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UNCONDITIONAL SUPERCONVERGENT ANALYSIS OF QUASI-WILSON ELEMENT FOR BENJAMIN-BONA-MAHONEY EQUATION*

Xiangyu Shi[1)]
Business School, Zhengzhou University, Zhengzhou 450001, China;
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email: 19020170155426@stu.xmu.edu.cn
Linzhang Lu
School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, China;
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email: llz@gznu.edu.cn, lzlu@xmu.edu.cn

Abstract

This article aims to study the unconditional superconvergent behavior of nonconforming quadrilateral quasi-Wilson element for nonlinear *Benjamin Bona Mahoney* (BBM) equation. For the generalized rectangular meshes including rectangular mesh, deformed rectangular mesh and piecewise deformed rectangular mesh, by use of the special character of this element, that is, the conforming part(bilinear element) has high accuracy estimates on the generalized rectangular meshes and the consistency error can reach order $O(h^2)$, one order higher than its interpolation error, the superconvergent estimates with respect to mesh size h are obtained in the broken H^1-norm for the semi-/ fully-discrete schemes. A striking ingredient is that the restrictions between mesh size h and time step τ required in the previous works are removed. Finally, some numerical results are provided to confirm the theoretical analysis.

Mathematics subject classification: 65N15, 65N30.
Key words: BBM equations, Quasi-Wilson element, Superconvergent behavior, Semi-and fully-discrete schemes, Unconditionally.

1. Introduction

In this paper, we consider the following nonlinear BBM equation:

$$\begin{cases} u_t - \Delta u_t = \nabla \cdot \vec{f}(u), & (X,t) \in \Omega \times (0,T], \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega. \end{cases} \tag{1.1}$$

Where $0 < T < \infty$, $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with the boundary $\partial\Omega$, $X = (x,y)$, $u_t = \frac{\partial u}{\partial t}$, $u_0(X)$ is a known sufficiently smooth function and $\vec{f}(u) = (-(\frac{1}{2}u^2 + u), -(\frac{1}{2}u^2 + u))$.

As we know, there have been some studies about the theoretical analysis and numerical simulation of finite element methods (FEMs) for problem (1.1). For example, the convergence of conforming *Crank-Nicolson* (CN) fully-discrete Galerkin FEM was discussed in [1]. The superconvergence of Galerkin FEMs for conforming element and nonconforming rectangular

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1) Corresponding author

EQ_1^{rot} element (see [4]) were studied in [2] and [3], respectively. Recently, the superconvergent analysis of an H^1-Galerkin FEM with conforming element pair was presented in [5]. A new mixed FEM and its' superconvergent behavior with nonconforming constrained rotated Q_1 element and constant pair was developed in [6]. The two-grid method for BDF2 scheme with bilinear element was investigated in [7]. The main advantage of [6] and [7] is that there is no restriction between h and τ.

On the other hand, it has been proven in [8] that the consistency error of the famous rectangular Wilson element is of order $O(h)$ and cannot be improved anymore even the exact solution is smooth enough. It has been shown in [9] that the consistency errors of quadrilateral quasi-Wilson elements of [10] are of order $O(h^2)$. Later on, these elements and their modified forms of [11, 12] have been widely applied to some PDEs for superconvergent analysis (see [13–16]). But up to now, there is no report on the application to BBM equation.

In the present work, we will attempt to use the quasi-Wilson element of [9] to solve problem (1.1). Then, for generalized quadrilateral meshes including rectangular mesh, deformed rectangular mesh and piecewise deformed rectangular mesh(see [17, 18]), we derive the superconvergent estimates/ unconditional superconvergent estimates for the semi-discrete scheme/ the Backward Euler (BE) and CN schemes on quadrilateral meshes by proving the boundedness of the numerical solution in the broken H^1-norm instead of L^∞-norm, which improves the results of [2, 3].

The rest of this paper is organized as follows: In section 2, some important estimates of quasi-Wilson element are introduced. In section 3, the superclose estimate with order $O(h^2)$ for the semi-discrete scheme is derived. In sections 4-5, the superclose estimates are obtained for both BE and CN fully-discrete schemes with order $O(h^2+\tau)$ and $O(h^2+\tau^2)$ without the restriction between h and τ, respectively. In section 6, the unconditional global superconvergent results of the above three schemes are gained through interpolated post-processing technique. In the last section, some numerical results are given to show the performance of our method.

2. Some Estimates of Quasi-Wilson Element

Let $\hat{K}=[-1,1]^2$ be the reference element on $\xi-\eta$ plane with four vertices $\hat{A}_1=(-1,-1)$, $\hat{A}_2=(1,-1)$, $\hat{A}_3=(1,1)$ and $\hat{A}_4=(-1,1)$. We define the quasi-Wilson element $\{\hat{K},\hat{P},\hat{\Sigma}\}$ on $\hat{K}$ as [9, 15]:

$$\hat{P}=span\{N_i(\xi,\eta)\ (i=1,2,3,4),\ \hat{\psi}(\xi),\ \hat{\psi}(\eta)\},$$
$$\hat{\Sigma}=\{\hat{v}(\hat{a}_i), i=1,2,3,4; \frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi^2}d\xi d\eta, \frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\eta^2}d\xi d\eta\}.$$

where $N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$, $(\xi_1,\xi_2,\xi_3,\xi_4)=(-1,1,1,-1)$, $(\eta_1,\eta_2,\eta_3,\eta_4)=(-1,-1,1,1)$, $\hat{\psi}(s)=\frac{1}{2}(s^2-1)-\frac{5}{12}(s^4-1)$ and $\hat{v}_i=\hat{v}(\hat{A}_i)$, $i=1,2,3,4$.

Obviously, the only difference between this element and the classical Wilson element is the change of $\psi(\cdot)$. Let T_h be a family of regular convex quadrilateral subdivision of Ω, $K\in T_h$ be an element with vertices $A_i(x_i,y_i)$, $1\le i\le 4$, then there exists a mapping F_K given by

$$x^K=\sum N_i(\xi,\eta)x_i,\qquad y^K=\sum N_i(\xi,\eta)y_i,$$

such that

$$F_K(\hat{A}_i)=A_i,\qquad F_K(\hat{K})=K.$$

For any function $v(x,y)$ defined on K, we let

$$\hat{v}(\xi,\eta) = v(x^K(\xi,\eta), y^K(\xi,\eta)) \quad or \quad \hat{v} = v \circ F_K.$$

Then on K, we can define

$$P_K = \{p,\ p|_K = \hat{p} \circ F_K^{-1},\ \ \hat{p} \in \hat{P}\},$$

and the associated quasi-Wilson element space

$$V_h = \{v,\ v|_K \in P_K,\ \forall K \in T_h\}.$$

Let

$$V_h^0 = \{v \in V_h, v(a) = 0,\ \forall\ node\ a \in \partial\Omega\}.$$

Then, the following lemma can be found in [9,15] and will play a important role in our error analysis.

Lemma 2.1. *For each $v_h \in V_h^0$, $v_h = \underline{v_h} + v_h^1$ (where $\underline{v_h}$ and v_h^1 are the conforming and nonconforming parts, respectively), there hold*

$$\|v_h\|_h^2 = \|\underline{v_h}\|_h^2 + \|v_h^1\|_h^2, \quad \|v_h^1\|_0 \le Ch\|v_h^1\|_h, \tag{2.1}$$

$$\int_K q\frac{\partial v_h^1}{\partial x}dxdy = \int_K q\frac{\partial v_h^1}{\partial y}dxdy = 0, \qquad \forall q \in P_1(K), \tag{2.2}$$

$$\sum_{K\in\Gamma_h}\int_{\partial K}\frac{\partial u}{\partial n}v_h ds \le Ch^2|u|_3\|v_h\|_h, \qquad \forall u \in H^3(\Omega). \tag{2.3}$$

Here and later in this paper, $C > 0$ (with or without subscript) denotes a constant independent of h and maybe different at different places. $P_1(K)$ is the linear polynomial space on K, $\|\cdot\|_h = (\sum\limits_{K\in T_h}|\cdot|_{1,K}^2)^{\frac{1}{2}}$.

For the propose of using higher accuracy analysis of bilinear element, we should require T_h to be rectangular mesh/ or deformed rectangular mesh/ or piecewise deformed rectangular mesh, such that for $u \in H^3(\Omega)$ and $v_h \in V_h^0$, there holds

$$(\nabla(u - \underline{I_h}u), \nabla\underline{v_h}) \le Ch^2|u|_3\|\underline{v_h}\|_1,$$

where $\underline{I_h}u \in H_0^1(\Omega)$ is the bilinear interpolation of u (see page 165 of [17] and pages 17, 27-28 of [18] for details).

Now we are ready to state the following:

Lemma 2.2. *Let T_h be one of above three types of meshes, $u \in H^3(\Omega) \cap H_0^1(\Omega)$, then for $v_h \in V_h^0$, there holds*

$$(\nabla(u - \underline{I_h}u), \nabla v_h)_h \le Ch^2|u|_3\|v_h\|_h, \tag{2.4}$$

where I_h is the associated interpolation operator over V_h, $(*,**)_h = \sum\limits_{K\in T_h}(*,**)_K = \sum\limits_{K\in T_h}\int_K *(**)dxdy$.

Proof. Note that

$$(\nabla(u-\underline{I_h u}),\nabla v_h)_h=(\nabla(u-\underline{I_h u}),\nabla \underline{v_h})_h+(\nabla(u-\underline{I_h u}),\nabla v_h^1)_h=:I_1+I_2.$$

By the above estimate and (2.1) we have

$$I_1=(\nabla(u-\underline{I_h u}),\nabla \underline{v_h})_h=(\nabla(u-\underline{I_h u}),\nabla \underline{v_h})\le Ch^2|u|_3\|\underline{v_h}\|_h\le Ch^2|u|_3\|v_h\|_h.$$

On the other hand, by (2.2) we have

$$\begin{aligned}I_2=(\nabla u,\nabla v_h^1)_h&=\sum_{K\in T_h}(\nabla u-P_1^K(\nabla u),\nabla v_h^1)_K\\&\le Ch^2\sum_{K\in T_h}|u|_{3,K}|v_h^1|_{1,K}\le Ch^2|u|_3\|v_h^1\|_h\le Ch^2|u|_3\|v_h\|_h,\end{aligned}$$

where $P_1^K(\nabla u)$ is the linear interpolation of ∇u on K defined by [18]:

$$\int_K(P_1^K(\nabla u)-\nabla u)q\,dxdy=0,\ \forall q\in P_1(K).$$

Then, combining the estimates of I_1 and I_2 yields the desired result. □

3. Superclose Estimate of Semi-Discrete Scheme

The weak formulation of problem (1.1) is: to find $u:[0,T]\to H_0^1(\Omega)$, such that

$$\begin{cases}(u_t,v)+(\nabla u_t,\nabla v)=(\nabla\cdot\vec{f}(u),v), & \forall v\in H_0^1(\Omega),\\ u(X,0)=u_0(X), & \forall X\in\Omega.\end{cases}\tag{3.1}$$

We may pose the semi-discrete problem to find $u_h:[0,T]\to V_h^0$, such that

$$\begin{cases}(u_{ht},v_h)+(\nabla u_{ht},\nabla v_h)_h=(\nabla\cdot\vec{f}(u_h),v_h)_h, & \forall v_h\in V_h^0,\\ u_h(X,0)=\underline{I_h u_0(X)}, & \forall X\in\Omega.\end{cases}\tag{3.2}$$

Theorem 3.1. *Let u and u_h be the solutions of* (1.1) *and* (3.2), *respectively. Assume that $u,u_t\in L^2(0,T;H^3(\Omega))$, then for sufficiently small h, $t\in[0,T]$, we have*

$$\|\underline{I_h u}-u_h\|_h\le Ch^2.\tag{3.3}$$

Proof. Let $u-u_h=(u-\underline{I_h u})+(\underline{I_h u}-u_h)=:\alpha+\beta$. Then from (1.1) and (3.2), we have for $v_h\in V_h^0$,

$$\begin{aligned}(\beta_t,v_h)+(\nabla\beta_t,\nabla v_h)_h=&-(\alpha_t,v_h)-(\nabla\alpha_t,\nabla v_h)_h+\sum_{K\in T_h}\int_{\partial K}\frac{\partial u_t}{\partial n}v_h ds\\&+\sum_{K\in T_h}(\nabla\cdot f(u)-\nabla\cdot f(u_h),v_h)_K.\end{aligned}\tag{3.4}$$

Let $v_h=\beta$ in (3.4), there holds

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\beta\|_0^2+\|\beta\|_h^2)=&-(\alpha_t,\beta)-(\nabla\alpha_t,\nabla\beta)_h+\sum_{K\in T_h}\int_{\partial K}\frac{\partial u_t}{\partial n}\beta ds\\&+\sum_{K\in T_h}(\nabla\cdot(f(u)-f(u_h)),\beta)_K\triangleq\sum_{i=1}^4 A_i.\end{aligned}\tag{3.5}$$

Then, by Lemma 2.1, terms $A_1 \sim A_3$ could be bounded by:

$$A_1 \leq Ch^2\|u_t\|_2\|\beta\|_0 \leq Ch^4\|u_t\|_2^2 + C\|\beta\|_0^2,$$
$$A_2 \leq Ch^2|u|_3\|\beta\|_h \leq Ch^4\|u\|_3^2 + C\|\beta\|_h^2,$$

and

$$A_3 \leq Ch^2|u_t|_3\|\beta\|_h \leq Ch^4\|u_t\|_3^2 + C\|\beta\|_h^2,$$

respectively. For the sake of simplify, we define $\nabla * u = u_x + u_y$ for scalar function u. Then

$$\begin{aligned} A_4 &= \sum_{K\in T_h} (\nabla \cdot (f(u) - f(u_h)), \beta)_K \\ &= -\sum_{K\in T_h} (\nabla * (u - u_h), \beta)_K - \sum_{K\in T_h} (u\nabla * (u - u_h), \beta)_K - \sum_{K\in T_h} ((u - u_h)\nabla * u_h, \beta)_K \\ &=: B_1, B_2, B_3. \end{aligned} \tag{3.6}$$

Then, based on [17] and Lemma 2.1, we have

$$\begin{aligned} B_1 &= \sum_{K\in T_h} (\nabla * \alpha, \beta)_K + \sum_{K\in T_h} (\nabla * \beta, \beta)_K \\ &\leq \sum_{K\in T_h} [(\nabla * \alpha, \underline{\beta})_K + (\nabla * \alpha, \beta^1)_K] + C\|\beta\|_h^2 \\ &\leq Ch^2\|u\|_3\|\underline{\beta}\|_0 + \sum_{K\in T_h} \|\nabla\alpha\|_{0,K}\|\beta^1\|_{0,K} + C\|\beta\|_h^2 \\ &\leq Ch^2\|u\|_3\|\underline{\beta}\|_0 + \sum_{K\in T_h} Ch\|u\|_{2,K}\|\beta^1\|_{0,K} + C\|\beta\|_h^2 \\ &\leq Ch^2\|u\|_3\|\underline{\beta}\|_0 + Ch^2\|u\|_2\|\beta^1\|_h + C\|\beta\|_h^2 \\ &\leq Ch^4\|u\|_3^2 + C\|\beta\|_h^2. \end{aligned}$$

Now, for $u \in W^{1,\infty}(K)$, we define $\widetilde{u}|_K = \frac{1}{|K|}\int_K u dx dy$. Then, $|u - \widetilde{u}| \leq Ch|u|_{1,\infty,K}$. Thus, it follows that

$$\begin{aligned} B_2 &= \sum_{K\in T_h} ((u - \widetilde{u})\nabla * \alpha, \beta)_K + \sum_{K\in T_h} (\widetilde{u}\nabla * \alpha, \beta)_K + \sum_{K\in T_h} (u\nabla * \beta, \beta)_K \\ &\leq \sum_{K\in T_h} \|u - \widetilde{u}\|_{0,\infty,K}\|\nabla * \alpha\|_{0,K}\|\beta\|_{0,K} + \sum_{K\in T_h} \widetilde{u}|_K[(\nabla * \alpha, \underline{\beta})_K + (\nabla * \alpha, \beta^1)_K] \\ &\quad + \sum_{K\in T_h} \|u\|_{0,\infty,K}\|\nabla * \beta\|_{0,K}\|\beta\|_{0,K} \\ &\leq Ch|u|_{1,\infty}Ch|u|_2\|\beta\|_0 + Ch^2\|u\|_3\|\beta\|_h + C\|\beta\|_h^2 \\ &\leq Ch^4\|u\|_3^2 + C\|\beta\|_h^2. \end{aligned}$$

Now, let $u_h(t) = u_h(\cdot, t)$ as [6], we can prove that $\|u_h(t)\|_h \leq C_1$ with $C_1 = 1 + \max_{0\leq t\leq T} \|u(t)\|_1$. So B_3 can be estimated as

$$\begin{aligned} B_3 &\leq \|u - u_h\|_{0,4}\|\nabla * u_h\|_0\|\beta\|_{0,4} \\ &\leq C(\|\alpha\|_{0,4} + \|\beta\|_{0,4})\|\beta\|_{0,4} \leq Ch^4\|u\|_{2,4}^2 + C\|\beta\|_h^2, \end{aligned}$$

and it follows that

$$A_4 \le Ch^4\|u\|_3^2 + C\|\beta\|_h^2.$$

Consequently, based on the above estimates, (3.5) becomes

$$\frac{1}{2}\frac{d}{dt}(\|\beta\|_0^2 + \|\beta\|_h^2) \le Ch^4(\|u\|_3^2 + \|u_t\|_3^2) + C\|\beta\|_h^2.$$

Then, taking integral with respect to t, and noting that $\beta(0)=0$, there holds

$$\|\beta\|_h^2 \le Ch^4\int_0^t (\|u\|_3^2 ds + \|u_t\|_3^2 ds)ds + C\int_0^t \|\beta\|_h^2 ds.$$

By use of *Gronwall* inequality, we have

$$\|\beta\|_h^2 \le Ch^4\int_0^t (\|u\|_3^2 + \|u_t\|_3^2)ds. \tag{3.7}$$

This implies that

$$\begin{aligned}\|u_h(t)\|_h &\le \|u_h(t) - \underline{I_h u(t)}\|_h + \|\underline{I_h u(t)} - u(t)\|_h + \|u(t)\|_h \\ &\le Ch^2[\int_0^t (\|u\|_3^2 + \|u_t\|_3^2)ds]^{\frac{1}{2}} + Ch\|u(t)\|_2 + \|u(t)\|_1 \\ &\le C(h^2+h) + \|u(t)\|_1 \le C_1.\end{aligned}$$

The proof is completed. □

4. Superclose Estimate of BE Fully-Discrete Scheme

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a subdivision of $[0,T]$ with time step $\tau = T/N$ for some positive integer N, $t_n = n\tau$, and denote

$$\partial_t\psi^n = \frac{(\psi^n - \psi^{n-1})}{\tau}, \quad \psi^n = \psi(t_n), \quad \overline{\psi}^{n-\frac{1}{2}} = \frac{\psi^n + \psi^{n-1}}{2}.$$

Then, we consider the following BE scheme: find $U_h^n : [0,T] \to V_h^0$, such that for $n \ge 1$,

$$\begin{cases} (\partial_t U_h^n, v_h) + (\nabla\partial_t U_h^n, \nabla v_h)_h = (\nabla\cdot\vec{f}(U_h^n), v_h)_h, & \forall v_h \in V_h^0, \\ U_h^0(X,0) = \underline{I_h U_0(X)}, & \forall X \in \Omega. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $\{u^n\}$ and $\{U_h^n\}$ be the solutions of (1.1) and (4.1), respectively. Assume that $u, u_t \in L^\infty(0,T;H^3(\Omega))$ and $u_{tt} \in L^2(0,T;H^1(\Omega))$, then for $0 \le n \le N$, there holds*

$$\|\underline{I_h u^n} - U_h^n\|_h \le C(h^2 + \tau). \tag{4.2}$$

Proof. Let $u^n - U_h^n = (u^n - \underline{I_h u^n}) + (\underline{I_h u^n} - U_h^n) =: \alpha^n + \beta^n$. According to (1.1) and (4.1), we have

$$\begin{aligned}&(\partial_t\beta^n, v_h) + (\nabla\partial_t\beta^n, \nabla v_h)_h \\ =& -(\partial_t\alpha^n, v_h) - (\nabla\partial_t\alpha^n, \nabla v_h)_h + \sum_{K\in T_h}\int_{\partial K}\frac{\partial u_t^n}{\partial n}v_h ds + (R_1, v_h) \\ &+ (\nabla R_1, \nabla v_h)_h - (\nabla\cdot f(u^n) - \nabla\cdot f(U_h^n), v_h)_h =: \sum_{i=1}^6 D_i,\end{aligned} \tag{4.3}$$

where $R_1 = \partial_t u^n - u_t^n$ satisfies $\|R_1\|_0^2 \le C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 ds$. Let $v_h = \beta^n$ in (4.3), the left side of (4.3) is:

$$(\partial_t \beta^n, \beta^n) + (\nabla \partial_t \beta^n, \nabla \beta^n)_h \ge \frac{1}{2\tau}\Big(\|\beta^n\|_0^2 - \|\beta^{n-1}\|_0^2 + \|\beta^n\|_h^2 - \|\beta^{n-1}\|_h^2\Big).$$

By interpolation theory and Lemmas 2.1-2.2, the terms $D_1 \sim D_5$ on the right hand of (4.3) can be estimates as:

$$\begin{aligned}
&D_1 = (\partial_t \alpha^n, \beta^n) \le \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_2^2 ds + C\|\beta^n\|_0^2,\\
&D_2 = (\nabla \partial_t \alpha^n, \nabla \beta^n)_h \le Ch^2 \|\partial_t u^n\|_3 \|\beta^n\|_h \le \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_3^2 ds + C\|\beta^n\|_h^2,\\
&D_3 = \sum_{K\in T_h} \int_{\partial K} \frac{\partial u_t^n}{\partial n} \beta^n ds \le Ch^2 \|u_t^n\|_3 \|\beta^n\|_h \le Ch^4 \|u_t^n\|_3^2 + C\|\beta^n\|_h^2,\\
&D_4 = (R_1, \beta^n) \le C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 ds + C\|\beta^n\|_0^2,\\
&D_5 = (\nabla R_1, \nabla \beta^n)_h \le C\|R_1\|_h \|\beta^n\|_h \le C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_1^2 ds + C\|\beta^n\|_h^2.
\end{aligned}$$

As for the nonlinear term D_6, we rewrite it as

$$D_6 = (\nabla * (u^n - U_h^n), \beta^n)_h + (u^n \nabla * (u^n - U_h^n), \beta^n)_h + ((u^n - U_h^n)\nabla * U_h^n, \beta^n)_h =: \sum_{i=1}^{3} E_i.$$

Then, similar to the semi-discrete case, it is not difficult to check that

$$\begin{aligned}
&E_1 \le (\nabla * \alpha^n, \beta^n)_h + (\nabla * \beta^n, \beta^n)_h \le Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2,\\
&E_2 \le Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2.
\end{aligned}$$

Similar to [6], we can prove that $\|U_h^n\|_h \le C_2 (\forall\, n = 0, 1, \cdots, N)$ with $C_2 = 1 + \max_{0\le n\le N} \|u^n\|_1$. So, we have

$$E_3 \le \|u^n - U_h^n\|_{0,4} \|\nabla * U_h^n\|_0 \|\beta^n\|_{0,4} \le Ch^4 \|u^n\|_{2,4}^2 + C\|\beta^n\|_h^2,$$

which leads to

$$D_6 \le Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2.$$

Combining the estimates of $D_1 \sim D_6$, (4.3) becomes

$$\begin{aligned}
&\frac{1}{2\tau}(\|\beta^n\|_h^2 - \|\beta^{n-1}\|_h^2)\\
\le&\frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_3^2 ds + Ch^4 (\|u^n\|_3^2 + \|u_t^n\|_3^2) + C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_1^2 ds + C\|\beta^n\|_h^2.
\end{aligned}$$

Then, multiplying above inequality by 2τ, summing it from 1 to n and noting that $\beta^0 = 0$, we have

$$\begin{aligned}
&(1 - C\tau)\|\beta^n\|_h^2\\
\le&Ch^4 (\|u\|_{L^\infty(H^3(\Omega))} + \|u_t\|_{L^\infty H^3(\Omega)}^2) + C\tau^2 \int_0^{t_n} \|u_{tt}\|_1^2 ds + C\tau \sum_{i=1}^{n-1} \|\beta^i\|_h^2.
\end{aligned}$$

By use of discrete *Gronwall* inequality, when $1 - C\tau > 0$, we obtain

$$\|\beta^n\|_h^2 \le Ch^4(\|u\|_{L^\infty(H^3(\Omega))} + \|u_t\|^2_{L^\infty H^3(\Omega)}) + C\tau^2 \int_0^{t_n} \|u_{tt}\|_1^2 ds,$$

which yields the desired result (4.2). $\square$

5. Superclose Estimate of CN Fully-Discrete Scheme

We develop the CN scheme as: find $U_h^n : [0,T] \to V_h^0$, such that for $v_h \in V_h^0$, $n \ge 1$,

$$\begin{cases} (\partial U_h^n, v_h) + (\nabla \partial U_h^n, \nabla v_h)_h = (\nabla \cdot \vec{f}(\overline{U}_h^{n-\frac{1}{2}}), v_h)_h, & \forall v_h \in V_h^0, \\ U_h^0(X,0) = \underline{I_h U_0(X)}, & \forall X \in \Omega. \end{cases} \tag{5.1}$$

Theorem 5.1. *Let* $\{u^n\}$ *and* $\{U_h^n\}$ *be the solutions of* (1.1) *and* (5.1), *respectively. Assume that* $u, u_t \in L^\infty(0,T;H^3(\Omega))$, $u_{tt}, u_{ttt} \in L^2(0,T;H^1(\Omega))$ *then for* $0 \le n \le N$, *there holds*

$$\|\underline{I_h u^n} - U_h^n\|_h \le C(h^2 + \tau^2). \tag{5.2}$$

Proof. According to (1.1) and (5.1), we have the error equation:

$$\begin{aligned} &(\partial_t \beta^n, v_h) + (\nabla \partial_t \beta^n, \nabla v_h)_h \\ =& -(\partial_t \alpha^n, v_h) - (\nabla \partial_t \alpha^n, \nabla v_h)_h + \sum_{K \in T_h} \int_{\partial K} \frac{\partial u_t^{n-\frac{1}{2}}}{\partial n} v_h ds \\ &+ (R_2, v_h) + (\nabla R_2, \nabla v_h)_h + (\nabla \cdot f(\overline{u}^{n-\frac{1}{2}}) - \nabla \cdot f(\overline{U}_h^{n-\frac{1}{2}}), v_h)_h \\ &+ (\nabla \cdot f(u^{n-\frac{1}{2}}) - \nabla \cdot f(\overline{u}^{n-\frac{1}{2}}), v_h)_h = \sum_{i=1}^7 F_i, \end{aligned} \tag{5.3}$$

where $R_2 = \partial_t u^n - u_t^{n-\frac{1}{2}}$ satisfies $\|R_2\|_0^2 \le C\tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|_0^2 ds$. Let $v_h = \beta^n$ in (5.3), we have for the left side of (5.3) that

$$(\partial_t \beta^n, \beta^n) + (\nabla \partial_t \beta^n, \nabla \beta^n)_h \ge \frac{1}{2\tau}(\|\beta^n\|_0^2 - \|\beta^{n-1}\|_0^2 + \|\beta^n\|_h^2 - \|\beta^{n-1}\|_h^2).$$

Now, we start to estimate the terms on the right side of (5.3).

In fact, by interpolation theory and Lemmas 2.1-2.2, we can check that

$$\begin{aligned} F_1 &= (\partial_t \alpha^n, \beta^n) \le \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_2^2 ds + C\|\beta^n\|_0^2, \\ F_2 &= (\nabla \partial_t \alpha^n, \nabla \beta^n)_h \le \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_3^2 ds + C\|\beta^n\|_h^2, \\ F_3 &= \sum_{K \in T_h} \int_{\partial K} \frac{\partial u_t^{n-\frac{1}{2}}}{\partial n} \beta^n ds \le Ch^2 \|u_t^{n-\frac{1}{2}}\|_3 \|\beta^n\|_h \le Ch^4 \|u_t^{n-\frac{1}{2}}\|_3^2 + C\|\beta^n\|_h^2, \\ F_4 &= (R_2, \beta^n) \le C\tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|_0^2 ds + C\|\beta^n\|_0^2, \\ F_5 &= (\nabla R_2, \nabla \beta^n)_h \le C\|R_1\|_h \|\beta^n\|_h \le C\tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|_1^2 ds + C\|\beta^n\|_h^2, \end{aligned}$$

and

$$F_7 = (\nabla \cdot f(u^{n-\frac{1}{2}}) - \nabla \cdot f(\overline{u}^{n-\frac{1}{2}}), \beta^n)_h \leq C\tau^3 \int_{t_{n-1}}^{t_n} \|u_{tt}\|_1^2 ds + C\|\beta^n\|_0^2.$$

For the nonlinear term F_6, we rewrite it as

$$\begin{aligned}
&(\nabla \cdot f(\overline{u}^{n-\frac{1}{2}}) - \nabla \cdot f(\overline{U}_h^{n-\frac{1}{2}}), \beta^n)_h \\
=& -(\nabla * (\overline{u}^{n-\frac{1}{2}} - \overline{U}_h^{n-\frac{1}{2}}), \beta^n)_h - (\overline{u}^{n-\frac{1}{2}} \nabla * (\overline{u}^{n-\frac{1}{2}} - \overline{U}_h^{n-\frac{1}{2}}), \beta^n)_h \\
&- ((\overline{u}^{n-\frac{1}{2}} - \overline{U}_h^{n-\frac{1}{2}}) \nabla * \overline{U}_h^{n-\frac{1}{2}}, \beta^n)_h \triangleq \sum_{i=1}^{3} G_i.
\end{aligned} \tag{5.4}$$

Obviously, by Lemma 2.2, we have

$$\begin{aligned}
&G_1 \leq (\nabla * \overline{\alpha}^{n-\frac{1}{2}}, \beta^n)_h + (\nabla * \overline{\beta}^{n-\frac{1}{2}}, \beta^n)_h \leq Ch^4 \|\overline{u}^{n-\frac{1}{2}}\|_3^2 + C(\|\beta^n\|_h^2 + \|\beta^{n-1}\|_h^2), \\
&G_2 \leq Ch^4 \|\overline{u}^{n-\frac{1}{2}}\|_3^2 + C(\|\beta^n\|_h^2 + \|\beta^{n-1}\|_h^2).
\end{aligned}$$

Similarly, we can prove that for sufficiently small h,

$$\|\bar{U}_h^{n-\frac{1}{2}}\|_h \leq C_2, \quad \forall n \in [1, N]. \tag{5.5}$$

Thus, G_3 can be estimates as

$$\begin{aligned}
G_3 &\leq \|\overline{u}^{n-\frac{1}{2}} - \overline{U}_h^{n-\frac{1}{2}}\|_{0,4} \|\nabla * \overline{U}_h^{n-\frac{1}{2}}\|_0 \|\beta^n\|_{0,4} \\
&\leq Ch^4 \|\overline{u}^{n-\frac{1}{2}}\|_{2,4}^2 + C(\|\beta^n\|_h^2 + \|\beta^{n-1}\|_h^2),
\end{aligned}$$

which yields that

$$F_6 \leq Ch^4 \|\overline{u}^{n-\frac{1}{2}}\|_3^2 + C(\|\beta^n\|_h^2 + \|\beta^{n-1}\|_h^2).$$

Combining the above estimates, (5.3) becomes

$$\begin{aligned}
&\frac{1}{2\tau}(\|\beta^n\|_0^2 - \|\beta^{n-1}\|_0^2 + \|\beta^n\|_h^2 - \|\beta^{n-1}\|_h^2) \\
\leq& \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_3^2 ds + C\tau^3 \int_{t_{n-1}}^{t_n} (\|u_{ttt}\|_1^2 + \|u_{tt}\|_1^2) ds \\
&+ Ch^4(\|\overline{u}^{n-\frac{1}{2}}\|_3^2 + \|\overline{u_t}^{n-\frac{1}{2}}\|_3^2) + C(\|\beta^n\|_h^2 + \|\beta^{n-1}\|_h^2).
\end{aligned}$$

Then, multiplying above equality by 2τ, summing it from 1 to n and noting that $\beta^0 = 0$, we have

$$\begin{aligned}
(1 - C\tau)(\|\beta^n\|_0^2 + \|\beta^n\|_h^2) \leq& Ch^4(\|u_t\|_{L^\infty(H^3(\Omega))}^2 + \|u\|_{L^\infty(H^3(\Omega))}^2) \\
&+ C\tau^4 \int_0^{t_n} (\|u_{ttt}\|_1^2 + \|u_{tt}\|_1^2) ds + C\tau \sum_{i=1}^{n-1} \|\beta^i\|_0^2 + C\tau \sum_{i=1}^{n-1} \|\beta^i\|_h^2,
\end{aligned}$$

by *Gronwall*'s inequality, when $1 - C\tau > 0$, we obtain

$$\|\beta^n\|_h^2 \leq Ch^4(\|u_t\|_{L^\infty(H^3(\Omega))}^2 + \|u\|_{L^\infty(H^3(\Omega))}^2) + C\tau^4 \int_0^{t_n} (\|u_{ttt}\|_1^2 + \|u_{tt}\|_1^2) ds, \tag{5.6}$$

which is the desired result.

6. Global Superconvergent Estimates of the Above Three Schemes

For propose of getting the global superconvergent results, we employ the interpolated post-processing operator Π_{2h} constructed in [17] satisfying

$$\begin{cases} \Pi_{2h}\underline{I_h u} = \Pi_{2h}u, \quad |u-\Pi_{2h}u|_1 \le Ch^2|u|_3, & \forall u \in H^3(\Omega), \\ \|\Pi_{2h}v\|_1 \le C\|v\|_h, & \forall v \in V_h^0. \end{cases} \tag{6.1}$$

Theorem 6.1. *Under the conditions of Theorem 3.1, Theorem 4.1 and Theorem 5.1, respectively, we have*

$$\|u-\Pi_{2h}u_h\|_h \le Ch^2 \quad for\ semi-discrete\ scheme, \tag{6.2}$$

and

$$\|u^n-\Pi_{2h}U_h^n\|_h \le \begin{cases} C(h^2+\tau) & for\ BE\ scheme, \\ C(h^2+\tau^2) & for\ CN\ scheme, \end{cases} \tag{6.3}$$

respectively.

Proof. We only prove (6.2), and (6.3) can be treated in the similar way.

In fact, by employing Theorem 3.1, (6.1) and triangle inequality, there holds

$$\begin{aligned} \|u-\Pi_{2h}u_h\|_h &= \|u-\Pi_{2h}\underline{I_h u}+\Pi_{2h}\underline{I_h u}-\Pi_{2h}u_h\|_h \le \|u-\Pi_{2h}\underline{I_h u}\|_h + \|\Pi_{2h}\underline{I_h u}-\Pi_{2h}u_h\|_h \\ &\le \|u-\Pi_{2h}u\|_1 + \|\underline{I_h u}-u_h\|_h \le Ch^2(\|u\|_3 + (\int_0^t (\|u\|_3^2+\|u_t\|_3^2)ds)^{\frac{1}{2}}) \le Ch^2. \end{aligned}$$

The proof is completed.

Remark 6.1. It can be checked that the conditions such as $\tau = O(h^{1+\alpha})$ (for some $\alpha > 0$) used implicity in [2,3] for BBM equation are indeed removed by showing $\|u_h\|_h \le C$ instead of $\|u_h\|_{0,\infty} \le C$ (see [19,20]). At the same time, our results also hold true for the modified quasi-Wilson element studied in [11,12].

Remark 6.2. It should be mentioned that the main reason why we can get the superconvergent estimates of this paper is that we modify the shape functions of $\hat{\psi}(\xi)$ and $\hat{\psi}(\eta)$ of the classical rectangular Wilson element, which lead to the important properties of (2.2) and (2.3) for quasi-Wilson element on the quadrilateral meshes. This idea comes from the plate bending element (see [21]). Moreover, there also have been some very important and valuable results about the Wilson element on quadrilateral mesh, such as three kinds of nonconforming quadrilateral elements established from different approaches were studied intensively in [22] for incompressible elasticity, and the uniform convergence rate was derived for both displacement and stresses when the incompressible limit equals to 0.5. Thus how to extend the results of [22] and to get the super-convergence result of these elements is a very interesting topic in the future study.

7. Numerical Experiments

In order to confirm our theoretical analysis, we consider the BBM equation as [2,6] with $\Omega = (0,1)\times(0,1)$, $T=1$:

$$\begin{cases} u_t - \Delta u_t = \nabla\cdot\vec{f}(u) + g, & (X,t)\in\Omega\times(0,1], \\ u(X,t) = 0, & (X,t)\in\partial\Omega\times(0,1], \\ u(X,0) = u_0(X), & X\in\Omega. \end{cases} \tag{7.1}$$

Table 7.1: Numerical results for BE scheme at $t = 1$.

mesh	4×4	8×8	16×16	32×32
$\|u^n - U_h^n\|_h$	0.010711632	0.004991097	0.002421478	0.001200197
Order	/	1.1017	1.0435	1.0126
$\|u^n - U_h^n\|_0$	0.001037618	0.000286990	0.000073508	0.000018487
Order	/	1.8542	1.9650	1.9914
$\|\underline{I_h u^n} - U_h^n\|_h$	0.003624803	0.001020460	0.000263965	0.000066638
Order	/	1.8287	1.9508	1.9859
$\|u^n - \Pi_{2h} U_h^n\|_h$	0.006540113	0.001676318	0.000412113	0.000102465
Order	/	1.9640	2.0242	2.0079

Table 7.2: Numerical results for CN scheme at $t = 1$.

mesh	4×4	8×8	16×16	32×32
$\|u^n - U_h^n\|_h$	0.011132405	0.005056957	0.002430280	0.001201317
Order	/	1.1384	1.0571	1.0165
$\|u^n - U_h^n\|_0$	0.001182465	0.000324567	0.000082982	0.000020860
Order	/	1.8652	1.9676	1.9921
$\|\underline{I_h u^n} - U_h^n\|_h$	0.004420070	0.001221855	0.000314161	0.000079162
Order	/	1.8550	1.9595	1.9886
$\|u^n - \Pi_{2h} U_h^n\|_h$	0.007281603	0.001823046	0.000446984	0.000111084
Order	/	1.9979	2.0281	2.0086

Where g and u_0 are obtained by the exact solution $u = e^{-t}(x^4 - x^3)(y^2 - y)$.

For simplicity, in the computation, we use the asymptotically regular parallelogram meshes (see [4] for details), and take $\tau = h^2$ for BE scheme and $\tau = h$ for CN scheme, respectively. Fig.7.1 and Fig.7.2 describe the graphs of exact solution and quasi-Wilson element solution with mesh 32×32 at time $t = 1$, respectively. Fig.7.3 and Fig.7.4 show the mesh subdivisions with numbers 8×8, 16×16 and 32×32, respectively.

From Tables 7.1–7.2 we can see that for both BE and CN schemes the convergent rates in the broken H^1-norm are of order $O(h)$, the superclose and superconvergent rates are of order $O(h^2)$, which confirm the theoretical analysis and show the good performance of the proposed methods.

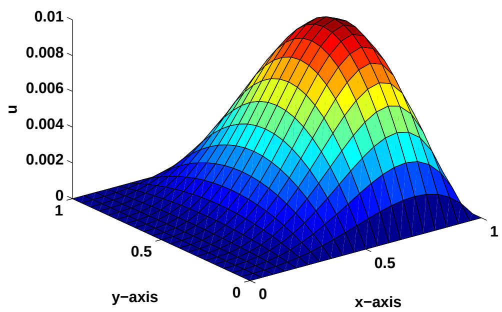

Fig.7.1. The exact solution u at time $t = 1$.

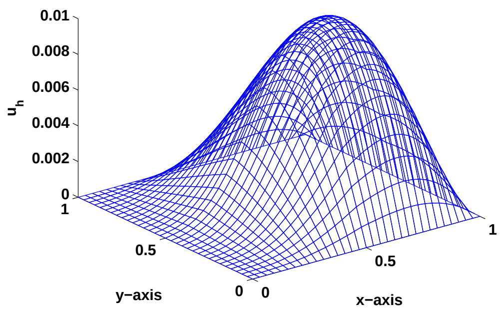

Fig.7.2. The quasi-Wilson element solution with mesh 32×32 at time $t = 1$.

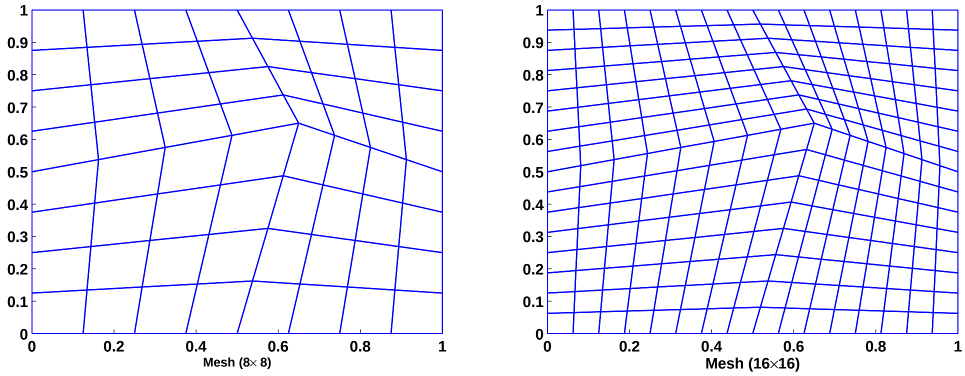

Fig.7.3. The asymptotically regular parallelogram meshes for 8×8 and 16×16.

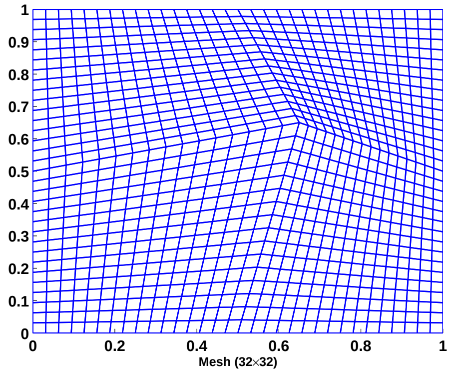

Fig.7.4. The asymptotically regular parallelogram meshes for 32×32.

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