UNCONDITIONAL SUPERCONVERGENT ANALYSIS OF QUASI-WILSON ELEMENT FOR BENJAMIN-BONA-MAHONEY EQUATION

Xiangyu Shi
Business School, Zhengzhou University, Zhengzhou, China
School of Mathematical Sciences, Xiamen University, Xiamen, China
Email: 1902017015526@stu.xmu.edu.cn

Linzhang Lu
School of Mathematical Sciences, Guizhou Normal University, Guiyang, China
School of Mathematical Sciences, Xiamen University, Xiamen, China
Email: llz@gznu.edu.cn, lzlu@xmu.edu.cn

Abstract

This article aims to study the unconditional superconvergent behavior of nonconforming quadrilateral quasi-Wilson element for nonlinear Benjamin Bona Mahoney (BBM) equation. For the generalized rectangular meshes including rectangular mesh, deformed rectangular mesh and piecewise deformed rectangular mesh, by use of the special character of this element, that is, the conforming part(bilinear element) has high accuracy estimates on the generalized rectangular meshes and the consistency error can reach order \(O(h^2)\), one order higher than its interpolation error, the superconvergent estimates with respect to mesh size \(h\) are obtained in the broken \(H^1\)-norm for the semi-/ fully-discrete schemes. A striking ingredient is that the restrictions between mesh size \(h\) and time step \(\tau\) required in the previous works are removed. Finally, some numerical results are provided to confirm the theoretical analysis.


Key words: BBM equations, Quasi-Wilson element, Superconvergent behavior, Semi-and fully-discrete schemes, Unconditionally.

1. Introduction

In this paper, we consider the following nonlinear BBM equation:

\[
\begin{align*}
    &u_t - \Delta u_t = \nabla \cdot \vec{f}(u), \quad (X, t) \in \Omega \times (0, T], \\
    &u(X, t) = 0, \quad (X, t) \in \partial \Omega \times (0, T], \\
    &u(X, 0) = u_0(X), \quad X \in \Omega.
\end{align*}
\]

Where \(0 < T < \infty, \Omega \subset \mathbb{R}^2\) is a bounded convex domain with the boundary \(\partial \Omega\), \(X = (x, y)\), \(u_t = \frac{\partial u}{\partial t}\), \(u_0(X)\) is a known sufficiently smooth function and \(\vec{f}(u) = (-\frac{1}{2}u^2 + u, -\frac{1}{2}u^2 + u))\).

As we know, there have been some studies about the theoretical analysis and numerical simulation of finite element methods (FEMs) for problem (1.1). For example, the convergence of conforming Crank-Nicolson (CN) fully-discrete Galerkin FEM was discussed in [1]. The superconvergence of Galerkin FEMs for conforming element and nonconforming rectangular
Unconditional Superconvergent Analysis of Quasi-Wilson Element for BBM Equation

95

EQ1 of element (see [4]) were studied in [2] and [3], respectively. Recently, the superconvergent analysis of an $H^1$-Galerkin FEM with conforming element pair was presented in [5]. A new mixed FEM and its' superconvergent behavior with nonconforming constrained rotated $Q_1$ element and constant pair was developed in [6]. The two-grid method for BDF2 scheme with bilinear element was investigated in [7]. The main advantage of [6] and [7] is that there is no restriction between $h$ and $\tau$.

On the other hand, it has been proven in [8] that the consistency error of the famous rectangular Wilson element is of order $O(h)$ and cannot be improved anymore even the exact solution is smooth enough. It has been shown in [9] that the consistency errors of quadrilateral quasi-Wilson elements of [10] are of order $O(h^2)$. Later on, these elements and their modified forms of [11, 12] have been widely applied to some PDEs for superconvergent analysis (see [13–16]). But up to now, there is no report on the application to BBM equation.

In the present work, we will attempt to use the quasi-Wilson element of [9] to solve problem (1.1). Then, for generalized quadrilateral meshes including rectangular mesh, deformed rectangular mesh and piecewise deformed rectangular mesh (see [17, 18]), we derive the superconvergent estimates/ unconditional superconvergent estimates for the semi-discrete scheme/ the Backward Euler (BE) and CN schemes on quadrilateral meshes by proving the boundedness of the numerical solution in the broken $H^1$-norm instead of $L^\infty$-norm, which improves the results of [2,3].

The rest of this paper is organized as follows: In section 2, some important estimates of quasi-Wilson element are introduced. In section 3, the superclose estimate with order $O(h^2)$ for the semi-discrete scheme is derived. In sections 4-5, the superclose estimates are obtained for both BE and CN fully-discrete schemes with order $O(h^2 + \tau)$ and $O(h^2 + \tau^2)$ without the restriction between $h$ and $\tau$, respectively. In section 6, the unconditional global superconvergent results of the above three schemes are gained through interpolated post-processing technique. In the last section, some numerical results are given to show the performance of our method.

2. Some Estimates of Quasi-Wilson Element

Let $\hat{K} = [-1,1]^2$ be the reference element on $\xi - \eta$ plane with four vertices $\hat{A}_1 = (-1,-1)$, $\hat{A}_2 = (1,-1)$, $\hat{A}_3 = (1,1)$ and $\hat{A}_4 = (-1,1)$. We define the quasi-Wilson element $\{\hat{K}, \hat{P}, \hat{\Sigma}\}$ on $\hat{K}$ as [9,15]:

\[
\hat{P} = \text{span}\{N_i(\xi,\eta) | i = 1,2,3,4\}, \quad \hat{\psi}(\xi), \quad \hat{\psi}(\eta)\},
\]

\[
\hat{\Sigma} = \{\hat{\psi}(\hat{a}_i), i = 1,2,3,4; \quad \frac{1}{|K|} \int_{\hat{K}} \frac{\partial^2 \hat{\psi}}{\partial \xi^2} d\xi d\eta, \quad \frac{1}{|K|} \int_{\hat{K}} \frac{\partial^2 \hat{\psi}}{\partial \eta^2} d\xi d\eta\},
\]

where $N_i(\xi,\eta) = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta)$, $(\xi_1,\xi_2,\xi_3,\xi_4) = (-1,1,1,-1)$, $(\eta_1,\eta_2,\eta_3,\eta_4) = (-1,-1,1,1)$, $\hat{\psi}(s) = \frac{1}{2}(s^2 - 1) - \frac{\dot{\nu}_i}{12}(s^4 - 1)$ and $\dot{\nu}_i = \hat{\nu}(\hat{A}_i)$, $i = 1,2,3,4$.

Obviously, the only difference between this element and the classical Wilson element is the change of $\hat{\psi}(\cdot)$. Let $T_h$ be a family of regular convex quadrilateral subdivision of $\Omega$, $K \in T_h$ be an element with vertices $A_i(x_i,y_i)$, $1 \leq i \leq 4$, then there exists a mapping $F_K$ given by

\[
x^K = \sum N_i(\xi,\eta)x_i, \quad y^K = \sum N_i(\xi,\eta)y_i,
\]

such that

\[F_K(\hat{A}_i) = A_i, \quad F_K(\hat{K}) = K.\]
For any function \( v(x, y) \) defined on \( K \), we let 
\[
\hat{v}(\xi, \eta) = v(x^K(\xi, \eta), y^K(\xi, \eta)) \quad \text{or} \quad \hat{v} = v \circ F_K.
\]
Then on \( K \), we can define 
\[
P_K = \{ p, p|_K = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P} \},
\]
and the associated quasi-Wilson element space 
\[
V_h = \{ v, v|_K \in P_K, \forall K \in T_h \}.
\]
Let 
\[
V_0^h = \{ v \in V_h, v(a) = 0, \forall \text{node } a \in \partial \Omega \}.
\]
Then, the following lemma can be found in [9,15] and will play a important role in our error analysis.

**Lemma 2.1.** For each \( v_h \in V_0^h \), \( v_h = v_h^0 + v_h^1 \) (where \( v_h^0 \) and \( v_h^1 \) are the conforming and nonconforming parts, respectively), there hold
\[
\| v_h \|_h^2 = \| v_h^0 \|_h^2 + \| v_h^1 \|_h^2, \quad \| v_h^1 \|_0 \leq Ch\| v_h^1 \|_h, \quad (2.1)
\]
\[
\int_K q \frac{\partial v_h^1}{\partial x} \, dx \, dy = \int_K q \frac{\partial v_h^1}{\partial y} \, dx \, dy = 0, \quad \forall q \in P_1(K), \quad (2.2)
\]
\[
\sum_{K \in T_h} \int_{\partial K} \frac{\partial u}{\partial n} v_h \, ds \leq Ch^2 \| u \|_3 \| v_h \|_h, \quad \forall u \in H^3(\Omega). \quad (2.3)
\]

Here and later in this paper, \( C > 0 \) (with or without subscript) denotes a constant independent of \( h \) and maybe different at different places. \( P_1(K) \) is the linear polynomial space on \( K \), \( \| \cdot \|_h = (\sum_{K \in T_h} | \cdot |^2_K)^{\frac{1}{2}} \).

For the propose of using higher accuracy analysis of bilinear element, we should require \( T_h \) to be rectangular mesh/ or deformed rectangular mesh/ or piecewise deformed rectangular mesh, such that for \( u \in H^3(\Omega) \) and \( v_h \in V_0^h \), there holds
\[
(\nabla (u - I_h u), \nabla v_h) \leq Ch^2 \| u \|_3 \| v_h \|_h, \quad (\star \star, (\star \star) = \sum_{K \in T_h} \int_K \star (\star) \, dx \, dy).
\]
where \( I_h \) is the bilinear interpolation of \( u \) (see page 165 of [17] and pages 17, 27-28 of [18] for details).

Now we are ready to state the following:

**Lemma 2.2.** Let \( T_h \) be one of above three types of meshes, \( u \in H^3(\Omega) \cap H^1_0(\Omega) \), then for \( v_h \in V_0^h \), there holds
\[
(\nabla (u - I_h u), \nabla v_h)_h \leq Ch^2 \| u \|_3 \| v_h \|_h, \quad (2.4)
\]
where \( I_h \) is the associated interpolation operator over \( V_h \), \( (\star, (\star)_h = \sum_{K \in T_h} (\star, (\star)_K = \sum_{K \in T_h} \int_K \star (\star) \, dx \, dy. \)
Proof. Note that
\[(\nabla (u - I_h u), \nabla v_h)_h = (\nabla (u - I_h u), \nabla v_h)_h + (\nabla (u - I_h u), \nabla v_h^I)_h =: I_1 + I_2.\]

By the above estimate and (2.1) we have
\[I_1 = (\nabla (u - I_h u), \nabla v_h)_h = (\nabla (u - I_h u), \nabla v_h) \leq Ch^2 |u|_3 \|v_h\|_h \leq Ch^2 |u|_3 \|v_h\|_h.
\]

On the other hand, by (2.2) we have
\[I_2 = (\nabla u, \nabla v_h^I)_h = \sum_{K \in T_h} (\nabla u - P^K_1 (\nabla u), \nabla v_h^I)_K \leq Ch^2 \sum_{K \in T_h} |u|_3, K \|v_h^I\|_{1,K} \leq Ch^2 |u|_3 \|v_h\|_h \leq Ch^2 |u|_3 \|v_h\|_h,
\]

where \(P^K_1(\nabla u)\) is the linear interpolation of \(\nabla u\) on \(K\) defined by [18]:
\[
\int_K (P^K_1(\nabla u) - \nabla u) q dxdy = 0, \ \forall q \in P_1(K).
\]

Then, combining the estimates of \(I_1\) and \(I_2\) yields the desired result. \(\Box\)

3. Superclose Estimate of Semi-Discrete Scheme

The weak formulation of problem (1.1) is: to find \(u : [0,T] \rightarrow H^1_0(\Omega)\), such that
\[
\begin{cases}
(u_t + \nabla v, \nabla v) = (\nabla \cdot \bar{f}(u), v), & \forall v \in H^1_0(\Omega), \\
u(X, 0) = u_0(X), & \forall X \in \Omega.
\end{cases}
\tag{3.1}
\]

We may pose the semi-discrete problem to find \(u_h : [0,T] \rightarrow V^0_h\), such that
\[
\begin{cases}
(u_{ht}, v_h) + (\nabla u_{ht}, \nabla v_h) = (\nabla \cdot \bar{f}(u_h), v_h), & \forall v_h \in V^0_h, \\
u_h(X, 0) = I_h u_0(X), & \forall X \in \Omega.
\end{cases}
\tag{3.2}
\]

**Theorem 3.1.** Let \(u\) and \(u_h\) be the solutions of (1.1) and (3.2), respectively. Assume that \(u, u_t \in L^2(0,T; H^3(\Omega))\), then for sufficiently small \(h, \ t \in [0,T]\), we have
\[\|I_h u - u_h\|_h \leq Ch^2.\]  

**Proof.** Let \(u - u_h = (u - I_h u) + (I_h u - u_h) =: \alpha + \beta\). Then from (1.1) and (3.2), we have for \(v_h \in V^0_h\),
\[
(\beta_t, v_h) + (\nabla \beta_t, \nabla v_h)_h = - (\alpha_t, v_h) - (\nabla \alpha_t, \nabla v_h)_h + \sum_{K \in T_h} \int_{\partial K} \frac{\partial u_t}{\partial n} v_h ds \\
+ \sum_{K \in T_h} (\nabla \cdot f(u) - \nabla \cdot f(u_h), v_h)_K.
\tag{3.4}
\]

Let \(v_h = \beta\) in (3.4), there holds
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|\beta\|^2_h + \|\beta\|^2_h) &= - (\alpha_t, \beta) - (\nabla \alpha_t, \nabla \beta)_h + \sum_{K \in T_h} \int_{\partial K} \frac{\partial u_t}{\partial n} \beta ds \\
&+ \sum_{K \in T_h} (\nabla \cdot (f(u) - f(u_h)), \beta)_K \triangleq \sum_{i=1}^4 A_i.
\tag{3.5}
\end{align*}
\]
Then, based on [17] and Lemma 2.1, we have
\[
A_1 \leq Ch^2\|u_t\|_2\|\beta\|_0 \leq Ch^4\|u_t\|_2^2 + C\|\beta\|_0^2,
\]
\[
A_2 \leq Ch^2\|u_3\|\beta\|_h \leq Ch^4\|u\|_3^2 + C\|\beta\|_h^2,
\]
and
\[
A_3 \leq Ch^2\|u_t\|_3\|\beta\|_h \leq Ch^4\|u_t\|_3^2 + C\|\beta\|_h^2,
\]
respectively. For the sake of simplify, we define \(\nabla^* = (\nabla \cdot (f(u) - f(u_h)))\). Then
\[
A_4 = \sum_{K \in T_h} (\nabla \cdot (f(u) - f(u_h)))\beta_K
= -\sum_{K \in T_h} (\nabla^* (u - u_h), \beta)_K - \sum_{K \in T_h} (u\nabla^* (u - u_h), \beta)_K - \sum_{K \in T_h} ((u - u_h)\nabla^* u_h, \beta)_K
=: B_1, B_2, B_3.
\]
(3.6)

Then, based on [17] and Lemma 2.1, we have
\[
B_1 = \sum_{K \in T_h} (\nabla \cdot (f(u) - f(u_h)))\alpha_K + \sum_{K \in T_h} (\nabla \cdot (f(u) - f(u_h)))\beta_K
\leq \sum_{K \in T_h} [(\nabla \cdot (f(u) - f(u_h)))\alpha_K + (\nabla \cdot (f(u) - f(u_h)))\beta_K] + C\|\beta\|_h^2
\leq Ch^2\|u_3\|\beta\|_0 + \sum_{K \in T_h} \|\nabla \alpha\|_{0,K}\|\beta\|_{0,K} + C\|\beta\|_h^2
\leq Ch^2\|u_3\|\beta\|_0 + \sum_{K \in T_h} Ch\|u_2,K\|\beta_1\|_{0,K} + C\|\beta\|_h^2
\leq Ch^2\|u_3\|\beta\|_0 + Ch^2\|u_2\|\beta_1\|_h + C\|\beta\|_h^2
\leq Ch^4\|u_3\|_2 + C\|\beta\|_h^2.
\]

Now, for \(u \in W^{1,\infty}(K)\), we define \(\tilde{u}|_K = \frac{1}{|K|} \int_K u \, dx\). Then, \(|u - \tilde{u}| \leq Ch|u|_{1,\infty,K}\). Thus, it follows that
\[
B_2 = \sum_{K \in T_h} ((u - \tilde{u})\nabla \cdot (f(u) - f(u_h)))\alpha_K + \sum_{K \in T_h} (\tilde{u}\nabla \cdot (f(u) - f(u_h)))\alpha_K + \sum_{K \in T_h} (u\nabla (f(u) - f(u_h)))\beta_K
\leq \sum_{K \in T_h} \|u - \tilde{u}\|_{0,K}\|\nabla \cdot (f(u) - f(u_h))\|_{0,K} + \sum_{K \in T_h} \|\tilde{u}|_K [(\nabla \cdot (f(u) - f(u_h)))\alpha_K + (\nabla \cdot (f(u) - f(u_h)))\beta_K]
\leq Ch|u|_{1,\infty,K}Ch|u_1|\|\beta\|_0 + Ch^2\|u_2\|\beta_1\|_h + C\|\beta\|_h^2
\leq Ch^4\|u_3\|_2 + C\|\beta\|_h^2.
\]

Now, let \(u_h(t) = u_h(\cdot, t)\) as [6], we can prove that \(|u_h(t)|_h \leq C_1\) with \(C_1 = 1 + \max_{0 \leq t \leq T} \|u(t)\|_1\).
So \(B_3\) can be estimated as
\[
B_3 \leq \|u - u_h\|_{0,4}\|\nabla \cdot (f(u) - f(u_h))\|_{0,4}
\leq C(\|\alpha\|_{0,4} + \|\beta\|_{0,4})\|\beta\|_{0,4} \leq Ch^4\|u\|_{2,4}^2 + C\|\beta\|_h^2,
\]
and it follows that

\[ A_4 \leq Ch^4\|u\|_h^2 + C\|\beta\|_h^2. \]

Consequently, based on the above estimates, (3.5) becomes

\[ \frac{1}{2} \frac{d}{dt}(\|\beta\|_h^2 + \|\beta\|_h^2) \leq Ch^4(\|u\|_h^2 + \|u_t\|_h^2) + C\|\beta\|_h^2. \]

Then, taking integral with respect to \(t\), and noting that \(\beta(0) = 0\), there holds

\[ \|\beta\|_h^2 \leq Ch^4 \int_0^t (\|u\|_h^2 + \|u_t\|_h^2)ds + C \int_0^t \|\beta\|_h^2 ds. \]

By use of Gronwall inequality, we have

\[ \|\beta\|_h^2 \leq Ch^4 \int_0^t (\|u\|_h^2 + \|u_t\|_h^2)ds. \quad (3.7) \]

This implies that

\[
\begin{align*}
\|u_h(t)\|_h & \leq \|u_h(t) - I_hu(t)\|_h + \|I_hu(t) - u(t)\|_h + \|u(t)\|_h \\
& \leq C\|h\|^2 \int_0^t (\|u\|_h^2 + \|u_t\|_h^2)ds + C\|u(t)\|_2 + \|u(t)\|_1 \\
& \leq C(h^2 + h) + \|u(t)\|_1 \leq C_1.
\end{align*}
\]

The proof is completed. \(\Box\)

4. Superclose Estimate of BE Fully-Discrete Scheme

Let \(0 = t_0 < t_1 < \cdots < t_N = T\) be a subdivision of \([0, T]\) with time step \(\tau = T/N\) for some positive integer \(N\), \(t_n = n\tau\), and denote

\[ \partial_t \psi^n = \frac{(\psi^n - \psi^{n-1})}{\tau}, \quad \psi^n = \psi(t_n), \quad \psi^{n-\frac{1}{2}} = \frac{\psi^n + \psi^{n-1}}{2}. \]

Then, we consider the following BE scheme: find \(U^n_h : [0, T] \to V^n_h\), such that for \(n \geq 1\),

\[
\begin{align*}
\{ (\partial_t U^n_h, v_h) + (\nabla \partial_t U^n_h, \nabla v_h) \} &= (\nabla \cdot f(U^n_h), v_h)_h, \quad \forall v_h \in V^n_h, \\
U^0_h(X, 0) &= I_hU^0_h(X), \quad \forall X \in \Omega.
\end{align*}
\]

Theorem 4.1. Let \(\{u^n\}\) and \(\{U^n_h\}\) be the solutions of (1.1) and (4.1), respectively. Assume that \(u, u_t \in L^\infty(0, T; H^3(\Omega))\) and \(u_t \in L^2(0, T; H^1(\Omega))\), then for \(0 \leq n \leq N\), there holds

\[ \|I_hu^n - U^n_h\|_h \leq C(h^2 + \tau). \quad (4.2) \]

Proof. Let \(u^n - U^n_h = (u^n - I_hu^n) + (I_hu^n - U^n_h) =: \alpha^n + \beta^n\). According to (1.1) and (4.1), we have

\[
\begin{align*}
(\partial_t \beta^n, v_h) + (\nabla \partial_t \beta^n, \nabla v_h)_h &= - (\partial_t \alpha^n, v_h) - (\nabla \partial_t \alpha^n, \nabla v_h)_h + \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n}{\partial n} v_h ds + (R_1, v_h) \\
&+ (\nabla R_1, \nabla v_h) - (\nabla \cdot f(u^n) - \nabla \cdot f(U^n_h), v_h) =: \sum_{i=1}^6 D_i,
\end{align*}
\]

(4.3)
where \( R_1 = \partial_t u^n - u^n_t \) satisfies \( \|R_1\|_0 \leq C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 ds \). Let \( v_h = \beta^n \) in (4.3), the left side of (4.3) is:

\[
(\partial_t \beta^n, \beta^n) + (\nabla \partial_t \beta^n, \nabla \beta^n)_h \geq \frac{1}{2\tau} \left( \|\beta^n\|_0^2 - \|\beta^{n-1}\|_0^2 + \|\beta^n\|_h^2 - \|\beta^{n-1}\|_h^2 \right).
\]

By interpolation theory and Lemmas 2.1-2.2, the terms \( D_1 \sim D_5 \) on the right hand of (4.3) can be estimated as:

\[
D_1 = (\partial_t \alpha^n, \beta^n) \leq \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_0^2 ds + \|\beta^n\|_0^2,
\]

\[
D_2 = (\nabla \partial_t \alpha^n, \nabla \beta^n)_h \leq Ch^2 \|\partial_t u^n\|_3 \|\beta^n\|_h \leq \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|_0^2 ds + \|\beta^n\|_0^2,
\]

\[
D_3 = \sum_{K \in T_n} \int_{\partial K} \frac{\partial u^n}{\partial n} \beta^n ds \leq Ch^3 \|u^n\|_3 \|\beta^n\|_h \leq Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2,
\]

\[
D_4 = (R_1, \beta^n) \leq C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 ds + \|\beta^n\|_0^2,
\]

\[
D_5 = (\nabla R_1, \nabla \beta^n)_h \leq C\|R_1\|_h \|\beta^n\|_h \leq C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 ds + \|\beta^n\|_h^2.
\]

As for the nonlinear term \( D_6 \), we rewrite it as

\[
D_6 = (\nabla \ast (u^n - U^n_h), \beta^n)_h + (u^n \nabla \ast (u^n - U^n_h), \beta^n)_h + ((u^n - U^n_h) \nabla \ast U^n_h, \beta^n)_h =: \sum_{i=1}^3 E_i.
\]

Then, similar to the semi-discrete case, it is not difficult to check that

\[
E_1 \leq (\nabla \ast \alpha^n, \beta^n)_h + (\nabla \ast \beta^n, \beta^n)_h \leq Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2,
\]

\[
E_2 \leq Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2.
\]

Similar to [6], we can prove that \( \|U^n_h\|_h \leq C_2 (\forall \ n = 0, 1, \cdots, N) \) with \( C_2 = 1 + \max_{0 \leq n \leq N} \|u^n\|_1 \). So, we have

\[
E_3 \leq \|u^n - U^n_h\|_{0,4} \|\nabla \ast U^n_h\|_{0,4} \|\beta^n\|_{0,4} \leq Ch^4 \|u^n\|_{2,4}^2 + C\|\beta^n\|_h^2,
\]

which leads to

\[
D_6 \leq Ch^4 \|u^n\|_3^2 + C\|\beta^n\|_h^2.
\]

Combining the estimates of \( D_1 \sim D_6, \) (4.3) becomes

\[
\frac{1}{2\tau} (\|\beta^n\|^2_h - \|\beta^{n-1}\|_h^2) \leq Ch^4 \int_{t_{n-1}}^{t_n} \|u_t\|_0^2 ds + Ch^4 (\|u^n\|_3^2 + \|u^n_t\|_3^2) + C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 ds + C\|\beta^n\|_h^2.
\]

Then, multiplying above inequality by \( 2\tau \), summing it from 1 to \( n \) and noting that \( \beta^0 = 0 \), we have

\[
(1 - C\tau) \|\beta^n\|_h^2 \leq Ch^4 (\|u\|_{L^\infty(H^1(\Omega))} + \|u_t\|_L^2 + \|u_{tt}\|_{H^1(\Omega)} + C\tau^2 \int_0^{t_n} \|u_{tt}\|_1^2 ds + C\tau \sum_{i=1}^{n-1} \|\beta^i\|_h^2).
\]
By use of discrete Gronwall inequality, when \(1 - C\tau > 0\), we obtain
\[
\|\beta^n\|_h^2 \leq Ch^4 (\|u^n\|_{L^\infty(H^3(\Omega))} + \|u_t\|_{L^\infty(H^3(\Omega))}) + C\tau^2 \int_0^{t_n} \|u_{tt}\|^2_t ds,
\]
which yields the desired result (4.2).

5. Superclose Estimate of CN Fully-Discrete Scheme

We develop the CN scheme as: find \(U^n_h : [0, T] \to V^0_h\), such that for \(v_h \in V^0_h, n \geq 1\),
\[
\begin{cases}
\left( \partial_t \beta^n, v_h \right) + (\nabla \partial_t \alpha^n, \nabla v_h)_h = (\nabla \cdot f(U_h^{n-\frac{1}{2}}), v_h)_h, & \forall v_h \in V^0_h, \\
U^n_h(X, 0) = I_h U^n(X), & \forall X \in \Omega.
\end{cases}
\]

**Theorem 5.1.** Let \(\{u^n\}\) and \(\{U^n_h\}\) be the solutions of (1.1) and (5.1), respectively. Assume that \(u, u_t \in L^\infty(0, T; H^3(\Omega)), u_{tt}, u_{ttt} \in L^2(0, T; H^1(\Omega))\) then for \(0 \leq n \leq N\), there holds
\[
\|I_h u^n - U^n_h\|_h \leq C(h^2 + \tau^2).
\]

**Proof.** According to (1.1) and (5.1), we have the error equation:
\[
\begin{align*}
(\partial_t \beta^n, v_h) + (\nabla \partial_t \alpha^n, \nabla v_h)_h &= - (\partial_t \alpha^n, v_h) - (\nabla \partial_t \alpha^n, \nabla v_h)_h + \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n_{i^n} - \frac{1}{2}}{\partial n} v_h ds \\
&
\quad + (R_2, v_h) + (\nabla R_2, \nabla v_h)_h + (\nabla \cdot f(U_h^{n-\frac{1}{2}}) - \nabla \cdot f(U_h^{n-\frac{1}{2}}), v_h)_h \\
&
\quad + (\nabla \cdot f(u^n_{i^n} - \frac{1}{2}) - \nabla \cdot f(u^n_{i^n} - \frac{1}{2}), v_h)_h = \sum_{i=1}^{7} F_i,
\end{align*}
\]
where \(R_2 = \partial_t u^n - u^n_{i^n} - \frac{1}{2}\) satisfies \(|R_2|^2 \leq C \tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|^2 ds\). Let \(v_h = \beta^n\) in (5.3), we have for the left side of (5.3) that
\[
(\partial_t \beta^n, \beta^n) + (\nabla \partial_t \beta^n, \nabla \beta^n)_h \geq \frac{1}{2} (\|\beta^n\|^2_0 - \|\beta^{n-1}\|^2_0 + \|\beta^n\|^2_0 - \|\beta^{n-1}\|^2_0).
\]

Now, we start to estimate the terms on the right side of (5.3).

In fact, by interpolation theory and Lemmas 2.1-2.2, we can check that
\[
\begin{align*}
F_1 &= (\partial_t \alpha^n, \beta^n) \leq \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|^2 ds + C\|\beta^n\|^2_0, \\
F_2 &= (\nabla \partial_t \alpha^n, \nabla \beta^n)_h \leq \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|^2 ds + C\|\beta^n\|^2_0, \\
F_3 &= \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n_{i^n} - \frac{1}{2}}{\partial n} \beta^n ds \leq Ch^4 \|u^n_{i^n} - \frac{1}{2}\|_3 \|\beta^n\|_h \leq Ch^4 \|u^n_{i^n} - \frac{1}{2}\|_3^2 + C\|\beta^n\|^2_0, \\
F_4 &= (R_2, \beta^n) \leq C \tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|^2 ds + C\|\beta^n\|^2_0, \\
F_5 &= (\nabla R_2, \nabla \beta^n)_h \leq C\|R_2\|_h \|\beta^n\|_h \leq C \tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|^2 ds + C\|\beta^n\|^2_0.
\]
and
\[ F_7 = (\nabla \cdot f(u_h^{-\frac{1}{2}}) - \nabla \cdot f(\pi_h^{n-\frac{1}{2}}), \beta^n)_h \leq C \tau^3 \int_{t_{n-1}}^{t_n} \|u_{tt}\|^2_1 ds + C\|\beta^n\|^2_0. \]

For the nonlinear term \( F_6 \), we rewrite it as
\[
(\nabla \cdot f(\pi_h^{n-\frac{1}{2}}) - \nabla \cdot f(\bar{u}_h^{n-\frac{1}{2}}), \beta^n)_h \\
= - (\nabla \cdot (\pi_h^{n-\frac{1}{2}} - \bar{u}_h^{n-\frac{1}{2}}), \beta^n)_h - (\pi_h^{n-\frac{1}{2}} \nabla \cdot (\pi_h^{n-\frac{1}{2}} - \bar{u}_h^{n-\frac{1}{2}}), \beta^n)_h \\
- ((\pi_h^{n-\frac{1}{2}} - \bar{u}_h^{n-\frac{1}{2}}) \nabla \cdot \bar{u}_h^{n-\frac{1}{2}}, \beta^n)_h \triangleq \sum_{i=1}^{3} G_i. \tag{5.4}
\]

Obviously, by Lemma 2.2, we have
\[
G_1 \leq (\nabla \cdot \pi_h^{n-\frac{1}{2}}, \beta^n)_h + (\nabla \cdot \bar{u}_h^{n-\frac{1}{2}}, \beta^n)_h \leq C h^4 \|\pi_h^{n-\frac{1}{2}}\|^2_3 + C(\|\beta^n\|^2_2 + \|\beta^{n-1}\|^2_2),
\]
\[
G_2 \leq Ch^4 \|\pi_h^{n-\frac{1}{2}}\|^2_3 + C(\|\beta^n\|^2_2 + \|\beta^{n-1}\|^2_2).
\]

Similarly, we can prove that for sufficiently small \( h \),
\[
\|\pi_h^{n-\frac{1}{2}}\|_h \leq C_2, \quad \forall n \in [1, N]. \tag{5.5}
\]

Thus, \( G_3 \) can be estimates as
\[
G_3 \leq \|\pi_h^{n-\frac{1}{2}} - \bar{u}_h^{n-\frac{1}{2}}\|_{0,4} \|\nabla \pi_h^{n-\frac{1}{2}}\|_h \|\beta^n\|_{0,4} \\
\leq C h^4 \|\pi_h^{n-\frac{1}{2}}\|^2_2 + C(\|\beta^n\|^2_2 + \|\beta^{n-1}\|^2_2),
\]

which yields that
\[
F_6 \leq Ch^4 \|\pi_h^{n-\frac{1}{2}}\|^2_3 + C(\|\beta^n\|^2_2 + \|\beta^{n-1}\|^2_2).
\]

Combining the above estimates, (5.3) becomes
\[
\frac{1}{2\tau}(\|\beta^n\|^2_0 + \|\beta^{n-1}\|^2_0 - \|\beta^n\|^2_0 - \|\beta^{n-1}\|^2_0) \\
\leq \frac{Ch^4}{\tau} \int_{t_{n-1}}^{t_n} \|u_t\|^2_1 ds + C \tau^3 \int_{t_{n-1}}^{t_n} (\|u_{ttt}\|^2_1 + \|u_{tt}\|^2_1) ds \\
+ Ch^4 \|\pi_h^{n-\frac{1}{2}}\|^2_2 + \|\bar{u}_h^{n-\frac{1}{2}}\|^2_2 + C(\|\beta^n\|^2_2 + \|\beta^{n-1}\|^2_2).
\]

Then, multiplying above equality by \( 2\tau \), summing it from 1 to \( n \) and noting that \( \beta^0 = 0 \), we have
\[
(1 - C\tau)(\|\beta^n\|^2_0 + \|\beta^{n-1}\|^2_0) \leq Ch^4(\|u_t\|^2_{L^\infty(H^1(\Omega))} + \|u\|^2_{L^\infty(H^2(\Omega))}) \\
+ C \tau^2 \int_0^{t_n} (\|u_{tt}\|^2_1 + \|u_{ttt}\|^2_1) ds + C \tau \sum_{i=1}^{n-1} \|\beta^i\|^2_0 + C \tau \sum_{i=1}^{n-1} \|\beta^i\|^2_2,
\]

by Gronwall's inequality, when \( 1 - C\tau > 0 \), we obtain
\[
\|\beta^n\|^2_2 \leq Ch^4(\|u_t\|^2_{L^\infty(H^1(\Omega))} + \|u\|^2_{L^\infty(H^2(\Omega))}) + C \tau^4 \int_0^{t_n} (\|u_{tt}\|^2_1 + \|u_{ttt}\|^2_1) ds, \tag{5.6}
\]

which is the desired result.
6. Global Superconvergent Estimates of the Above Three Schemes

For propose of getting the global superconvergent results, we employ the interpolated post-processing operator $\Pi_{2h}$ constructed in [17] satisfying

$$\left\{ \begin{array}{l}
\Pi_{2h} f = \Pi_{2h} u, \\
\|\Pi_{2h} v\|_1 \leq C\|v\|_h,
\end{array} \right. \quad \forall u \in H^3(\Omega), \quad \forall v \in V_h^0. \quad (6.1)$$

**Theorem 6.1.** Under the conditions of Theorem 3.1, Theorem 4.1 and Theorem 5.1, respectively, we have

$$\|u - \Pi_{2h} u_h\|_h \leq Ch^2 \text{ for semi - discrete scheme,} \quad (6.2)$$

and

$$\|u^n - \Pi_{2h} U^h_n\|_h \leq \left\{ \begin{array}{ll}
C(h^2 + \tau) & \text{for BE scheme,} \\
C(h^2 + \tau^2) & \text{for CN scheme,}
\end{array} \right. \quad (6.3)$$

respectively.

**Proof.** We only prove (6.2), and (6.3) can be treated in the similar way. In fact, by employing Theorem 3.1, (6.1) and triangle inequality, there holds

$$\|u - \Pi_{2h} u_h\|_h = \|u - \Pi_{2h} f + \Pi_{2h} f - \Pi_{2h} u - \Pi_{2h} u_h\|_h \leq \|u - \Pi_{2h} f\|_h + \|\Pi_{2h} f - \Pi_{2h} u\|_h$$

$$\leq \|u - \Pi_{2h} f\|_1 + \|\Pi_{2h} f - u_h\|_h \leq C(h^2 + \tau^2) + (\int_0^T (\|u\|_3^2 + \|u_t\|_3^2)ds)^{1/2} \leq Ch^2.$$

The proof is completed.

**Remark 6.1.** It can be checked that the conditions such as $\tau = O(h^{1+\alpha})$ (for some $\alpha > 0$) used implicity in [2,3] for BBM equation are indeed removed by showing $\|u_h\|_h \leq C$ instead of $\|u_h\|_{0, \infty} \leq C$ (see [19,20]). At the same time, our results also hold true for the modified quasi-Wilson element studied in [11,12].

**Remark 6.2.** It should be mentioned that the main reason why we can get the superconvergent estimates of this paper is that we modify the shape functions of $\hat{\psi}(\xi)$ and $\hat{\psi}(\eta)$ of the classical rectangular Wilson element, which lead to the important properties of (2.2) and (2.3) for quasi-Wilson element on the quadrilateral meshes. This idea comes from the plate bending element (see [21]). Moreover, there also have been some very important and valuable results about the Wilson element on quadrilateral mesh, such as three kinds of nonconforming quadrilateral elements established from different approaches were studied extensively in [22] for incompressible elasticity, and the uniform convergence rate was derived for both displacement and stresses when the incompressible limit equals to 0.5. Thus how to extend the results of [22] and to get the super-convergence result of these elements is a very interesting topic in the future study.

7. Numerical Experiments

In order to confirm our theoretical analysis, we consider the BBM equation as [2,6] with $\Omega = (0,1) \times (0,1), T = 1$:

$$\left\{ \begin{array}{l}
u_t - \Delta u_t = \nabla \cdot \bar{f}(u) + g, \\
u(X, t) = 0, \quad (X, t) \in \Omega \times (0,1), \\
u(X, 0) = u_0(X), \quad X \in \Omega.
\end{array} \right. \quad (7.1)$$

Table 7.1: Numerical results for BE scheme at $t = 1$.

<table>
<thead>
<tr>
<th>mesh</th>
<th>$4 \times 4$</th>
<th>$8 \times 8$</th>
<th>$16 \times 16$</th>
<th>$32 \times 32$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u^n - U^n_h|_h$</td>
<td>0.010711632</td>
<td>0.004991097</td>
<td>0.002421478</td>
<td>0.001200197</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.1017</td>
<td>1.0435</td>
<td>1.0126</td>
</tr>
<tr>
<td>$|u^n - U^n_h|_0$</td>
<td>0.001037618</td>
<td>0.000286990</td>
<td>0.000073508</td>
<td>0.000018487</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.8542</td>
<td>1.9650</td>
<td>1.9914</td>
</tr>
<tr>
<td>$|I_hu^n - U^n_h|_h$</td>
<td>0.003624803</td>
<td>0.001020460</td>
<td>0.000263965</td>
<td>0.000066638</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.8287</td>
<td>1.9508</td>
<td>1.9859</td>
</tr>
<tr>
<td>$|u^n - \Pi_2U^n_h|_h$</td>
<td>0.006540113</td>
<td>0.001676318</td>
<td>0.000412113</td>
<td>0.000102465</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.9640</td>
<td>2.0242</td>
<td>2.0079</td>
</tr>
</tbody>
</table>

Table 7.2: Numerical results for CN scheme at $t = 1$.

<table>
<thead>
<tr>
<th>mesh</th>
<th>$4 \times 4$</th>
<th>$8 \times 8$</th>
<th>$16 \times 16$</th>
<th>$32 \times 32$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u^n - U^n_h|_h$</td>
<td>0.011132405</td>
<td>0.005056957</td>
<td>0.002430280</td>
<td>0.001201317</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.1384</td>
<td>1.0571</td>
<td>1.0165</td>
</tr>
<tr>
<td>$|u^n - U^n_h|_0$</td>
<td>0.001182465</td>
<td>0.000324567</td>
<td>0.000082982</td>
<td>0.000020860</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.8652</td>
<td>1.9676</td>
<td>1.9921</td>
</tr>
<tr>
<td>$|I_hu^n - U^n_h|_h$</td>
<td>0.004420070</td>
<td>0.001221855</td>
<td>0.000314161</td>
<td>0.000079162</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.8550</td>
<td>1.9595</td>
<td>1.9886</td>
</tr>
<tr>
<td>$|u^n - \Pi_2U^n_h|_h$</td>
<td>0.007281603</td>
<td>0.001823046</td>
<td>0.000446984</td>
<td>0.00011084</td>
</tr>
<tr>
<td>Order</td>
<td>/</td>
<td>1.9979</td>
<td>2.0281</td>
<td>2.0086</td>
</tr>
</tbody>
</table>

Where $g$ and $u_0$ are obtained by the exact solution $u = e^{-t}(x^4 - x^3)(y^2 - y)$.

For simplicity, in the computation, we use the asymptotically regular parallelogram meshes (see [4] for details), and take $\tau = h^2$ for BE scheme and $\tau = h$ for CN scheme, respectively. Fig.7.1 and Fig.7.2 describe the graphs of exact solution and quasi-Wilson element solution with mesh $32 \times 32$ at time $t = 1$, respectively. Fig.7.3 and Fig.7.4 show the mesh subdivisions with numbers $8 \times 8$, $16 \times 16$ and $32 \times 32$, respectively.

From Tables 7.1–7.2 we can see that for both BE and CN schemes the convergent rates in the broken $H^1$-norm are of order $O(h)$, the superclose and superconvergent rates are of order $O(h^2)$, which confirm the theoretical analysis and show the good performance of the proposed
methods.

**Acknowledgments.** This work is supported by the National Natural Science Foundation of China (No.11671105).

**References**


