum_{l=1}^{n}\left[\sum_{k=1}^{l}\left(\tilde{b}_{\alpha_1,l-k}\partial_t u_h^{k-\frac{1}{2}},\partial_t u_h^{l-\frac{1}{2}}\right)+\sum_{k=1}^{l-1}\left(\tilde{b}_{\alpha_1,l-k-1}\partial_t u_h^{k-\frac{1}{2}},\partial_t u_h^{l-\frac{1}{2}}\right)\right].\end{aligned}\tag{3.5}$$

It follows from Lemma 2.3 that

$$\lambda\tau\sum_{l=1}^{n}\tilde{b}_{\alpha,l-1}\|\tilde{u}^0\|^2\le\frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|\tilde{u}^0\|^2,\tag{3.6}$$

$$\sum_{l=1}^{n}\frac{\tau C_2^2\|u_H^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}\le\frac{C_2^2\Gamma(3-\alpha)T^{\alpha}}{2-\alpha}\max_{0\le t\le T}\|u_H(x,t)\|^2.\tag{3.7}$$

By (2.4), we obtain that

$$-\lambda_1\tau\sum_{l=1}^{n}\left[\sum_{k=1}^{l}\left(\tilde{b}_{\alpha_1,l-k}\partial_t u_h^{k-\frac{1}{2}},\partial_t u_h^{l-\frac{1}{2}}\right)+\sum_{k=1}^{l-1}\left(\tilde{b}_{\alpha_1,l-k-1}\partial_t u_h^{k-\frac{1}{2}},\partial_t u_h^{l-\frac{1}{2}}\right)\right]\le 0.\tag{3.8}$$

Then, substituting (3.6)∼(3.8) into (3.5) yields

$$\begin{aligned}\|\nabla u_h^n\|^2 \leq & \|\nabla u_h^0\|^2 + \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|\tilde{u}^0\|^2 + \frac{C_2^2\Gamma(3-\alpha)T^\alpha}{2-\alpha}\max_{0\leq t\leq T}\|u_H(x,t)\|^2 \\ & + \sum_{l=1}^{n}\frac{\tau C_1^2\|u_h^{l-\frac{1}{2}}\|^2}{\lambda\tilde{b}_{\alpha,n-l}}. \end{aligned} \tag{3.9}$$

Using Poincaré inequality and Lemma 2.5, we obain

$$\|u_h^n\|^2 \leq C\left(\|\nabla u_h^0\|^2 + \|\tilde{u}^0\|^2 + \max_{0\leq t\leq T}\|u_H(x,t)\|^2 + \|u_h^0\|^2\right). \tag{3.10}$$

Considering (3.1), by (2.23), there exists

$$\|u_H^n\|^2 \leq C\left(\|\nabla u_H^0\|^2 + \|\tilde{u}_H^0\|^2 + \|u_H^0\|^2\right). \tag{3.11}$$

According to (3.11) and (3.10), we have

$$\|u_h^n\|^2 \leq C\left(\|\nabla u_H^0\|^2 + \|\nabla u_h^0\|^2 + \|\tilde{u}^0\|^2 + \|u^0\|^2\right). \tag{3.12}$$

This completes the proof. □

Now, we are ready to give error estimate for TGM. In the following analysis, we need to prove $\|u(t_n) - U_h^n\|$.

Theorem 3.2. *Let $u(t_n) \in H_0^1(\Omega)\cap H^{r+1}(\Omega)$, $u_t \in H^2(\Omega)$, $u_{tt} \in L^2(\Omega)$, $u_{ttt} \in L^2(\Omega)$, $U_h^n \in V_h^0$ and $U_h^0 = R_h u_0(\mathbf{x})$, we have*

$$\|u(t_n) - U_h^n\| \leq C\left(\tau^{\min\{2-\alpha_1, 3-\alpha\}} + h^{r+1} + H^{2r+2}\right). \tag{3.13}$$

Proof. Subtracting (2.3) from (3.2), it yields

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right), w_h\right) + \left(\tilde{D}_t^{\alpha}\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right), w_h\right) + \left(\nabla u^{n-\frac{1}{2}} - \nabla u_h^{n-\frac{1}{2}}, \nabla w_h\right) \\ =& \left(g(u^{n-\frac{1}{2}}) - g(u_H^{n-\frac{1}{2}}) - g'(u_H^{n-\frac{1}{2}})\left(u_h^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right), w_h\right) \\ & - \left(R_{\alpha_1}^{n-\frac{1}{2}}, w_h\right) - \left(R_{\alpha}^{n-\frac{1}{2}}, w_h\right). \end{aligned} \tag{3.14}$$

By Taylor expansion, we have

$$g(u^{n-\frac{1}{2}}) = g(u_H^{n-\frac{1}{2}}) + g'(u_H^{n-\frac{1}{2}})\left(u^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right) + \frac{g''(u_H^{n-\frac{1}{2}})}{2}\left(u^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right)^2. \tag{3.15}$$

Then, it follows from (3.15) that

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right), w_h\right) + \left(\tilde{D}_t^{\alpha}\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right), w_h\right) + \left(\nabla u^{n-\frac{1}{2}} - \nabla u_h^{n-\frac{1}{2}}, \nabla w_h\right) \\ =& \left(g'(u_H^{n-\frac{1}{2}})\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right) + \frac{g''(u_H^{n-\frac{1}{2}})\left(u^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right)^2}{2}, w_h\right) - \left(R_{\alpha_1}^{n-\frac{1}{2}}, w_h\right) - \left(R_{\alpha}^{n-\frac{1}{2}}, w_h\right) \\ \leq& \left(C\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right) + \frac{C_1}{2}\left(u^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right)^2, w_h\right) - \left(R_{\alpha_1}^{n-\frac{1}{2}}, w_h\right) - \left(R_{\alpha}^{n-\frac{1}{2}}, w_h\right). \end{aligned} \tag{3.16}$$

Let $u^{n-\frac{1}{2}} - R_h u^{n-\frac{1}{2}} = \rho^{n-\frac{1}{2}}$, $R_h u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}} = \sigma^{n-\frac{1}{2}}$ and $w_h = \partial_t \sigma^{n-\frac{1}{2}}$. Hence, (3.16) becomes

$$\begin{aligned}&\left(\tilde{D}_t^{\alpha_1}\sigma^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) + \left(\tilde{D}_t^{\alpha}\sigma^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) + \left(\nabla\sigma^{n-\frac{1}{2}}, \nabla\partial_t\sigma^{n-\frac{1}{2}}\right)\\ \leq &-\left(\tilde{D}_t^{\alpha_1}\rho^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) - \left(\tilde{D}_t^{\alpha}\rho^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) - \left(R_{\alpha_1}^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right)\\ &-\left(R_{\alpha}^{n-\frac{1}{2}}, \partial_t\sigma^{n-\frac{1}{2}}\right) + \left(C\left(u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\right) + \frac{C_1}{2}\left(u^{n-\frac{1}{2}} - u_H^{n-\frac{1}{2}}\right)^2, \partial_t\sigma^{n-\frac{1}{2}}\right). \end{aligned} \tag{3.17}$$

Let $\gamma^n = \|\nabla\sigma^n\|^2 + \lambda\tau\sum_{k=1}^n \tilde{b}_{\alpha,n-k}\|\partial_t\sigma^{k-\frac{1}{2}}\|^2$, $\gamma^0 = \|\nabla\sigma^0\|^2$ and we have

$$\begin{aligned}\gamma^n \leq &\gamma^0 + \lambda\tau\sum_{l=1}^n \tilde{b}_{\alpha,l-1}\|\tilde{\sigma}\|^2 - 2\tau\sum_{l=1}^n\left(\tilde{D}_t^{\alpha_1}\rho^{l-\frac{1}{2}}, \partial_t\sigma^{l-\frac{1}{2}}\right) - 2\tau\sum_{l=1}^n\left(\tilde{D}_t^{\alpha}\rho^{l-\frac{1}{2}}, \partial_t\sigma^{l-\frac{1}{2}}\right)\\ &- 2\tau\sum_{l=1}^n\left(R_{\alpha_1}^{l-\frac{1}{2}}, \partial_t\sigma^{l-\frac{1}{2}}\right) - 2\tau\sum_{l=1}^n\left(R_{\alpha}^{l-\frac{1}{2}}, \partial_t\sigma^{l-\frac{1}{2}}\right)\\ &+ 2\tau\sum_{l=1}^n\left(\frac{C_1}{2}\left(u^{l-\frac{1}{2}} - u_H^{l-\frac{1}{2}}\right)^2 + C\left(u^{l-\frac{1}{2}} - u_h^{l-\frac{1}{2}}\right), \partial_t\sigma^{l-\frac{1}{2}}\right). \end{aligned} \tag{3.18}$$

We use a similar process of proof to the estimate (2.47). By Poincaré inequality, Lemma 2.5, Lemma 2.1 and the estimate (2.47), it follows that

$$\begin{aligned}\|\sigma^n\|^2 &\leq C\left(\|\nabla\sigma^0\|^2 + \tau^{min\{4-2\alpha_1, 6-2\alpha\}} + \sum_{l=1}^n \|\left(u^{l-\frac{1}{2}} - u_H^{l-\frac{1}{2}}\right)^2\|^2 + h^{2r+2}\right)\\ &\leq C\left(\|\nabla\sigma^0\|^2 + \tau^{min\{4-2\alpha_1, 6-2\alpha\}} + H^{4r+4} + h^{2r+2}\right). \end{aligned} \tag{3.19}$$

which leads to

$$\|u - u_h\| \leq H^{2r+2} + h^{r+1} + \tau^{\min\{2-\alpha_1, 3-\alpha\}}. \tag{3.20}$$

This concludes the proof. □

Remark 3.2. In the estimate (3.20), we observe that TGM algorithm can achieve the convergence rate h^{r+1} as long as the mesh sizes satisfy $H = O(h^{\frac{1}{2}})$.

4. Numerical Examples

In this section, we present two numerical examples to demonstrate the theoretical analysis and illustrate the efficiency of the algorithm discussed in Section 3. To investigate the spatial and temporal convergence order, we use a bilinear finite element approximation and the computation is performed by using Matlab.

Example 4.1. The following equation has exact solution $u(x, y, t) = t^{3+\alpha+\alpha_1}\sin\pi x \sin\pi y$:

$${}_0^C D_t^{\alpha_1} u(x, y, t) + {}_0^C D_t^{\alpha} u(x, y, t) - \Delta u(x, y, t) = -u^2 + g_1, \quad (x, y, t) \in \Omega \times (0, T], \tag{4.1}$$

where $\Omega = (0, 1) \times (0, 1), T = 1, g_1$ is a known function.

Table 4.1: L^2-errors with $\alpha_1 = 0.25$, $\alpha = 1.75$ and $\tau = \frac{1}{1000}$.

TGM	H	h	$\|u^n - u_h^n\|$	Order	CPU time
	$\frac{1}{2}$	$\frac{1}{4}$	5.2253e-02	\	7.1870s
	$\frac{1}{3}$	$\frac{1}{9}$	9.6560e-03	2.0822	18.8962s
	$\frac{1}{4}$	$\frac{1}{16}$	2.8794e-03	2.1030	49.0644s
	$\frac{1}{5}$	$\frac{1}{25}$	1.1387e-03	2.0786	122.6577s
FEM		h	$\|u^n - U^n\|$	Order	CPU time
		$\frac{1}{4}$	5.2253e-02	\	7.4899s
		$\frac{1}{9}$	9.6560e-03	2.0822	21.0704s
		$\frac{1}{16}$	2.8794e-03	2.1030	66.0248s
		$\frac{1}{25}$	1.1387e-03	2.0786	165.8093s

Table 4.2: L^2-errors with $\alpha_1 = 0.9$, $\alpha = 1.3$ and $\tau = \frac{1}{1000}$.

TGM	H	h	$\|u^n - u_h^n\|$	Order	CPU time
	$\frac{1}{2}$	$\frac{1}{4}$	4.8519e-02	\	7.4024s
	$\frac{1}{3}$	$\frac{1}{9}$	8.7562e-03	2.1114	18.3025s
	$\frac{1}{4}$	$\frac{1}{16}$	2.5886e-03	2.1180	48.8763s
	$\frac{1}{5}$	$\frac{1}{25}$	1.0516e-03	2.0185	125.4723s
FEM		h	$\|u^n - U^n\|$	Order	CPU time
		$\frac{1}{4}$	4.8519e-02	\	12.2758s
		$\frac{1}{9}$	8.7562e-03	2.1114	26.9553s
		$\frac{1}{16}$	2.5886e-03	2.1180	67.8842s
		$\frac{1}{25}$	1.0515e-03	2.0185	157.2231s

Table 4.3: L^2-errors with $\alpha_1 = 0.6$, $\alpha = 1.6$ and $\tau = \frac{1}{1000}$.

TGM	H	h	$\|u^n - u_h^n\|$	Order	CPU time
	$\frac{1}{2}$	$\frac{1}{4}$	4.8821e-02	\	7.066s
	$\frac{1}{3}$	$\frac{1}{9}$	8.9255e-03	2.0954	19.0976s
	$\frac{1}{4}$	$\frac{1}{16}$	2.6655e-03	2.1005	56.5094s
	$\frac{1}{5}$	$\frac{1}{25}$	1.0750e-03	2.0347	133.5581s
FEM		h	$\|u^n - U^n\|$	Order	CPU time
		$\frac{1}{4}$	4.8821e-02	\	9.1908s
		$\frac{1}{9}$	8.9255e-03	2.0954	25.4966s
		$\frac{1}{16}$	2.6655e-03	2.1005	65.4233s
		$\frac{1}{25}$	1.0750e-03	2.0347	160.2209s

Let $H_x = H_y = H$, $h_x = h_y = h$ and $h = H^2$. Tables 4.1−4.3 show that the spatial convergence rates in L^2-norm of FEM and Algorithm 3.1 are both equivalent to 2. The convergence results are consistent with the results $O(h^{r+1})$ of the theoretical analysis. We also compare the CPU time with FEM and Algorithm 3.1 in Tables 4.1–4.3. And the results show that Algorithm 3.1 is more efficient than FEM.

The temporal convergence rate of FEM and Algorithm 3.1 are given in Table 4.4–4.7, for fixed mesh $h = H^2 = \frac{1}{81}$. The numerical results confirm that both FEM and TGM have temporal convergence rate $\min\{2 - \alpha_1, 3 - \alpha\}$. We can see that the computing time required for the Algorithm 3.1 is much less than FEM.

Table 4.4: L^2-errors and temporal convergence rate with $\alpha_1 = 0.25$ and $\alpha = 1.75$ for FEM and Algorithm 3.1.

τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{12}$	1.8426e-02	\	40.9931s	1.8426e-02	\	32.3262s
$\frac{1}{14}$	1.5281e-02	1.2138	47.3536s	1.5281e-02	1.2138	34.6403s
$\frac{1}{16}$	1.2987e-02	1.2185	49.9114s	1.2987e-02	1.2185	38.7563s
$\frac{1}{18}$	1.1246e-02	1.2219	51.9629s	1.1246e-02	1.2219	40.6421s

Table 4.5: L^2-errors and temporal convergence with $\alpha_1 = 0.75$ and $\alpha = 1.25$ for FEM and Algorithm 3.1.

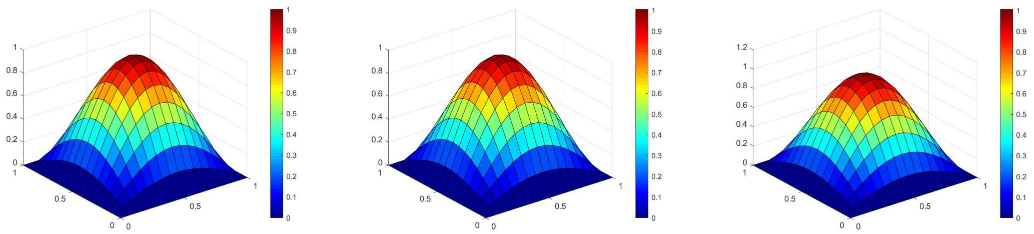
τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{12}$	7.0346e-03	\	44.8760s	7.0346e-03	\	33.997s
$\frac{1}{14}$	5.7610e-03	1.2956	51.8767s	5.7610e-03	1.2956	36.2469s
$\frac{1}{16}$	4.8493e-03	1.2903	58.0472s	4.8493e-03	1.2903	37.9648s
$\frac{1}{18}$	4.1684e-03	1.2845	63.9364s	4.1684e-03	1.2845	44.0979s

Table 4.6: L^2-errors and temporal convergence with $\alpha_1 = 0.4$ and $\alpha = 1.8$ for FEM and Algorithm 3.1.

τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{12}$	2.5335e-02	\	33.0361s	2.5335e-02	\	21.0652s
$\frac{1}{14}$	2.1169e-02	1.1653	39.143s	2.1169e-02	1.1653	23.8922s
$\frac{1}{16}$	1.8106e-02	1.1705	38.2849s	1.8106e-02	1.1705	28.1527s
$\frac{1}{18}$	1.5767e-02	1.1744	40.6447s	1.5767e-02	1.1744	29.707s

Table 4.7: L^2-errors and temporal convergence with $\alpha_1 = 0.8$ and $\alpha = 1.4$ for FEM and Algorithm 3.1.

τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{12}$	1.0859e-02	\	32.8911s	1.0859e-02	\	22.8447s
$\frac{1}{14}$	8.9067e-03	1.2859	39.262s	8.9067e-03	1.2859	26.5364s
$\frac{1}{16}$	7.5034e-03	1.2840	43.2461s	7.5034e-03	1.2840	29.9507s
$\frac{1}{18}$	6.4523e-03	1.2814	49.9362s	6.4523e-03	1.2814	32.6695s



(a) u (b) U (c) U_h

Fig. 4.1. The picture of exact solution, FEM solution and TGM solution when $\alpha_1 = 0.75, \alpha = 1.25$, and $H = \frac{1}{4}, h = \frac{1}{16}, \tau = \frac{1}{800}$.

Fig. 4.1 shows the exact solution, FEM solution, and TGM solution for Example 4.1. It is easy to see that both FEM and TGM can approximation the exact solution well.

Example 4.2. The following equation has exact solution $u(x, y, t) = (t^{3+\alpha+\alpha_1} + 1)sin\pi x sin\pi y$.

$$ {}_0^C D_t^{\alpha_1} u(x, y, t) + {}_0^C D_t^{\alpha} u(x, y, t) - \Delta u(x, y, t) = -e^u + g_2, \quad (x, y, t) \in \Omega \times (0, T], \tag{4.2} $$

where $\Omega = (0, 1) \times (0, 1), T = 1, g_2$ is a known function.

Table 4.8: L^2-errors with $\alpha_1 = 0.25$, $\alpha = 1.75$ and $\tau = \frac{1}{1500}$.

TGM	H	h	$\|u^n - u_h^n\|$	Order	CPU time
	$\frac{1}{3}$	$\frac{1}{9}$	1.0937e-02	\	25.2713s
	$\frac{1}{4}$	$\frac{1}{16}$	3.4311e-03	2.0148	63.3016s
	$\frac{1}{5}$	$\frac{1}{25}$	1.3842e-03	2.0341	134.0828s
	$\frac{1}{6}$	$\frac{1}{36}$	6.4958e-04	2.0747	294.0849s
FEM		h	$\|u^n - U^n\|$	Order	CPU time
		$\frac{1}{9}$	1.0937e-02	\	33.259s
		$\frac{1}{16}$	3.4311e-03	2.0148	79.5743s
		$\frac{1}{25}$	1.3842e-03	2.0341	191.0329s
		$\frac{1}{36}$	6.4958e-04	2.0747	411.2702s

Table 4.9: L^2-errors with $\alpha_1 = 0.9$, $\alpha = 1.3$ and $\tau = \frac{1}{1500}$.

TGM	H	h	$\|u^n - u_h^n\|$	Order	CPU time
	$\frac{1}{3}$	$\frac{1}{9}$	1.0313e-02	\	26.4137s
	$\frac{1}{4}$	$\frac{1}{16}$	3.2276e-03	2.0189	58.7831s
	$\frac{1}{5}$	$\frac{1}{25}$	1.2983e-03	2.0406	138.8660s
	$\frac{1}{6}$	$\frac{1}{36}$	6.0632e-04	2.0880	329.5642s
FEM		h	$\|u^n - U^n\|$	Order	CPU time
		$\frac{1}{9}$	1.0313e-02	\	32.7488s
		$\frac{1}{16}$	3.2276e-03	2.0189	88.3652s
		$\frac{1}{25}$	1.2983e-03	2.0406	212.0486s
		$\frac{1}{36}$	6.0632e-04	2.0880	455.5375s

Table 4.10: L^2-errors with $\alpha_1 = 0.6$, $\alpha = 1.6$ and $\tau = \frac{1}{1500}$.

TGM	H	h	$\|u^n - u_h^n\|$	Order	CPU time
	$\frac{1}{3}$	$\frac{1}{9}$	1.0269e-02	\	32.2237s
	$\frac{1}{4}$	$\frac{1}{16}$	3.2331e-03	2.0086	65.1752s
	$\frac{1}{5}$	$\frac{1}{25}$	1.3162e-03	2.0138	152.7707s
	$\frac{1}{6}$	$\frac{1}{36}$	6.2824e-04	2.0282	291.8133s
FEM		h	$\|u^n - U^n\|$	Order	CPU time
		$\frac{1}{9}$	1.0269e-02	\	36.1044s
		$\frac{1}{16}$	3.2331e-03	2.0086	86.7139s
		$\frac{1}{25}$	1.3162e-03	2.0138	208.6208s
		$\frac{1}{36}$	6.2824e-04	2.0282	463.0938s

Tables 4.8–4.10 shows that the spatial convergence rates in L^2-norm of FEM and Algorithm 3.1 are both equivalent to 2. The convergence results are both consistent with the results $O(h^{r+1})$, but Algorithm 3.1 takes much less computational time.

In Tables 4.11–4.14 with fixed mesh $h = H^2 = \frac{1}{121}$, it is observed that both the finite element method and Algorithm 3.1 generate $\min\{2 - \alpha_1, 3 - \alpha\}$ temporal convergence order, and the computing time required for Algorithm 3.1 is much less than FEM. Therefore, the TGM is indeed a very effective algorithm for solving nonlinear time fracitonal equation.

Fig. 4.2 shows the exact solution, FEM solution, and TGM solution for Example 4.2 when $\alpha_1 = 0.6, \alpha = 1.4$, and $h = H^2 = \frac{1}{25}, \tau = \frac{1}{1000}$. It is apparent that the numerical solutions of FEM and TGM are both in good agreement with the exact solution.

Table 4.11: L^2-errors and temporal convergence rate with $\alpha_1 = 0.25$ and $\alpha = 1.75$ for FEM and Algorithm 3.1.

τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{16}$	1.2417e-02	\	120.7335s	1.2417e-02	\	80.0768s
$\frac{1}{18}$	1.0744e-02	1.2287	134.1846s	1.0744e-02	1.2287	97.7199s
$\frac{1}{20}$	9.4352e-03	1.2331	146.1238s	9.4352e-03	1.2331	98.0350s
$\frac{1}{22}$	8.3861e-03	1.2366	160.3282s	8.3861e-03	1.2366	110.3966s

Table 4.12: L^2-errors and temporal convergence rate with $\alpha_1 = 0.75$ and $\alpha = 1.25$ for FEM and Algorithm 3.1.

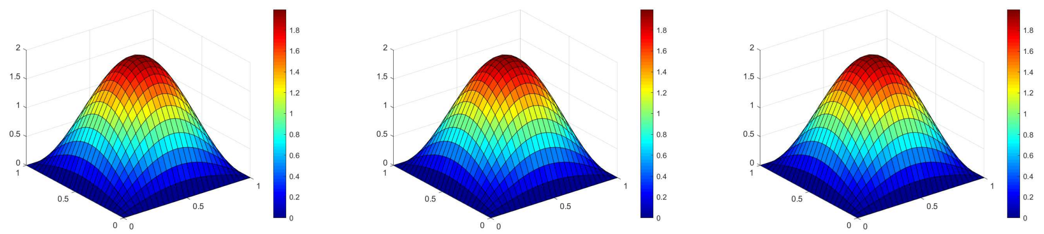
τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{16}$	4.4781e-03	\	150.4282s	4.4781e-03	\	81.3188s
$\frac{1}{18}$	3.8291e-03	1.3292	169.5714s	3.8291e-03	1.3292	90.9513s
$\frac{1}{20}$	3.3282e-03	1.3308	171.9142s	3.3282e-03	1.3308	100.6286s
$\frac{1}{22}$	2.9312e-03	1.3325	186.3463s	2.9312e-03	1.3325	110.4703s

Table 4.13: L^2-errors and temporal convergence rate with $\alpha_1 = 0.8$ and $\alpha = 1.2$ for FEM and Algorithm 3.1.

τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{16}$	5.0510e-03	\	154.846s	4.8963e-03	\	77.3909s
$\frac{1}{18}$	4.3576e-03	1.2538	183.5584s	4.1164e-03	1.2538	84.6573s
$\frac{1}{20}$	3.8179e-03	1.2549	154.6465s	3.5223e-03	1.2549	94.2824s
$\frac{1}{22}$	3.3871e-03	1.2561	216.7968s	3.0574e-03	1.2561	106.484s

Table 4.14: L^2-errors and temporal convergence rate with $\alpha_1 = 0.2$ and $\alpha = 1.8$ for FEM and Algorithm 3.1.

τ	$\|u^n - U^n\|$	Order	CPU time	$\|u^n - U_h^n\|$	Order	CPU time
$\frac{1}{16}$	1.5347e-02	\	163.2933s	1.5347e-02	\	74.7262s
$\frac{1}{18}$	1.3367e-02	1.1724	172.1797s	1.3367e-02	1.1724	81.2139s
$\frac{1}{20}$	1.1809e-02	1.1767	189.5842s	1.1809e-02	1.1767	100.4537s
$\frac{1}{22}$	1.0552e-02	1.1803	217.4856s	1.0552e-02	1.1803	104.0739s



(a) u (b) U (c) U_h

Fig. 4.2. The picture of exact solution, FEM solution, TGM solution when $\alpha_1 = 0.6, \alpha = 1.4$, and $H = \frac{1}{5}, h = \frac{1}{25}, \tau = \frac{1}{1000}$.

5. Conclusion

With detailed theoretical analysis and numerical experiments, we construct TGM for 2D nonlinear time fractional two-term mixed sub-diffusion and diffusion wave equations. It is easy to extend our estimates and algorithm to other nonlinear fractional equations. Future research will be performed to consider the more complicated two-grid algorithms, and propose a new stability estimate and convergence for L1-CN scheme with nonsmooth data [9, 13], when the $C^3[0,T]$ assumption does not hold for problem (2.3).

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