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A HYBRID VISCOSITY APPROXIMATION METHOD FOR A COMMON SOLUTION OF A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES, AN EQUILIBRIUM PROBLEM, AND FIXED POINT PROBLEMS*

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Abstract

In this paper, we introduce a new iterative method based on the hybrid viscosity approximation method for finding a common element of the set of solutions of a general system of variational inequalities, an equilibrium problem, and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space. We prove a strong convergence theorem of the proposed iterative scheme under some suitable conditions on the parameters. Furthermore, we apply our main result for W-mappings. Finally, we give two numerical results to show the consistency and accuracy of the scheme.

Mathematics subject classification: 47H10, 47J25, 47H09, 65J15, 46T20.
Key words: Equilibrium problem, Iterative method, Fixed point, Variational inequality.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx-Ty\|\le\|x-y\|$ for all $x,y\in C$. We use $Fix(T)$ to denote the set of fixed points T, i.e., $Fix(T)=\{x\in C: Tx=x\}$. Also, $f:C\to C$ is a contraction if $\|f(x)-f(y)\|\le\kappa\|x-y\|$ for all $x,y\in C$ and some constant $\kappa\in[0,1)$. In this case, f is said to be a κ-contraction.

Consider an equilibrium problem (EP) which is to find a point $x\in C$ satisfying the property:

$$\phi(x,y)\ge 0 \quad \text{for all } y\in C, \tag{1.1}$$

where $\phi: C\times C\to\mathbb{R}$ is a bifunction of C. We use $EP(\phi)$ to denote the set of solutions of EP (1.1), that is, $EP(\phi)=\{x\in C: (1.1) \text{ holds}\}$. The EP (1.1) includes, as special cases, numerous problems in physics, optimization and economics. Some authors (e.g., [12–14, 17–20, 22–24]) have proposed some useful methods for solving the EP (1.1). Set $\phi(x,y)=\langle Ax,y-x\rangle$ for all $x,y\in C$, where $A:C\to H$ is a nonlinear mapping. Then, $x^*\in EP(\phi)$ if and only if

$$\langle Ax^*,y-x^*\rangle\ge 0 \quad \text{for all } y\in C, \tag{1.2}$$

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that is, x^* is a solution of the variational inequality. The (1.2) is well known as the classical variational inequality. The set of solutions of (1.2) is denoted by $VI(A, C)$.

In 2008, Ceng et al. [5] considered the following problem of finding $(x^*, y^*) \in C \times C$ satisfying

$$\begin{cases} \langle \nu Ay^* + x^* - y^*, x - x^* \rangle \geq 0 & \text{for all } x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0 & \text{for all } x \in C, \end{cases} \tag{1.3}$$

which is called a general system of variational inequalities, where $A, B : C \to H$ are two nonlinear mappings, $\nu > 0$ and $\mu > 0$ are two fixed constants. Precisely, they introduced the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences, S is a nonexpansive mapping on C, P_C is the metric projection of H onto C and obtained strong convergence theorem.

The implicit midpoint rules for solving fixed point problems of nonexpansive mappings are a powerful numerical method for solving ordinary differential equations. So, many authors have studied them; see [2, 7, 10, 16, 21] and the references therein. In 2015, Xu et al. [21] applied the viscosity technique to the implicit midpoint rule for nonexpansive mappings and proposed the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence. They proved the sequence $\{x_n\}$ converges strongly to a fixed point of T which is the unique solution of a certain variational inequality.
Also, Ke and Ma [10] studied the following generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n x_n + (1 - t_n)x_{n+1}), \quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are real sequences. They showed the sequence $\{x_n\}$ converges strongly to a fixed point of T which is the unique solution of a certain variational inequality.

Recently, Cai et al. [4] introduced the following modified viscosity implicit rules

$$\begin{cases} x_1 \in C, \\ u_n = t_n x_n + (1 - t_n)y_n, \\ z_n = P_C(I - \mu B)u_n, \\ y_n = P_C(I - \lambda A)z_n, \\ x_{n+1} = P_C(\alpha_n f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \rho F)Ty_n), \quad n \geq 1, \end{cases}$$

where F is a Lipschitzian and strongly monotone map, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are real sequences, P_C is the metric projection of H onto C. Under some suitable assumptions imposed on the parameters, they obtained some strong convergence theorems.

In this paper, motivated by the above results, we propose a new composite iterative scheme for finding a common element of the set of solutions of a general system of variational inequalities, an equilibrium problem and the set of common fixed points of a countable family of nonexpansive mappings in Hilbert spaces. Then, we prove a strong convergence theorem and apply our main result for W-mappings. Finally, we give two numerical examples for supporting our main result.

2. Preliminaries

Let H be a real Hilbert space. We use $\rightharpoonup$ and $\to$ to denote the weak and strong convergence in H, respectively. The following identity holds:

$$\|\alpha x + \beta y\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta\|x-y\|^2,$$

for all $x, y \in H$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle\| \geq \|P_C(x) - P_C(y)\|^2 \quad \text{for all } x, y \in H. \tag{2.1}$$

Further, for $x \in H$ and $z \in C$, we have

$$z = P_C(x) \iff \langle x - z, z - y\rangle \geq 0 \quad \text{for all } y \in C.$$

Definition 2.1 ([4]). *A mapping $T : H \to H$ is called firmly nonexpansive if for any $x, y \in H$,*

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

Lemma 2.1 ([3]). *Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(A_1) *$\phi(x,x) = 0$ for all $x \in C$;*

(A_2) *ϕ is monotone, i.e., $\phi(x,y) + \phi(y,x) \leq 0$ for all $x, y \in C$;*

(A_3) *for each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y)$;*

(A_4) *for each $x \in C, y \mapsto \phi(x,y)$ is convex and weakly lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } y \in C.$$

Lemma 2.2 ([6]). *Assume $\phi : C \times C \to \mathbb{R}$ satisfies the conditions (A_1)-(A_4). For $r > 0$, define a mapping $Q_r : H \to C$ by*

$$Q_r x := \{z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C\}, \tag{2.2}$$

for all $x \in H$. Then, the following hold:

(i) *Q_r is single-valued;*

(ii) *Q_r is firmly nonexpansive;*

(iii) *$Fix(Q_r) = EP(\phi)$;*

(iv) $EP(\phi)$ *is closed and convex.*

Definition 2.2 ([4]). *A nonlinear operator A whose domain $D(A) \subseteq H$ and the range $R(A) \subseteq H$ is said to be $\alpha-$ inverse strongly monotone (for short, $\alpha-$ism) if there exists $\alpha > 0$ such that*

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2 \quad \text{for all } x, y \in D(A).$$

Lemma 2.3 ([8]). *Let C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

Lemma 2.4 ([1]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n v_n + \mu_n,$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$, $\{\mu_n\}$ is a sequence of nonnegative real numbers, and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} \gamma_n = \infty, \limsup_{n\to\infty} v_n \leq 0$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([5]). *For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of problem (1.3) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C(P_C(x - \mu Bx) - \nu AP_C(x - \mu Bx)) \quad \text{for all } x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Lemma 2.6 ([1]). *Let C be a nonempty closed bounded subset of H. Suppose*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

3. Main Results

Theorem 3.1. *Let C be a closed convex subset of H, $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A_1) - (A_4)$ of Lemma 2.1, $A, B : C \to H$ be α-ism and β-ism, respectively, $\{T_n\}$ be an infinite family of nonexpansive self-mappings on C and f be a κ-contraction on C for some $\kappa \in [0, 1)$. Set $\Gamma := \cap_{n=1}^{\infty} Fix(T_n) \cap Fix(G) \cap EP(\phi)$, where G is a mapping defined by Lemma 2.5 and assume $\Gamma \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{t_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(B_2) $\{r_n\} \subset (a, \infty)$ *for some* $a > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*;*

(B_3) $\{t_n\} \subset (b, 1]$ *for some* $b > 0$ *and* $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = t_n x_n + (1-t_n)y_n, \\ \phi(v_n, y) + \dfrac{1}{r_n}\langle y - v_n, v_n - u_n\rangle \geq 0 \quad \text{for all} \ \ y \in C, \\ z_n = P_C(I - \mu B)v_n, \\ y_n = P_C(I - \nu A)z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n y_n, \qquad n \geq 0, \end{cases} \tag{3.1}$$

where the initial guess $x_0 \in C$ is arbitrary, $\nu \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. Suppose $\sum_{n=1}^{\infty} \sup\{\|T_{n+1} z - T_n z\| : z \in K\} < \infty$ for any bounded subset K of C. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and $Fix(T) = \bigcap_{n=1}^{\infty} Fix(T_n)$. Then, the sequence $\{x_n\}$ converges strongly to $q \in \Gamma$, where $q = P_\Gamma f(q)$, which solves the variational inequality (VI):

$$\langle (I - f)q, q - x\rangle \leq 0 \quad \text{for all } x \in \Gamma. \tag{3.2}$$

To prove Theorem 3.1 we first establish some lemmas.

Lemma 3.1. *Let $C \subseteq H$, $A : C \to H$ be an α-ism and $\nu \in (0, 2\alpha)$. Then $I - \nu A$ is nonexpansive.*

Proof. For $x, y \in C$, we have

$$\begin{aligned} \|(I-\nu A)x - (I-\nu A)y\|^2 &= \|x - y - \nu(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\nu\langle x - y, Ax - Ay\rangle + \nu^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - 2\alpha\nu\|Ax - Ay\|^2 + \nu^2\|Ax - Ay\|^2 \\ &= \|x-y\|^2 + \nu(\nu - 2\alpha)\|Ax - Ay\|^2 \qquad (3.3) \\ &\leq \|x-y\|^2. \end{aligned}$$

□

Lemma 3.2. *Let $\{x_n\}$ be a sequence in H and $\{r_n\} \subset (a, \infty)$ for some $a > 0$. Suppose $u_n = Q_{r_n} x_n$ and $u_{n+1} = Q_{r_{n+1}} x_{n+1}$. Then*

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|.$$

Proof. By definition of Q_r, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \tag{3.4}$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \text{for all } y \in C. \tag{3.5}$$

Set $y = u_{n+1}$ in (3.4) and $y = u_n$ in (3.5). Then by adding these two inequalities and using (A_2), we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0,$$

and hence

$$\langle u_{n+1}-u_n, u_n-u_{n+1}+u_n-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle \geq 0.$$

This implies

$$\begin{aligned}\|u_{n+1}-u_n\|^2 \leq &\langle u_{n+1}-u_n, x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle\\ \leq &\|u_{n+1}-u_n\|\{\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|\}.\end{aligned}$$

Therefore

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|.$$

□

Lemma 3.3. *Let C be a closed convex subset of H, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences in C defined by*

$$\begin{cases} u_n = t_n x_n + (1-t_n)y_n,\\ v_n = Q_{r_n}u_n,\\ y_n = Gv_n,\end{cases}$$

where $0<b<t_n<1$ and $0<a<r_n$ for all $n\in N$ and G defined by Lemma 2.5. Suppose the sequences $\{x_n\}$ and $\{v_n\}$ are bounded. Then

$$\|y_{n+1}-y_n\| \leq \|x_{n+1}-x_n\|+(|t_{n+1}-t_n|+|r_{n+1}-r_n|)M,$$

where $M=\sup\left\{\frac{1}{ab}\|v_n-u_n\|, \frac{1}{b}\|x_n-y_n\| : n\in\mathbb{N}\right\}$.

Proof. From Lemma 3.2, we have

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|Q_{r_{n+1}}u_{n+1}-Q_{r_n}u_n\|\\ &\leq \|u_{n+1}-u_n\|+|1-\frac{r_n}{r_{n+1}}|\|v_{n+1}-u_{n+1}\|\\ &\leq \|u_{n+1}-u_n\|+|\frac{r_{n+1}-r_n}{r_{n+1}}|\|v_{n+1}-u_{n+1}\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\|y_{n+1}-y_n\|\\ &= \|P_C(I-\nu A)P_C(I-\mu B)v_{n+1}-P_C(I-\nu A)P_C(I-\mu B)v_n\|\\ &\leq \|v_{n+1}-v_n\|\\ &\leq \|u_{n+1}-u_n\|+|\frac{r_{n+1}-r_n}{r_{n+1}}|\|v_{n+1}-u_{n+1}\|\\ &= \|t_{n+1}x_{n+1}+(1-t_{n+1})y_{n+1}-t_nx_n-(1-t_n)y_n\|\\ &\qquad +|\frac{r_{n+1}-r_n}{r_{n+1}}|\|v_{n+1}-u_{n+1}\|\\ &= \|t_{n+1}(x_{n+1}-x_n)+(t_{n+1}-t_n)x_n+(1-t_{n+1})(y_{n+1}-y_n)-(t_{n+1}-t_n)y_n\|\\ &\qquad +|\frac{r_{n+1}-r_n}{r_{n+1}}|\|v_{n+1}-u_{n+1}\|\\ &\leq t_{n+1}\|x_{n+1}-x_n\|+(1-t_{n+1})\|y_{n+1}-y_n\|+|t_{n+1}-t_n|\|x_n-y_n\|\\ &\qquad +|\frac{r_{n+1}-r_n}{r_{n+1}}|\|v_{n+1}-u_{n+1}\|,\end{aligned}$$

which implies

$$
\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\le \|x_{n+1} - x_n\| + \frac{|t_{n+1} - t_n|}{t_{n+1}}\|x_n - y_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}t_{n+1}}\|v_{n+1} - u_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \frac{|t_{n+1} - t_n|}{b}\|x_n - y_n\| + \frac{|r_{n+1} - r_n|}{ab}\|v_{n+1} - u_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + (|t_{n+1} - t_n| + |r_{n+1} - r_n|)M.
\end{aligned} \tag{3.6}
$$

□

Lemma 3.4. *Let C, f and $\{T_n\}$ be defined by Theorem 3.1 satisfying $F := \cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences in C and $p \in F$. Define*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n y_n, \;\; n \ge 0, \tag{3.7}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\|y_n - p\| \le \|x_n - p\|$ for all $n \in N$. Then

(i) *$\{x_n\}$ is bounded;*

(ii) *Setting $M' = \sup\{\|f(x_n)\|, \|T_n y_n\| : n \in \mathbb{N}\}$, we get*

$$
\begin{aligned}
\|x_{n+1} - x_{n+2}\| \le{}& \alpha_n\kappa\|x_n - x_{n+1}\| + 2M'|\alpha_n - \alpha_{n+1}| \\
&+ (1 - \alpha_n)(\|T_n y_n - T_{n+1}y_n\| + \|y_n - y_{n+1}\|);
\end{aligned} \tag{3.8}
$$

(iii) *For all $n \in N$, we have*

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& (1 - (1 - \kappa)\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n^2\kappa\|x_n - p\|\|f(p) - p\| \\
&+ \alpha_n^2\|f(p) - p\|^2 + 2\alpha_n(1 - \alpha_n)\langle f(p) - p, T_n y_n - p\rangle.
\end{aligned}
$$

Proof. We prove the statements in order as the following:
Proof of (i). From (3.7), we get

$$
\begin{aligned}
\|x_{n+1} - p\| &\le \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(T_n y_n - p)\| \\
&\le \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1 - \alpha_n)\|y_n - p\| \\
&\le \alpha_n\kappa\|x_n - p\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\le (1 - (1 - \kappa)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\le \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\}.
\end{aligned}
$$

By induction, we obtain $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\kappa}\}$ for all $n \ge 1$. Hence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{T_n y_n\}$.
Proof of (ii). From (3.7), we have

$$
\begin{aligned}
&\|x_{n+1} - x_{n+2}\| \\
&= \|\alpha_n f(x_n) + (1 - \alpha_n)T_n y_n - \alpha_{n+1}f(x_{n+1}) - (1 - \alpha_{n+1})T_{n+1}y_{n+1}\| \\
&= \|\alpha_n(f(x_n) - f(x_{n+1})) + (\alpha_n - \alpha_{n+1})f(x_{n+1}) \\
&\qquad + (1 - \alpha_n)(T_n y_n - T_{n+1}y_{n+1}) + (\alpha_{n+1} - \alpha_n)T_{n+1}y_{n+1}\| \\
&\le \alpha_n\kappa\|x_n - x_{n+1}\| + 2M'|\alpha_n - \alpha_{n+1}| + (1 - \alpha_n)(\|T_n y_n - T_{n+1}y_n\| \\
&\qquad + \|T_n y_n - T_n y_{n+1}\|) \\
&\le \alpha_n\kappa\|x_n - x_{n+1}\| + 2M'|\alpha_n - \alpha_{n+1}| + (1 - \alpha_n)(\|T_n y_n - T_{n+1}y_n\| \\
&\qquad + \|y_n - y_{n+1}\|),
\end{aligned}
$$

for all $n \in \mathbb{N}$.
Proof of (iii). By using (3.7), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(T_ny_n-p)\|^2 \\
&= \alpha_n^2\|(f(x_n)-f(p))+(f(p)-p)\|^2+(1-\alpha_n)^2\|T_ny_n-p\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-p, T_ny_n-p\rangle \\
&\le \alpha_n^2(\kappa^2\|x_n-p\|^2+\|f(p)-p\|^2)+(1-\alpha_n)^2\|x_n-p\|^2 \\
&\quad +2\alpha_n^2\langle f(x_n)-f(p), f(p)-p\rangle \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-f(p), T_ny_n-p\rangle \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(p)-p, T_ny_n-p\rangle \\
&\le (\alpha_n^2\kappa^2+(1-\alpha_n)^2)\|x_n-p\|^2+2\alpha_n^2\kappa\|x_n-p\|\|f(p)-p\| \\
&\quad +\alpha_n^2\|f(p)-p\|^2+2\alpha_n(1-\alpha_n)\kappa\|x_n-p\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(p)-p, T_ny_n-p\rangle \\
&\le (1-(1-\kappa)\alpha_n)^2\|x_n-p\|^2+2\alpha_n^2\kappa\|x_n-p\|\|f(p)-p\| \\
&\quad +\alpha_n^2\|f(p)-p\|^2+2\alpha_n(1-\alpha_n)\langle f(p)-p, T_ny_n-p\rangle \\
&= (1-(1-\kappa)\alpha_n)^2\|x_n-p\|^2+\alpha_n(1-\kappa)[\frac{1}{1-\kappa}(\alpha_n\|f(p)-p\|^2 \\
&\quad +2\alpha_n\kappa\|x_n-p\|\|f(p)-p\|+2(1-\alpha_n)\langle f(p)-p, T_ny_n-p\rangle)]. \qquad \square
\end{aligned}$$

Lemma 3.5. *Let all the assumptions of Theorem* 3.1 *hold and* $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. *Then* $\lim_{n\to\infty}\|y_n-v_n\|=0$.

Proof. Suppose $x^*\in\Gamma$ and $y^*=P_C(x^*-\mu Bx^*)$. So from (3.1), we obtain

$$\|u_n-x^*\|=\|t_nx_n+(1-t_n)y_n-x^*\|\le t_n\|x_n-x^*\|+(1-t_n)\|y_n-x^*\|. \tag{3.9}$$

Noticing $v_n=Q_{r_n}u_n$ and $Q_{r_n}x^*=x^*$, we get

$$\|v_n-x^*\|\le\|u_n-x^*\|. \tag{3.10}$$

Then from (3.9), we have

$$\begin{aligned}
\|y_n-x^*\| &= \|Gv_n-x^*\|=\|Gv_n-Gx^*\|\le\|u_n-x^*\| \\
&\le t_n\|x_n-x^*\|+(1-t_n)\|y_n-x^*\|.
\end{aligned}$$

Hence $\|y_n-x^*\|\le\|x_n-x^*\|$. Therefore by using Lemma 3.4 (i), we obtain $\{x_n\}$ is bounded and by (3.9), we have $\|u_n-x^*\|\le\|x_n-x^*\|$. So from (3.10), we get $\|v_n-x^*\|\le\|x_n-x^*\|$. Hence by (3.3), we have

$$\begin{aligned}
\|z_n-y^*\|^2 &= \|P_C(I-\mu B)v_n-P_C(I-\mu B)x^*\|^2 \\
&\le \|(I-\mu B)v_n-(I-\mu B)x^*\|^2 \\
&\le \|v_n-x^*\|^2-\mu(2\beta-\mu)\|Bv_n-Bx^*\|^2 \\
&\le \|x_n-x^*\|^2-\mu(2\beta-\mu)\|Bv_n-Bx^*\|^2.
\end{aligned} \tag{3.11}$$

In a similar way, we get

$$\|y_n - x^*\|^2 \leq \|z_n - y^*\|^2 - \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2. \tag{3.12}$$

Substituting (3.11) into (3.12), we obtain

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \mu(2\beta - \mu)\|Bv_n - Bx^*\|^2 - \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2. \tag{3.13}$$

It follows from (3.13) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(T_n y_n - x^*)\|^2 \\
&\leq \alpha_n\|f(x_n) - f(x^*) + f(x^*) - x^*\|^2 \\
&\quad + (1-\alpha_n)\|T_n y_n - x^*\|^2 \\
&\leq \alpha_n\kappa^2\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2 \\
&\quad + (1-\alpha_n)\|y_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), f(x^*) - x^*\rangle \\
&\leq \alpha_n\kappa^2\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2 \\
&\quad + 2\alpha_n\kappa\|x_n - x^*\|\|f(x^*) - x^*\| \\
&\quad + (1-\alpha_n)(\|x_n - x^*\|^2 - \mu(2\beta - \mu)\|Bv_n - Bx^*\|^2 \\
&\quad - \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2) \\
&\leq (1 - (1-\kappa^2)\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2 \\
&\quad + 2\alpha_n\kappa\|x_n - x^*\|\|f(x^*) - x^*\| + (1-\alpha_n) \\
&\quad (-\mu(2\beta - \mu)\|Bv_n - Bx^*\|^2 - \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2),
\end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned}
&(1-\alpha_n)(\mu(2\beta - \mu)\|Bv_n - Bx^*\| + \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2) \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_2 \\
&\leq (\|x_n - x^*\| - \|x_{n+1} - x^*\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + \alpha_n M_2 \\
&\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + \alpha_n M_2,
\end{aligned}$$

where $M_2 = \sup\{\|f(x^*) - x^*\|^2 + 2\kappa\|x_n - x^*\|\|f(x^*) - x^*\| : n \in \mathbb{N}\}$. From (B_1) and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have

$$\lim_{n\to\infty}\|Bv_n - Bx^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Az_n - Ay^*\| = 0. \tag{3.15}$$

On the other hand by (2.1), we get

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(I - \nu A)z_n - P_C(I - \nu A)y^*\|^2 \\
&\leq \langle (I - \nu A)z_n - (I - \nu A)y^*, y_n - x^*\rangle \\
&= \frac{1}{2}[\|(I - \nu A)z_n - (I - \nu A)y^*\|^2 + \|y_n - x^*\|^2 \\
&\qquad - \|z_n - y_n + x^* - y^* - \nu(Az_n - Ay^*)\|^2].
\end{aligned}$$

This implies

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|z_n - y^*\|^2 - \|z_n - y_n + x^* - y^* - \nu(Az_n - Ay^*)\|^2\\ &= \|z_n - y^*\|^2 - [\|z_n - y_n + x^* - y^*\|^2 + \nu^2\|Az_n - Ay^*\|^2\\ &\quad - 2\nu\langle z_n - y_n + x^* - y^*, Az_n - Ay^*\rangle]\\ &\le \|z_n - y^*\|^2 - \|z_n - y_n + x^* - y^*\|^2\\ &\quad + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\|. \end{aligned} \tag{3.16}$$

Again by (2.1), we obtain

$$\begin{aligned}\|z_n - y^*\|^2 &= \|P_C(I-\mu B)v_n - P_C(I-\mu B)x^*\|^2\\ &\le \langle (I-\mu B)v_n - (I-\mu B)x^*, z_n - y^*\rangle\\ &= \frac{1}{2}[\|(I-\mu B)v_n - (I-\mu B)x^*\|^2 + \|z_n - y^*\|^2\\ &\quad - \|v_n - z_n + y^* - x^* - \mu(Bv_n - Bx^*)\|^2],\end{aligned}$$

which implies

$$\begin{aligned}&\|z_n - y^*\|^2\\ &\le \|v_n - x^*\|^2 - \|v_n - z_n + y^* - x^* - \mu(Bv_n - Bx^*)\|^2\\ &= \|v_n - x^*\|^2 - [\|v_n - z_n + y^* - x^*\|^2\\ &\quad - 2\mu\langle v_n - z_n + y^* - x^*, Bv_n - Bx^*\rangle + \mu^2\|Bv_n - Bx^*\|^2]\\ &\le \|x_n - x^*\|^2 - \|v_n - z_n + y^* - x^*\|^2\\ &\quad + 2\mu\|v_n - z_n + y^* - x^*\|\ \|Bv_n - Bx^*\|. \end{aligned} \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\begin{aligned}\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 &- \|v_n - z_n + y^* - x^*\|\\ &- \|z_n - y_n + x^* - y^*\|^2\\ &+ 2\mu\|v_n - z_n + y^* - x^*\|\ \|Bv_n - Bx^*\|\\ &+ 2\nu\|z_n - y_n + x^* - y^*\|\ \|Az_n - Ay^*\|. \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.14), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 = \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(T_n y_n - x^*)\|^2\\ &\le \alpha_n\|f(x_n) - f(x^*) + f(x^*) - x^*\|^2 + (1-\alpha_n)\|T_n y_n - x^*\|^2\\ &\le \alpha_n\kappa^2\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2\\ &\quad + (1-\alpha_n)\|y_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), f(x^*) - x^*\rangle\\ &\le \alpha_n\kappa^2\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2 + 2\alpha_n\kappa\|x_n - x^*\|\|f(x^*) - x^*\|\\ &\quad + (1-\alpha_n)(\|x_n - x^*\|^2 - \|v_n - z_n + y^* - x^*\|^2 - \|z_n - y_n + x^* - y^*\|^2\\ &\quad + 2\mu\|v_n - z_n + y^* - x^*\|\|Bv_n - Bx^*\|\\ &\quad + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\|)\\ &\le \|x_n - x^*\|^2 + (1-\alpha_n)(-\|v_n - z_n + y^* - x^*\|^2 - \|z_n - y_n + x^* - y^*\|^2\\ &\quad + 2\alpha_n M_3 + 2\mu\|v_n - z_n + y^* - x^*\|\|Bv_n - Bx^*\|\\ &\quad + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\|,\end{aligned}$$

where $M_3 = \sup\{\|f(x^*) - x^*\|^2, 2\kappa\|x_n - x^*\|\|f(x^*) - x^*\| : n \in N\}$. This implies

$$\begin{aligned}
&(1-\alpha_n)\|v_n - z_n + y^* - x^*\|^2 + (1-\alpha_n)\|z_n - y_n + x^* - y^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu\|v_n - z_n + y^* - x^*\|\|Bv_n - Bx^*\| \\
&\quad + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\| + 2\alpha_n M_3 \\
&\le \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad + 2\mu\|v_n - z_n + y^* - x^*\|\|Bv_n - Bx^*\| \\
&\quad + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\| + 2\alpha_n M_3.
\end{aligned}$$

From (B_1), $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and (3.15), we get

$$\lim_{n\to\infty}\|v_n - z_n + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|z_n - y_n + x^* - y^*\| = 0. \tag{3.19}$$

By (3.19) and

$$\|v_n - y_n\| \le \|v_n - z_n + y^* - x^*\| + \|z_n - y_n + x^* - y^*\|,$$

we obtain $\lim_{n\to\infty}\|v_n - y_n\| = 0$. □

Proof of Theorem 3.1

Since $P_\Gamma f$ is a contraction on Γ, there exists a unique element $q \in \Gamma$ such that $q = P_\Gamma f(q)$; equivalently, q is the unique solution of VI (3.2). Now, we proceed with the following steps:

Step 1. We claim $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Suppose $x^* \in \Gamma$ and $y^* = P_C(x^* - \mu Bx^*)$. As in the proof of Lemma 3.5, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{T_n y_n\}$. Set

$$M_1 = \sup\left\{\|f(x_n)\|, \|T_n y_n\|, \frac{1}{ab}\|v_n - u_n\|, \frac{1}{b}\|x_n - y_n\| : n \in \mathbb{N}\right\}.$$

Hence, substituting (3.6) into (3.8), we have

$$\begin{aligned}
&\|x_{n+1} - x_{n+2}\| \\
&\le \alpha_n\kappa\|x_n - x_{n+1}\| + 2M_1|\alpha_n - \alpha_{n+1}| + (1-\alpha_n)(\sup\{\|T_n z - T_{n+1}z\| : z \in K\} \\
&\quad + \|x_{n+1} - x_n\| + (|t_{n+1} - t_n| + |r_{n+1} - r_n|)M_1), \\
&\le (1 - (1-\kappa)\alpha_n)\|x_n - x_{n+1}\| + M_1(2|\alpha_n - \alpha_{n+1}| + |t_{n+1} - t_n| + |r_{n+1} - r_n|) \\
&\quad + \sup\{\|T_n z - T_{n+1}z\| : z \in K\},
\end{aligned}$$

where $K = \{y_n : n \in N\}$. So, from Lemma 2.4, we obtain $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 2. We claim $\lim_{n\to\infty}\|x_n - v_n\| = 0$. From Lemma 2.2, we have

$$\begin{aligned}
\|v_n - x^*\|^2 &= \|Q_{r_n}u_n - Q_{r_n}x^*\|^2 \le \langle u_n - x^*, v_n - x^*\rangle \\
&= \frac{1}{2}(\|u_n - x^*\|^2 + \|v_n - x^*\|^2 - \|v_n - u_n\|^2).
\end{aligned}$$

This implies

$$\|v_n - x^*\|^2 \le \|u_n - x^*\|^2 - \|u_n - v_n\|^2. \tag{3.20}$$

So, we derive from (3.20) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(T_ny_n-x^*)\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\|v_n-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)(\|u_n-x^*\|^2-\|u_n-v_n\|^2)\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\|u_n-x^*\|^2-\|u_n-v_n\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|u_n-v_n\|^2 &\le \|u_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|f(x_n)-x^*\|^2\\ &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|f(x_n)-x^*\|^2\\ &\le \|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+\alpha_n\|f(x_n)-x^*\|^2.\end{aligned}$$

Therefore from Step 1, we obtain $\lim_{n\to\infty}\|u_n-v_n\|=0$. So, by using Lemma 3.5, we get $\lim_{n\to\infty}\|u_n-y_n\|=0$. From (3.1), we have $\|u_n-y_n\|=t_n\|x_n-y_n\|$. Hence

$$\|x_n-y_n\|=\frac{\|u_n-y_n\|}{t_n}\le\frac{\|u_n-y_n\|}{b}.$$

Therefore $\lim_{n\to\infty}\|x_n-y_n\|=0$. So, from Lemma 3.5, we get

$$\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{3.21}$$

Also, from (3.1), we have

$$\lim_{n\to\infty}\|x_{n+1}-T_ny_n\|=\lim_{n\to\infty}\alpha_n\|f(x_n)-T_ny_n\|=0.$$

Hence

$$\lim_{n\to\infty}\|x_n-T_ny_n\|=0. \tag{3.22}$$

Since

$$\begin{aligned}\|v_n-T_nv_n\| &\le \|T_nv_n-T_ny_n\|+\|T_ny_n-x_n\|+\|x_n-v_n\|\\ &\le \|v_n-y_n\|+\|T_ny_n-x_n\|+\|x_n-v_n\|,\end{aligned}$$

from Lemma 3.5, (3.21) and (3.22), we obtain $\lim_{n\to\infty}\|v_n-T_nv_n\|=0$. Therefore from

$$\begin{aligned}\|v_n-Tv_n\| &\le \|T_nv_n-Tv_n\|+\|v_n-T_nv_n\|\\ &\le \sup\{\|T_nz-Tz\|:z\in K'\}+\|v_n-T_nv_n\|,\end{aligned}$$

where $K'=\{v_n:n\in N\}$ and Lemma 2.6, we have

$$\lim_{n\to\infty}\|v_n-Tv_n\|=0. \tag{3.23}$$

Step 3. We claim $\limsup_{n\to\infty}\langle(I-f)q,q-T_ny_n\rangle\le 0$. To show this, choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle(I-f)q,q-v_n\rangle=\lim_{i\to\infty}\langle(I-f)q,q-v_{n_i}\rangle.$$

Since $\{v_{n_i}\}$ is bounded, without loss of generality, we assume $v_{n_i}\rightharpoonup z$. We show $z\in\Gamma$. From (3.23) and Lemma 2.3, we get $z\in Fix(T)$. Now, we show $z\in EP(\phi)$. Since $v_n=Q_{r_n}u_n$, we obtain

$$\phi(v_n,y)+\frac{1}{r_n}\langle y-v_n,v_n-u_n\rangle\ge 0 \quad \text{for all } y\in C.$$

From (A_2), we get $\frac{1}{r_n}\langle y - v_n, v_n - u_n\rangle \geq \phi(y, v_n)$ for all $y \in C$. Replacing n by n_i, we have

$$\frac{1}{r_{n_i}}\langle y - v_{n_i}, v_{n_i} - u_{n_i}\rangle \geq \phi(y, v_{n_i}) \text{ for all } y \in C.$$

Since $v_{n_i} \rightharpoonup z$ and $\lim_{i\to\infty} \|v_{n_i} - u_{n_i}\| = 0$, it follows from (A_4) and (B_2) that $\phi(y, z) \leq 0$ for all $y \in C$. Set $y_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in C$. Then $y_t \in C$ and hence $\phi(y_t, z) \leq 0$. From (A_1) and (A_2), we obtain

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, z) \leq t\phi(y_t, y).$$

Therefore $\phi(y_t, y) \geq 0$. Letting $t \to 0$, we get $\phi(z, y) \geq 0$ for all $y \in C$. This implies $z \in EP(\phi)$. Moreover, we know

$$\lim_{i\to\infty} \|v_{n_i} - Gv_{n_i}\| = \lim_{i\to\infty} \|v_{n_i} - y_{n_i}\| = 0.$$

From Lemma 2.3, we have $z \in Fix(G)$. So $z \in \Gamma$. From $q = P_\Gamma f(q)$, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)q, q - T_n y_n\rangle &= \lim_{i\to\infty}\langle (I-f)q, q - T_{n_i} y_{n_i}\rangle \\ &= \lim_{i\to\infty}\langle (I-f)q, q - T_{n_i} v_{n_i}\rangle \\ &= \lim_{i\to\infty}\langle (I-f)q, q - v_{n_i}\rangle \\ &= \lim_{i\to\infty}\langle (I-f)q, q - z\rangle \leq 0.\end{aligned}$$

Step 4. We claim $\{x_n\}$ converges strongly to q. By using lemma 3.4 (iii), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq (1 - (1-\kappa)\alpha_n)\|x_n - q\|^2 + \alpha_n(1-\kappa)[\frac{1}{1-\kappa}(\alpha_n\|f(q) - q\|^2 \\ + 2\alpha_n\kappa\|x_n - q\|\|f(q) - q\| + 2(1-\alpha_n)\langle f(q) - q, T_n y_n - q\rangle)].\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq &(1-\gamma_n)\|x_n - q\|^2 + \gamma_n[\frac{1}{1-\kappa}(\alpha_n\|f(q) - q\|^2 \\ &+ 2\alpha_n\kappa\|x_n - q\|\|f(q) - q\| + 2(1-\alpha_n)\langle f(q) - q, T_n y_n - q\rangle)],\end{aligned} \tag{3.24}$$

where $\gamma_n = \alpha_n(1-\kappa)$, we may apply Lemma 2.4 to (3.24) to obtain that $\|x_n - q\| \to 0$, that is, $x_n \to q$ in norm.

Corollary 3.1. *Let all the assumptions of Theorem* 3.1 *hold except the bifunction* $\phi = 0$ *and* $\Gamma := \cap_{n=1}^\infty Fix(T_n) \cap Fix(G)$ *[instead of* $\Gamma := \cap_{n=1}^\infty Fix(T_n) \cap Fix(G) \cap EP(\phi)$*]. Then, the sequences* $\{x_n\}$ *defined by*

$$\begin{cases} u_n = t_n x_n + (1-t_n)y_n, \\ z_n = P_C(I - \mu B)u_n, \\ y_n = P_C(I - \nu A)z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n y_n, \quad n \geq 0, \end{cases}$$

where the initial guess $x_0 \in C$ *is arbitrary, converges strongly to* $q \in \Gamma$*, where* $q = P_\Gamma f(q)$*, which solves the variational inequality* (3.2).

Remark 3.1. Corollary 3.1 is a generalization of [4, Theorem 3.1] in the sense that the old theorem establishes just for a single nonexpansive mapping, but Corollary 3.1 establishes for a sequence of nonexpansive mappings.

4. Applications

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{4.1}$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, T_2, \cdots, T_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$; see [11].

Lemma 4.1 ([15]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 4.1, one can define mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \tag{4.2}$$

for every $x \in C$. Such a W is called the $W-$mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Throughout this section, we assume $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$.

Lemma 4.2 ([15]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, $Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$.*

Theorem 4.1. *Let C be a closed convex subset of H, $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A_1) - (A_4)$ of Lemma 2.1, $A, B : C \to H$ be α-ism and β-ism, respectively and f a κ-contraction on C for some $\kappa \in [0,1)$. Set $\Gamma := \cap_{n=1}^{\infty} Fix(T_n) \cap Fix(G) \cap EP(\phi)$, where G is a mapping defined by Lemma 2.5 and assume $\Gamma \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{t_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B$_1$) *$\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(B$_2$) *$\{r_n\} \subset (a,\infty)$ for some $a > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;*

(B$_3$) *$\{t_n\} \subset (c,1]$ for some $b > 0$ and $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.*

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = t_n x_n + (1-t_n)y_n, \\ \phi(v_n, y) + \dfrac{1}{r_n}\langle y - v_n, v_n - u_n\rangle \geq 0 \quad \textit{for all} \;\; y \in C, \\ z_n = P_C(I - \mu B)v_n, \\ y_n = P_C(I - \nu A)z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)W_n y_n, \quad n \geq 0, \end{cases}$$

where the initial guess $x_0 \in C$ is arbitrary, $\nu \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. Then, the sequence $\{x_n\}$ converges strongly to $q \in \Gamma$, where $q = P_\Gamma f(q)$, which solves the variational inequality (3.2).

Proof. From (4.2) and Lemma 4.2, we have $\cap_{n=1}^{\infty} Fix(W_n) = \cap_{n=1}^{\infty} Fix(T_n) = Fix(W)$. So by Theorem 3.1, it is suffices to show $\sum_{n=1}^{\infty} \sup\{\|W_{n+1}z - W_n z\| : z \in K\} < \infty$ for any bounded subset K of C. Let K be a bounded subset of C and $z \in K$. From (4.1), since T_i and $U_{n,i}$ are nonexpansive, we obtain

$$\begin{aligned} \|W_{n+1}z - W_n z\| &= \|\lambda_1 T_1 U_{n+1,2}z - \lambda_1 T_1 U_{n,2}z\| \\ &\leq \lambda_1 \|U_{n+1,2}z - U_{n,2}z\| \\ &= \lambda_1 \|\lambda_2 T_2 U_{n+1,3}z - \lambda_2 T_2 U_{n,3}z\| \\ &\leq \lambda_1 \lambda_2 \|U_{n+1,3}z - U_{n,3}z| \\ &\leq \ldots \\ &\leq \lambda_1 \lambda_2 \ldots \lambda_n \|U_{n+1,n+1}z - U_{n,n+1}z\| \\ &\leq M \prod_{i=1}^{n} \lambda_i \leq M b^n, \end{aligned} \tag{4.3}$$

where $M \geq 0$ is a constant such that $M = \sup\{\|U_{n+1,n+1}z - U_{n,n+1}z\| : z \in K\}$. Since $0 < b < 1$, we have

$$\sum_{n=1}^{\infty} \sup\{\|W_{n+1}z - W_n z\| : z \in K\} \leq M \sum_{n=1}^{\infty} b^n < \infty. \qquad \square$$

5. Numerical Test

In this section, first we give a numerical example which satisfies all assumptions in Theorem 3.1 in order to illustrate the convergence of the sequence generated by the iterative process defined by (3.1). Next, we give another numerical example for (3.1) to compare its behavior with iterative method (1.4) of Ke and Ma [10].

Example 5.1. Let $H = l^2$ be a real Hilbert space with the inner product $\langle x, y\rangle = \sum x^i y^i$ and the norm $\|x\| = \sqrt{\langle x, x\rangle}$ where $x = (x^1, x^2, x^3, ...)$ and $y = (y^1, y^2, y^3, ...)$ are two real sequences. Let $C = \{x : \|x\| \leq 10\}$. Define $\phi(x, y) = -4\|x\|^2 + 3\langle x, y\rangle + \|y\|^2$. First, we verify that ϕ satisfies the conditions $(A_1) - (A_4)$ as follows:
(A_1) $\phi(x, x) = -4\|x\|^2 + 3\|x\|^2 + \|x\|^2 = 0$ for all $x \in l^2$;

(A_2) $\phi(x,y)+\phi(y,x)=-\|x-y\|^2\le 0$ for all $x,y\in l^2$;
(A_3) For all $x,y,z\in l^2$,

$$\begin{aligned}\limsup_{t\to 0^+}\phi(tz+(1-t)x,y)=&\limsup_{t\to 0^+}(-4\|(tz+(1-t)x)\|^2+\langle tz+(1-t)x,y\rangle+\|y\|^2)\\ =&\phi(x,y).\end{aligned}$$

(A_4) For all $x\in l^2$, $\Phi(y)=\phi(x,y)=-4\|x\|^2+3\langle x,y\rangle+\|y\|^2$ is a lower semicontinuous and convex function.

From Lemma 2.2, Q_r is single-valued for all $r>0$. Now, we deduce a formula for $Q_r(x)$ where $x=\{x^i\}$. For any $y=\{y^i\}$ and $r>0$, we have

$$\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0\Leftrightarrow$$
$$r\sum(y^i)^2+\sum((3r+1)z^i-x^i)y^i+\sum x^iz^i-(4r+1)\sum(z^i)^2\ge 0,$$

where $z=\{z^i\}$. Set $G(y^i)=r(y^i)^2+((3r+1)z^i-x^i)y^i+x^iz^i-(4r+1)(z^i)^2$. Then $G(y^i)$ is a quadratic function of y^i with coefficients $a=r, b=(3r+1)z^i-x^i$ and $c=(x^i)(z^i)-(4r+1)(z^i)^2$. So its discriminate $\Delta=b^2-4ac$ is

$$\Delta=\|(5r+1)z-x\|^2.$$

Since $G(y^i)\ge 0$ for all $y^i\in\mathbb{R}$, this is true if and only if $\Delta\le 0$. That is, $[(5r+1)z^i-x^i]^2\le 0$. Therefore, $z^i=\frac{x^i}{5r+1}$, which yields $Q_r(x)=\frac{x^i}{5r+1}$. So, from Lemma 2.2, we get $EP(\phi)=\{0\}$. Let $\alpha_n=\frac{1}{2n}$, $r_n=\frac{n}{4n-1}$, $t_n=\frac{1}{2}$, and $T_nx=\frac{x}{n}$ for all $n\in\mathbb{N}$. Suppose $f(x)=\frac{x}{5}$, $Ax=\frac{x}{6}$ is 3−ism , $Bx=\frac{x}{3}$ is 2−ism, $\nu=1$, and $\mu=2$. Hence $\Gamma=\cap_{n=1}^{\infty}Fix(T_n)\cap EP(\phi)\cap Fix(G)=\{0\}$. Then, from Theorem 3.1, the sequence $\{x_n\}$, generated iteratively by

$$\begin{cases}u_n=\frac{1}{2}x_n+\frac{1}{2}y_n,\\ v_n=Q_{r_n}u_n=\frac{4n-1}{9n-1}u_n,\\ z_n=P_C(I-\mu B)v_n=P_C(\frac{1}{3}v_n)=\frac{1}{3}v_n,\\ y_n=P_C(I-\nu A)z_n=P_C(\frac{5}{6}z_n)=\frac{5}{18}v_n,\\ x_{n+1}=\frac{1}{10n}x_n+(1-\frac{1}{2n})\frac{y_n}{n}=\frac{1}{10n}x_n+\frac{2n-1}{2n^2}y_n,\end{cases}\tag{5.1}$$

converges strongly to $0\in\Gamma$, where $0=P_\Gamma(f)(0)$.

In the following, we provide numerical results for two suitable initial point.

Now, we will compare the effectiveness of our algorithm with the algorithm (1.4) by a numerical example. In fact, Ke and Ma [10] proved the following strong convergence theorem.

Theorem 5.1. *Let C be a closed convex subset of H, T be a nonexpansive self-mappings on C with $Fix(T)\ne\emptyset$ and f be a κ-contraction on C for some $\kappa\in[0,1)$. Pick any $x_0\in H$, let $\{x_n\}$ be a sequence generated by (1.4), where $\{\alpha_n\}$ and $\{t_n\}$ are real sequences satisfying the following conditions:*

(B$_1$) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(B$_2$) $0<\varepsilon\le t_n\le t_{n+1}<1$.

Then, the sequence $\{x_n\}$ converges strongly to $q\in Fix(T)$ which solves the variational inequality:

$$\langle(I-f)q,q-x\rangle\le 0\quad \textit{for all } x\in Fix(T).$$

Example 5.2. Let all the assumptions of Example 5.1 hold except the mappings $T_n x = Tx = x$ for all $n \in N$. First, suppose the sequence $\{x_n\}$ be generated by (3.1). Then, the scheme (3.1) can be simplified as

$$\begin{cases} u_n = \frac{1}{2}x_n + \frac{1}{2}y_n, \\ v_n = Q_{r_n}u_n = \frac{4n-1}{9n-1}u_n, \\ z_n = P_C(I-\mu B)v_n = P_C(\frac{1}{3}v_n) = \frac{1}{3}v_n, \\ y_n = P_C(I-\nu A)z_n = P_C(\frac{5}{6}z_n) = \frac{5}{18}v_n, \\ x_{n+1} = \frac{1}{10n}x_n + (1-\frac{1}{2n})\frac{y_n}{n} = \frac{1}{10n}x_n + \frac{2n-1}{2n}y_n. \end{cases} \tag{5.2}$$

Therefore, the sequence $\{x_n\}$ converges strongly to 0 by Theorem 3.1. Next, let the sequence

Table 5.1: Comparison between Algorithm (5.2) and Algorithm (5.3).

n	$\|x_n - 0\|$ for (5.2)	$\|x_n - 0\|$ for (5.3)
1	3.1623	3.1623
2	0.4031	1.4757
3	0.038494	1.0035
4	0.0032857	0.77413
5	0.00026411	0.6365
⋮	⋮	⋮
26	$1.7197e^{-28}$	0.16131
27	$1.1694e^{-29}$	0.15644
28	$7.9425e^{-31}$	0.15189
29	$5.3885e^{-32}$	0.14762
30	$3.6519e^{-33}$	0.14362

Numerical results for $x_1 = (1,1,1,1,1,1,1,1,1,1,0,0,\cdots)$

Table 5.2: The values of the sequence $\{\|x_n\|\}$ for Algorithm (5.1).

n	$\|x_n - 0\|$	$\|x_n - 0\|$
1	1.2825	0.57735
2	0.16349	0.073596
3	0.011893	0.0053539
4	0.00060269	0.00027131
5	$2.3412e^{-5}$	$1.0539e^{-5}$
⋮	⋮	⋮
26	$9.4505e^{-46}$	$4.2542e^{-46}$
27	$5.9668e^{-48}$	$2.686e^{-48}$
28	$3.629e^{-50}$	$1.6336e^{-50}$
29	$2.1291e^{-52}$	$9.5843e^{-53}$
30	$1.2064e^{-54}$	$5.4308e^{-55}$

Numerical results for $x_1 = (1, 1/2, 1/3/, \cdots)$ and $x_1 = (1/2, 1/4, 1/8, \cdots)$

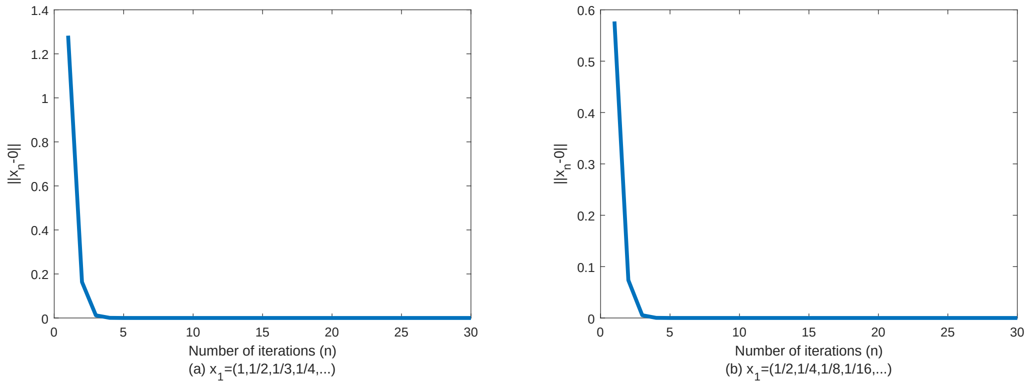

Fig. 5.1. The convergence of $\{x_n\}$ with different initial values x_1.

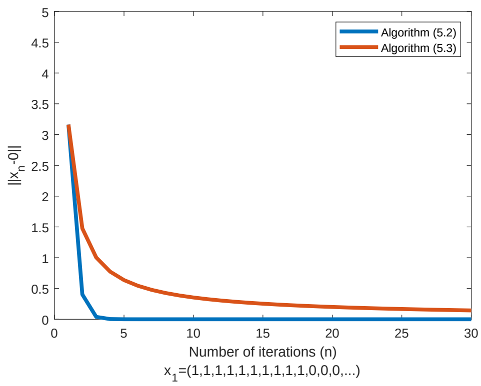

Fig. 5.2. Comparison between Algorithm (5.2) and Algorithm (5.3).

$\{x_n\}$ be generated by (1.4). Then, the scheme (1.4) can be simplified as

$$x_{n+1} = \frac{1}{10n}x_n + (1 - \frac{1}{2n})(\frac{1}{2}x_n + \frac{1}{2}x_{n+1}). \tag{5.3}$$

Therefore, the sequence $\{x_n\}$ converges strongly to 0 by Theorem 5.1.

Next, the numerical comparison of algorithms (5.2) and (5.3) is provided.

Tables 5.2-5.1 and Figs. 5.1-5.2 show that the sequence $\{x_n\}$ generated by the above algorithms converges to 0.

Remark 5.1. The Table 5.1 shows that the convergent rate of iterative algorithm (3.1) is faster than that of iterative algorithm (1.4) of Ke and Ma.

6. Conclusions

Regarding our main theorem, we introduced an iterative method to find a common element of the set of solutions of a general system of variational inequalities, an equilibrium problem, and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space, which is a generalization of the past method, introduced by Cai in [4] based on a single nonexpansive mapping. It is important to note that we reduced using projection in our method to avoid the unsuitable error of projecting, which increases the rate of convergence.

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