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A DIRECT DISCONTINUOUS GALERKIN METHOD FOR TIME FRACTIONAL DIFFUSION EQUATIONS WITH FRACTIONAL DYNAMIC BOUNDARY CONDITIONS*

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Abstract

This paper deals with the numerical approximation for the time fractional diffusion problem with fractional dynamic boundary conditions. The well-posedness for the weak solutions is studied. A direct discontinuous Galerkin approach is used in spatial direction under the uniform meshes, together with a second-order Alikhanov scheme is utilized in temporal direction on the graded mesh, and then the fully discrete scheme is constructed. Furthermore, the stability and the error estimate for the full scheme are analyzed in detail. Numerical experiments are also given to illustrate the effectiveness of the proposed method.

Mathematics subject classification: 65M12.
Key words: Time fractional diffusion equation, Numerical stability, Convergence.

1. Introduction

Classical partial differential equations (PDE) with the dynamic boundary conditions (DBC) have been considered in [1, 2] by some physicists to simulate the interaction among fluids with the walls of domain. Recently, there are also some reports on their numerical solutions [3–5]. However, the research of efficient numerical methods for the time fractional partial differential equations (TFPDE) with the fractional dynamic boundary conditions (FDBC) has not been developed very much. To the best of our knowledge, the only work [6] on this topic presented a Rothe's approach for the source identification of the time fractional wave equations with FDBC. In this paper, we will study the following time fractional diffusion equations with FDBC:

$$
\begin{aligned}
&{}_0^C D_t^\alpha u(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t), && (\boldsymbol{x},t) \in \Omega \times (0,T], && \text{(1.1a)}\\
&u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), && \boldsymbol{x} \in \bar{\Omega}, && \text{(1.1b)}\\
&\mu\, {}_0^C D_t^\alpha u(\boldsymbol{x},t) = -\lambda u(\boldsymbol{x},t) + \varsigma \Delta_{\partial\Omega} u(\boldsymbol{x},t) - \partial_{\mathbf{n}} u(\boldsymbol{x},t) + f_{\partial\Omega}(\boldsymbol{x},t), && (\boldsymbol{x},t) \in \partial\Omega \times (0,T], && \text{(1.1c)}
\end{aligned}
$$

where $0 < \alpha < 1$, $\boldsymbol{x} := (x,y) \in \Omega$ with Ω is a bounded domain in $\mathbb{R}^2$, $\bar{\Omega}$ is the closure of Ω, $\partial\Omega$ is the boundary of Ω, μ, λ, ς are non-negative constants, f_Ω and $f_{\partial\Omega}$ are given functions, and ${}_0^C D_t^\alpha$ denotes the left Caputo fractional derivative operator

$$
{}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}s} u(s) \right) \mathrm{d}s.
$$

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Here, Δ is the classical Laplace operator along with $\Delta_{\partial\Omega}$ is Laplace-Beltrami operator [7], $\partial_{\mathbf{n}}u$ represents the exterior normal derivative of u on $\partial\Omega$.

Anomalous heat diffusion appears in many problems, and can be modeled by the fractional diffusion equation (1.1a). Here, u is the temperature, $\partial_{\mathbf{n}}u$ is used to represent the heat flux, f and $f_{\partial\Omega}$ represent the heat source in the domain and on the boundary respectively. It is worth noting that the anomalous rate of change for the heat with the time should be taken into account both in the region and on the boundary. The feature of anomalous diffusion is the nonlinear growth of the mean squared displacement with time $\langle \boldsymbol{x}^2(t)\rangle \sim t^{\alpha}$, which deviates from the well-known property $\langle \boldsymbol{x}^2(t)\rangle \sim t$ of the Brownian motion [8]. Therefore, we use the time fractional derivative operator in (1.1c). Because the Laplace-Beltrami operator contains the tangential derivative, the heat flow along the boundary is allowed. The effect of the boundary condition (1.1c) is to send a "heat wave" into the region and an infinitesimal layer near the boundary. In fact, FDBC with the Caputo derivative is an extension of typical DBC. Especially, when $\alpha = 1$, (1.1) reduces to the parabolic problems with DBC in [5], whose derivation and physical interpretation can be found in [9].

Fractional differential equations have recently drawn growing attention because the diverse utilizations, such as dynamics in self-similar structures, engineering science, biological systems and so on [10–12]. In particular, as a valuable tool to model complex systems for anomalous diffusion transport, TFPDE with initial-boundary value conditions have been investigated by many scholars [13, 14]. Recently, the construction and analysis of discretization approach for TFPDE has obtained a series of results in numerical methods, see [15–18].

Discontinuous Galerkin (DG) method has been widely applied for solving PDE numerically owing to its flexibility and higher accuracy. However, DG method is still difficult for solving diffusion equations, because it is not easy to define numerical fluxes about the diffusion terms. Subsequently, [19] researched the local discontinuous Galerkin (LDG) method to deal with this problem. The main idea of LDG method is to change the original equation into a first-order system by introducing some auxiliary variables, which may result in high computational expense. To avoid this drawback, the hybridizable discontinuous Galerkin (HDG) method and the direct discontinuous Galerkin (DDG) method are proposed. So far as we know, [20] studied the HDG method for the spatial discretization of time fractional diffusion equation. The DDG method has also been applied in space approximation for TFPDE with the initial and boundary conditions, including time fractional reaction-diffusion problem with periodic boundary [21], time fractional reaction-diffusion problem with homogeneous Dirichlet conditions [22] and time fractional diffusion problem with Robin boundary [23]. It is worth noting that, DDG method can compel the weak form directly of PDE into the DG function space for the test functions and the numerical solutions, see [24, 25]. Since no new variables are introduced in the weak formulation of the problem, the DDG method has a practical advantage over LDG method. Thus, we will utilize the DDG method for the spatial discretization of (1.1) with FDBC.

Now we turn to consider the time discretization. L1 formula is a primary numerical formula to approximate time fractional derivatives with accuracy $2-\alpha$ when the order of fractional derivative is α $(0 < \alpha < 1)$ [26, 27]. Actually, L1 formula is obtained by applying a piecewise linear interpolation at every small interval for the integrand. This guides us to replace linear interpolation with another higher-order interpolation to improve the accuracy. In the following literatures, several numerical Caputo formulae with higher precision are given, for example, L1-2 scheme [28] and L2-1_{σ} formula [29] which used the piecewise quadratic polynomial interpolation. The numerical solutions converge temporally with order 2 for subdiffusion equation (1.1a) with sufficiently smooth solutions. Here, we will design a second order Alikhanov type (L2-1_σ) scheme for time fractional derivative of (1.1) under graded mesh, which is effective in resolving the initial singularity of time fractional diffusion problems.

The structure is as follows. In Section 2, we obtain the well-posedness of weak solution. In Section 3, we present DDG method for spatial discretization on uniform meshes and second order L2-1_σ formula for time by using graded mesh. Moreover, we prove that the fully discrete scheme is stable in Section 4 and convergent in Section 5. Finally, we provide numerical experiments to validate the availability of our method.

2. Well-Posedness of Weak Problem

2.1. Preliminaries

Let $L^2(\Omega)$ be Lebesgue space with square integrable functions in Ω. The inner product is written by $(\cdot,\cdot)_\Omega$ and the norm by $\|\cdot\|_\Omega$. For a real number $s>0$, the Sobolev space is written by $H^s(\Omega)$, where the norm and seminorm are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, see [30]. Hereafter, the domain Ω can be removed from the notations if no confusion would arise, and we use $A\lesssim B$ to denote $A\leq CB$, where C is a positive real number independent of the spatial and temporal mesh sizes.

We also introduce Laplace-Beltrami operator [7] on $\partial\Omega$. Firstly, we define a projection matrix $\mathcal{J}:=I-\mathbf{n}\otimes\mathbf{n}=(\delta_{ij}-\mathbf{n}_i\mathbf{n}_j)$ for $i,j=1,2$, where I is the identity matrix, $\mathbf{n}$ is the unit external normal vector, $a\otimes b=(a_ib_j)_{ij}$ and δ_{ij} is Kroneker delta. Furthermore, for $w:\partial\Omega\to\mathbb{R}$, we set the tangential gradient as $\nabla_{\partial\Omega}w:=\mathcal{J}\nabla w$. In addition, for $E:\partial\Omega\to\mathbb{R}^2$, we define the tangential divergence as $\mathrm{div}_{\partial\Omega}(E):=\mathrm{tr}((\nabla E)\mathcal{J})$, and $\mathrm{tr}(\cdot)$ is the trace operator. Consequently, we now define the Laplace-Beltrami operator $\Delta_{\partial\Omega}w:=\mathrm{div}_{\partial\Omega}(\nabla_{\partial\Omega}u)$.

Introduce Sobolev surface space [7] as follows:

$$H^s(\partial\Omega):=\left\{w\in H^s(\partial\Omega):\nabla_{\partial\Omega}w\in\left[H^{s-1}(\partial\Omega)\right]^2\right\},\quad s\geq 1$$

with $H^0(\partial\Omega)=L^2(\partial\Omega)$. Then, for $\mu>0$, we introduce the following space:

$$H^s_\mu(\Omega,\partial\Omega):=\{w\in H^s(\Omega):\mu w|_{\partial\Omega}\in H^s(\partial\Omega)\}$$

with the norm

$$\|w\|_{H^s_\mu(\Omega,\partial\Omega)}=\sqrt{\|w\|^2_{H^s(\Omega)}+\mu\|w\|^2_{H^s(\partial\Omega)}}.$$

When $s=0$, we write $H^0_\mu(\Omega,\partial\Omega)$ as $L^2_\mu(\Omega,\partial\Omega)$, and the norm

$$\|w\|_{L^2_\mu(\Omega,\partial\Omega)}=\sqrt{\|w\|^2_\Omega+\mu\|w\|^2_{\partial\Omega}},$$

and we will omit the subscript when $\mu=1$.

Now, we introduce some useful results of fractional calculus.

Lemma 2.1 (see [31]). *If $w:[0,T]\to Y$ has a derivative, where Y is a real Hilbert space, then*

$$\left(w(t),{}_0^CD_t^\alpha w(t)\right)\geq\frac{1}{2}{}_0^CD_t^\alpha\|w(t)\|^2.$$

Lemma 2.2 (see [32]). *Let $c(t) \geq 0$ be a nondecreasing function and locally integrable on $(0,T)$, and $d(t) \geq 0$ be a nondecreasing continuous function on $(0,T)$. If $w(t) \geq 0$ and locally integrable along with*

$$w(t) \leq c(t) + d(t)\int_0^t (t-s)^{\alpha-1} w(s)\mathrm{d}s, \quad 0 \leq t < T.$$

Then

$$w(t) \leq c(t)E_\alpha\big(d(t)\Gamma(\alpha)t^\alpha\big), \quad t \in [0,T),$$

where E_α is the Mittag-Leffler function, and $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+1)}$.

2.2. Existence and uniqueness

Introduce a few important spaces

$$C_s([0,T],H) := \{w \in L^\infty([0,T],H) : w \,\text{is continuous weakly}\},$$

and

$$\begin{aligned} H &:= L^2_\mu(\Omega,\partial\Omega) = \big\{w \in L^2(\Omega) : \mu w|_{\partial\Omega} \in L^2(\partial\Omega)\big\}, \\ V &:= H^1_\mu(\Omega,\partial\Omega) = \big\{w \in H^1(\Omega) : \mu w|_{\partial\Omega} \in H^1(\partial\Omega)\big\}. \end{aligned}$$

Similar to [5], by multiplying (1.1a) and (1.1c) by the test function $v \in V$, using the Green's formula in the domain and on the boundary, adding the two equations, we can obtain the variational form of (1.1): Find $u \in L^2([0,T],V) \cap C_s([0,T],H)$ such that

$$\begin{aligned} &\big({}_0^C D_t^\alpha u, v\big)_\Omega + \mu\big({}_0^C D_t^\alpha u, v\big)_{\partial\Omega} + \mathbf{a}(u,v) = (f,v)_\Omega + (f_{\partial\Omega},v)_{\partial\Omega}, \\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

where

$$\mathbf{a}(u,v) = (\nabla u, \nabla v)_\Omega + \varsigma\,(\nabla_{\partial\Omega} u, \nabla_{\partial\Omega} v)_{\partial\Omega} + \lambda(u,v)_{\partial\Omega}.$$

Now, the existence and uniqueness of the weak solution for (1.1) can be acquired.

Theorem 2.1. *Let $f \in L^{\frac{2}{\alpha_0}}\big([0,T],L^2(\Omega)\big)$, $f_{\partial\Omega} \in L^{\frac{2}{\alpha_0}}\big([0,T],L^2(\partial\Omega)\big)$ for any $\alpha_0 \in (0,\alpha]$ and $u_0 \in H$. Then there is a unique function*

$$u \in L^2([0,T],V) \cap C_s([0,T],H),$$

which satisfies (2.1).

Proof. According to the fundamental approach of [33, Chapter III, Theorem 3.1], we divide the proof in the following five steps.

Step 1. Construction of discrete solution   For each m, let $H_m \subset H$ be a finite dimensional subspace generated by $\{\zeta_1,\cdots,\zeta_m\}$ and $u_m(t) := u_m(\cdot,t)$. There exists a unique sequence η_j to satisfy

$$u_m(t) = \sum_{j=1}^m \eta_j(t)\zeta_j,$$

and

$$\begin{aligned} &\big({}_0^C D_t^\alpha u_m(t), v\big)_\Omega + \mu\big({}_0^C D_t^\alpha u_m(t), v\big)_{\partial\Omega} + \mathbf{a}(u_m(t),v) = (f(t),v)_\Omega + (f_{\partial\Omega}(t),v)_{\partial\Omega}, \\ &u_m(0) = u_{m0}, \end{aligned} \tag{2.2}$$

where u_{m0} is an orthogonal projection from H of u_0 onto the space H_m formed by $\zeta_1,\cdots,\zeta_m$.

Step 2. Energy estimates Let $v=u_m(t)$ in (2.2), using Lemma 2.1, we get

$$\begin{aligned} &{}_0^C D_t^\alpha \left(\|u_m(t)\|_\Omega^2+\mu\|u_m(t)\|_{\partial\Omega}^2\right)+2\left(\|\nabla u_m(t)\|_\Omega^2+\varsigma\|\nabla_{\partial\Omega}u_m(t)\|_{\partial\Omega}^2+\lambda\|u_m(t)\|_{\partial\Omega}^2\right)\\ \le&\|u_m(t)\|_\Omega^2+\mu\|u_m(t)\|_{\partial\Omega}^2+\|f(t)\|_\Omega^2+\|f_{\partial\Omega}(t)\|_{\partial\Omega}^2. \end{aligned} \tag{2.3}$$

Applying Lemma 2.2, Hölder's inequality together with Young's inequality, for all $\alpha_0\in(0,\alpha]$ and $l=\frac{\alpha-\alpha_0}{1-\alpha_0}$, we obtain

$$\begin{aligned} &\|u_m(t)\|_\Omega^2+\mu\|u_m(t)\|_{\partial\Omega}^2\\ \lesssim&\left(\|u_m(0)\|_\Omega^2+\mu\|u_m(0)\|_{\partial\Omega}^2+\int_0^t(t-s)^{\alpha-1}\left(\|f(s)\|_\Omega^2+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^2\right)\mathrm{d}s\right)E_\alpha\left(\Gamma(\alpha)t^\alpha\right)\\ \lesssim&\|u_m(0)\|_\Omega^2+\mu\|u_m(0)\|_{\partial\Omega}^2+\int_0^t(t-s)^{\alpha-1}\left(\|f(s)\|_\Omega^2+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^2\right)\mathrm{d}s\\ \lesssim&\|u_m(0)\|_\Omega^2+\mu\|u_m(0)\|_{\partial\Omega}^2+\left[\left(\int_0^t\|f(s)\|_\Omega^{2/\alpha_0}\mathrm{d}s\right)^{\alpha_0}+\left(\int_0^t\|f_{\partial\Omega}(s)\|_{\partial\Omega}^{2/\alpha_0}\mathrm{d}s\right)^{\alpha_0}\right]\\ &\quad\times\left(\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_0}}\mathrm{d}s\right)^{1-\alpha_0}\\ \lesssim&\|u_m(0)\|_\Omega^2+\mu\|u_m(0)\|_{\partial\Omega}^2+\int_0^t\left(\|f(s)\|_\Omega^{2/\alpha_0}+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^{2/\alpha_0}\right)\mathrm{d}s+\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_0}}\mathrm{d}s\\ \lesssim&\|u_m(0)\|_\Omega^2+\mu\|u_m(0)\|_{\partial\Omega}^2+\int_0^T\left(\|f(s)\|_\Omega^{2/\alpha_0}+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^{2/\alpha_0}\right)\mathrm{d}s+\frac{T^{1+l}}{1+l}. \end{aligned} \tag{2.4}$$

This guarantees

$$\{u_m\}\ \text{ is bounded in }\ L^\infty([0,T],H). \tag{2.5}$$

Integrating (2.3) with order α, we have

$$\begin{aligned} &2\int_0^t(t-s)^{\alpha-1}\left(\|\nabla u_m(s)\|_\Omega^2+\varsigma\|\nabla_{\partial\Omega}u_m(s)\|_{\partial\Omega}^2+\lambda\|u_m(s)\|_{\partial\Omega}^2\right)\mathrm{d}s\\ \le&\|u_m(0)\|_\Omega^2+\mu\|u_m(0)\|_{\partial\Omega}^2\\ &+\int_0^t(t-s)^{\alpha-1}\Big(\|u_m(s)\|_\Omega^2+\mu\|u_m(s)\|_{\partial\Omega}^2+\|f(s)\|_\Omega^2+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^2\Big)\mathrm{d}s. \end{aligned} \tag{2.6}$$

For $s\in(0,t)$, (2.5) gives

$$\|u_m(s)\|_\Omega^2+\mu\|u_m(s)\|_{\partial\Omega}^2\lesssim\|f(s)\|_\Omega^2+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^2, \tag{2.7}$$

and by (2.4), we get

$$\begin{aligned} &\int_0^t(t-s)^{\alpha-1}\left(\|f(s)\|_\Omega^2+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^2\right)\mathrm{d}s\\ \lesssim&\int_0^T\left(\|f(s)\|_\Omega^{2/\alpha_0}+\|f_{\partial\Omega}(s)\|_{\partial\Omega}^{2/\alpha_0}\right)\mathrm{d}s+\frac{T^{1+l}}{1+l}. \end{aligned} \tag{2.8}$$

Thus, by using (2.6) to (2.8), we arrive at

$$\int_0^t(t-s)^{\alpha-1}\left(\|\nabla u_m(s)\|_\Omega^2+\mu\|\nabla_{\partial\Omega}u_m(s)\|_{\partial\Omega}^2\right)\mathrm{d}s$$

$$
\begin{aligned}
&\lesssim \|u_m(0)\|_\Omega^2 + \mu\|u_m(0)\|_{\partial\Omega}^2 + \int_0^t (t-s)^{\alpha-1}\left(\|f(s)\|_\Omega^2 + \|f_{\partial\Omega}(s)\|_{\partial\Omega}^2\right)\mathrm{d}s \\
&\lesssim \|u_m(0)\|_\Omega^2 + \mu\|u_m(0)\|_{\partial\Omega}^2 + \int_0^T \left(\|f(s)\|_\Omega^{2/\alpha_0} + \|f_{\partial\Omega}(s)\|_{\partial\Omega}^{2/\alpha_0}\right)\mathrm{d}s + \frac{T^{1+l}}{1+l}. \qquad (2.9)
\end{aligned}
$$

Moreover, $T^{\alpha-1} \leq (t-s)^{\alpha-1}$ for $s \in (0,t)$ yields

$$
\begin{aligned}
&T^{\alpha-1}\int_0^t \left(\|\nabla u_m(s)\|_\Omega^2 + \mu\|\nabla_{\partial\Omega} u_m(s)\|_{\partial\Omega}^2\right)\mathrm{d}s \\
&\leq \int_0^t (t-s)^{\alpha-1}\left(\|\nabla u_m(s)\|_\Omega^2 + \mu\|\nabla_{\partial\Omega} u_m(s)\|_{\partial\Omega}^2\right)\mathrm{d}s \\
&\lesssim \|u_m(0)\|_\Omega^2 + \mu\|u_m(0)\|_{\partial\Omega}^2 + \int_0^T \left(\|f(s)\|_\Omega^{2/\alpha_0} + \|f_{\partial\Omega}(s)\|_{\partial\Omega}^{2/\alpha_0}\right)\mathrm{d}s + \frac{T^{1+l}}{1+l}. \qquad (2.10)
\end{aligned}
$$

Hence,

$$
\{u_m\} \text{ is a bounded sequence in } L^2([0,T],V). \qquad (2.11)
$$

Step 3. Compactness. Write $\dot{u}_m : \mathbb{R} \to V$ as

$$
\dot{u}_m(t) = \begin{cases} u_m(t), & t \in [0,T], \\ 0, & t \in \mathbb{R} \setminus [0,T]. \end{cases}
$$

Indeed, Caputo derivative of $\dot{u}_m$ can be written as

$$
\begin{aligned}
{}_{-\infty}^{\ C}D_t^\alpha \dot{u}_m(t) &= {}_{-\infty}I_t^{1-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\dot{u}_m(t)\right) \\
&= {}_{-\infty}I_t^{1-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}u_m(t) + u_m(0)\delta_0 - u_m(T)\delta_T\right) \\
&= {}_0^C D_t^\alpha u_m(t) + {}_{-\infty}I_t^{1-\alpha}\left(u_m(0)\delta_0 - u_m(T)\delta_T\right), \qquad (2.12)
\end{aligned}
$$

where ${}_{-\infty}I_t^{1-\alpha}$ is the left Riemann-Liouville integral and

$$
{}_{-\infty}I_t^{1-\alpha} w(t) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^t (t-s)^{-\alpha} w(s)\mathrm{d}s.
$$

Consequently, from (2.2) and (2.12), we get

$$
\begin{aligned}
&\left({}_0^C D_t^\alpha \dot{u}_m(t), \zeta_j\right)_\Omega + \mu\left({}_0^C D_t^\alpha \dot{u}_m(t), \zeta_j\right)_{\partial\Omega} \\
&= \langle \dot{\chi}_m(t), \zeta_j\rangle + \left(u_m(0), \zeta_j\right)_\Omega {}_{-\infty}I_t^{1-\alpha}\delta_0 + \mu\left(u_m(0), \zeta_j\right)_{\partial\Omega} {}_{-\infty}I_t^{1-\alpha}\delta_0 \\
&\quad - \left(u_m(T), \zeta_j\right)_\Omega {}_{-\infty}I_t^{1-\alpha}\delta_T - \mu\left(u_m(T), \zeta_j\right)_{\partial\Omega} {}_{-\infty}I_t^{1-\alpha}\delta_T,
\end{aligned}
$$

where δ_0 and δ_T are the Dirac distributions at 0 and T, together with

$$
\dot{\chi}_m(t) = \begin{cases} \chi_m(t), & t \in [0,T], \\ 0, & t \in \mathbb{R} \setminus [0,T]. \end{cases}
$$

Here

$$
\chi_m(t) = f(t) + f_{\partial\Omega}(t) - \bar{G}u_m(t)|_\Omega - \varsigma\bar{G}u_m(t)|_{\partial\Omega} - \lambda u_m(t)|_{\partial\Omega} - \nabla u_m(t)\cdot\mathbf{n}|_{\partial\Omega}
$$

and

$$\bar{G}=\begin{cases}-\Delta & \text{in } \Omega,\\ -\Delta_{\partial\Omega} & \text{on } \partial\Omega.\end{cases}$$

Then, it is easy to obtain

$$\int_0^T \|\chi_m(t)\|_{V'}\,\mathrm{d}t \lesssim \int_0^T \left(\|f(t)\|_{V'}+\|f_{\partial\Omega}(t)\|_{V'}+\|u_m(t)\|_V\right)\mathrm{d}t,$$

which is bounded by (2.11), and V' is the dual space of V. By Parseval inequality, we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} |\tau|^{2\alpha-2}\,\|\hat{u}_m(\tau)\|_V^2\,\mathrm{d}\tau\\ =&\int_{-\infty}^{+\infty}\left\|{}_{-\infty}I_t^{1-\alpha}\dot{u}_m(t)\right\|_V^2\mathrm{d}t=\int_0^T\left\|{}_0I_t^{1-\alpha}u_m(t)\right\|_V^2\mathrm{d}t\\ \le&\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2\int_0^T\|u_m(t)\|_V^2\,\mathrm{d}t,\end{aligned}$$

where $\hat{u}_m$ is Fourier transform of $\dot{u}_m$. Thus, we can apply the compactness result in [33].

Step 4. Existence. The estimates (2.4) and (2.10) ensure the existence of $u\in L^2([0,T],V)\cap L^\infty([0,T],H)$ and a sub-sequence $u_{m'}$ such that $u_{m'}\to u$ in $L^2([0,T],V)$ weakly, and in $L^\infty([0,T],H)$ weak-star, as $m'\to\infty$, along with $u_{m'}\to u$ in $L^2([0,T],H)$ strongly.

Integrating (2.2), for $t_B\in[0,T]$, we obtain

$$\begin{aligned}&(u_{m'}(t),v)_\Omega-(u_{m'}(t_B),v)_\Omega+\mu\,(u_{m'}(t),v)_{\partial\Omega}-\mu\,(u_{m'}(t_B),v)_{\partial\Omega}\\ =&\int_0^{t_B}\left[(t_B-s)^{-1+\alpha}-(t-s)^{-1+\alpha}\right]\left(\mathbf{a}(u_{m'}(s),v)-\langle f(s),v\rangle_\Omega-\langle f_{\partial\Omega}(s),v\rangle_{\partial\Omega}\right)\mathrm{d}s\\ &-\int_{t_B}^t(t-s)^{-1+\alpha}\left(\mathbf{a}(u_{m'}(s),v)-\langle f(s),v\rangle_\Omega-\langle f_{\partial\Omega}(s),v\rangle_{\partial\Omega}\right)\mathrm{d}s.\end{aligned}\tag{2.13}$$

Since $u_{m'}\to u$ weakly in $L^2([0,T],V)$, we suppose that $u_{m'}(t_B)\to u(t_B)$ in V for any $t_B\in[0,T]\setminus\mathcal{G}$ and the measure of $\mathcal{G}$ is equal to 0. Hence, for $t_B\notin\mathcal{G}$, $\lim_{m'\to\infty}u_{m'}(t_B)=u(t_B)$ in H. So, (2.9) gives

$$\begin{aligned}&\lim_{m'\to\infty}\int_{t_B}^t(t-s)^{-1+\alpha}\big((\nabla u_{m'}(s),\nabla v)_\Omega+\mu\,(\nabla_{\partial\Omega}u_{m'}(s),\nabla_{\partial\Omega}v)_{\partial\Omega}\big)\mathrm{d}s\\ =&\int_{t_B}^t(t-s)^{-1+\alpha}\big((\nabla u(s),\nabla v)_\Omega+\mu\,(\nabla_{\partial\Omega}u(s),\nabla_{\partial\Omega}v)_{\partial\Omega}\big)\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}&\lim_{m'\to\infty}\int_0^{t_B}\left[(t_B-s)^{-1+\alpha}-(t-s)^{-1+\alpha}\right]\big((\nabla u_{m'}(s),\nabla v)_\Omega+\mu\,(\nabla_{\partial\Omega}u_{m'}(s),\nabla_{\partial\Omega}v)_{\partial\Omega}\big)\mathrm{d}s\\ =&\int_0^{t_B}\left[(t_B-s)^{-1+\alpha}-(t-s)^{-1+\alpha}\right]\big((\nabla u(s),\nabla v)_\Omega+\mu\,(\nabla_{\partial\Omega}u(s),\nabla_{\partial\Omega}v)_{\partial\Omega}\big)\mathrm{d}s.\end{aligned}$$

Taking the limit of (2.13) for the sequence m', for $t\ge t_B$ and t, $t_B\notin\mathcal{G}$, we get

$$(u(t),v)_\Omega+\mu\,(u(t),v)_{\partial\Omega}-\left[(u(t_B),v)_\Omega+\mu\,(u(t_B),v)_{\partial\Omega}\right]$$

$$
\begin{aligned}
&= \int_0^{t_B} \left[(t_B - s)^{-1+\alpha} - (t-s)^{-1+\alpha}\right] \left(\mathbf{a}(u(s), v) - \langle f(s), v\rangle_\Omega - \langle f_{\partial\Omega}(s), v\rangle_{\partial\Omega}\right) \mathrm{d}s \\
&\quad - \int_{t_B}^{t} (t-s)^{-1+\alpha} \left(\mathbf{a}(u(s), v) - \langle f(s), v\rangle_\Omega - \langle f_{\partial\Omega}(s), v\rangle_{\partial\Omega}\right) \mathrm{d}s. \qquad (2.14)
\end{aligned}
$$

When $t \to t_B$, we have

$$
(u(t), v)_\Omega + \mu\, (u(t), v)_{\partial\Omega} - \left[(u(t_B), v)_\Omega + \mu\, (u(t_B), v)_{\partial\Omega}\right] \;\to\; 0.
$$

From (2.14), it not difficult to obtain (2.1) for $t_B = 0$. Here, we denote $u(t) = u(\cdot, t)$ and the existence of the weak solution is proved.

Step 5. Uniqueness. Suppose that u_1 and u_2 are the solutions of (2.1). Take $v = \tilde{u}$ in (2.1), where $\tilde{u} = u_1 - u_2$. Thus, we get

$$
\left({}_0^C D_t^\alpha \tilde{u}, \tilde{u}\right)_\Omega + \mu \left({}_0^C D_t^\alpha \tilde{u}, \tilde{u}\right)_{\partial\Omega} + \mathbf{a}(\tilde{u}, \tilde{u}) = 0.
$$

Then, by using Lemma 2.1, we obtain

$$
{}_0^C D_t^\alpha \left(\|\tilde{u}\|_\Omega^2 + \mu\|\tilde{u}\|_{\partial\Omega}^2\right) + 2\left(\|\nabla\tilde{u}\|_\Omega^2 + \varsigma\|\nabla_{\partial\Omega}\tilde{u}\|_{\partial\Omega}^2 + \lambda\|\tilde{u}\|_{\partial\Omega}^2\right) \le 0,
$$

which implies $\tilde{u} = 0$ for $t \in [0, T]$. $\square$

3. Fully Discrete Method

3.1. Space discretization

Without loss of generality, we set $\Omega := (a, b) \times (c, d)$. In fact, the assumption of the rectangular spatial domain is not essential because the method can be designed for arbitrary bounded domain [24]. Assume that Ω is divided equally into $M_1 \times M_2$ nonoverlapping open rectangular meshes Q, $M_i \in \mathbb{N}_+$, $i = 1, 2$. $\mathscr{Q}_h$ is the set of all Q and $\bar{\Omega} = \cup_{Q\in\mathscr{Q}_h} \bar{Q}$. We set the mesh size $h = \mathrm{diam}(Q)$ for $Q \in \mathscr{Q}_h$. e denotes an interior edge, which is nonempty intersection of the closure between the adjacent elements. In the meantime, a boundary edge $e_{\partial\Omega}$ is defined by nonempty intersection of the closure for an element of $\mathscr{Q}_h$ and $\partial\Omega$. In addition, $\mathscr{E}_h^0$ and $\mathscr{E}_h^{\partial\Omega}$ are used to represent the set of interior edges and exterior edges respectively.

Introduce the broken Sobolev spaces (see [34])

$$
\begin{aligned}
&H^s(\mathscr{Q}_h) := \left\{w \in L^2(\Omega) : w|_Q \in H^s(Q), Q \in \mathscr{Q}_h\right\}, \\
&H^s(\mathscr{E}_h^{\partial\Omega}) := \left\{w \in L^2(\partial\Omega) : w|_{e_{\partial\Omega}} \in H^s(e_{\partial\Omega}), e_{\partial\Omega} \in \mathscr{E}_h^{\partial\Omega}\right\}, \quad s \ge 0
\end{aligned}
$$

with $H^0(\mathscr{Q}_h) = L^2(\mathscr{Q}_h)$ and $H^0(\mathscr{E}_h^{\partial\Omega}) = L^2(\mathscr{E}_h^{\partial\Omega})$. Moreover, we define

$$
H_\mu^s(\mathscr{Q}_h, \mathscr{E}_h^{\partial\Omega}) := \left\{w \in H^s(\mathscr{Q}_h) : \mu w|_{\mathscr{E}_h^{\partial\Omega}} \in H^s(\mathscr{E}_h^{\partial\Omega})\right\}.
$$

Define the finite dimensional space

$$
\mathcal{V}_h := \left\{w \in L^2(\Omega) : w|_{\bar{Q}} \in \mathbb{P}^p(\bar{Q}), Q \in \mathscr{Q}_h\right\}, \quad p \ge 1,
$$

where $\mathbb{P}^p(\bar{Q})$ as the set of polynomials and the degrees are not more than p on $\bar{Q}$. For each e, the jumps and the averages of $w \in H^1(\mathscr{Q}_h)$ are

$$[w]_e = \mathbf{n}_e^+ w^+ + \mathbf{n}_e^- w^-, \quad \{w\}_e = \frac{w^+ + w^-}{2}.$$

Now, we rewrite the variational formulation (2.1) as: for all $Q \in \mathscr{Q}_h$ and $t \in (0,T]$, find $u(\cdot,t) \in H^1(\mathscr{Q}_h, \mathscr{E}_h^{\partial\Omega})$ to satisfy

$$\begin{aligned}
&\int_Q {}_0^C D_t^\alpha u \cdot v \mathrm{d}x\mathrm{d}y + \int_Q \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\partial Q}\left(\frac{\partial u}{\partial x} v \mathrm{n}_x + \frac{\partial u}{\partial y} v \mathrm{n}_y\right)\mathrm{d}s = \int_Q f v \mathrm{d}x\mathrm{d}y, \\
&\mu \int_{\mathscr{E}_h^{\partial\Omega}} {}_0^C D_t^\alpha u \cdot v \mathrm{d}x\mathrm{d}y = -\lambda \int_{\mathscr{E}_h^{\partial\Omega}} u v \mathrm{d}s - \varsigma \int_{\mathscr{E}_h^{\partial\Omega}} \nabla_{\partial\Omega} u \nabla_{\partial\Omega} v \mathrm{d}s \\
&\qquad\qquad - \int_{\mathscr{E}_h^{\partial\Omega}} \partial_{\mathbf{n}} u \, v \mathrm{d}s + \int_{\mathscr{E}_h^{\partial\Omega}} f_{\partial\Omega} v \mathrm{d}x\mathrm{d}y, \\
&\int_{\bar{Q}} u(0) v \mathrm{d}x\mathrm{d}y = \int_{\bar{Q}} u_0 v \mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{3.1}$$

where $v \in H^1(\mathscr{Q}_h, \mathscr{E}_h^{\partial\Omega})$.

From (3.1), DDG semi-discrete scheme is constructed as: for all $Q \in \mathscr{Q}_h$ and $t \in (0,T]$, find $u_h(\cdot,t) \in \mathcal{V}_h$ to arrive at

$$\begin{aligned}
&\int_Q {}_0^C D_t^\alpha u_h \cdot v_h \mathrm{d}x\mathrm{d}y + \int_Q \left(\frac{\partial u_h}{\partial x}\frac{\partial v_h}{\partial x} + \frac{\partial u_h}{\partial y}\frac{\partial v_h}{\partial y}\right)\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\partial Q}\left(\widehat{\frac{\partial u_h}{\partial x}} v_h \mathrm{n}_x + \widehat{\frac{\partial u_h}{\partial y}} v_h \mathrm{n}_y\right)\mathrm{d}s = \int_Q f v_h \mathrm{d}x\mathrm{d}y, \\
&\int_{\bar{Q}} u(0) v_h \mathrm{d}x\mathrm{d}y = \int_{\bar{Q}} u_{h0} v_h \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.2}$$

Here, $v_h \in \mathcal{V}_h$, $u_{h0} \in \mathcal{V}_h$ is the L^2-projection of u_0 onto $\mathcal{V}_h$ and the numerical fluxes are defined by

$$\begin{aligned}
&\widehat{\frac{\partial u_h}{\partial x}}\Big|_{\partial Q} = \theta_0 h^{-1}[u_h]_{\partial Q}\mathrm{n}_x + \left\{\frac{\partial u_h}{\partial x}\right\}_{\partial Q} + \theta_1 h\left[\frac{\partial^2 u_h}{\partial x^2}\right]_{\partial Q}\mathrm{n}_x, \quad \partial Q \in \mathscr{E}_h^0, \\
&\widehat{\frac{\partial u_h}{\partial y}}\Big|_{\partial Q} = \theta_0 h^{-1}[u_h]_{\partial Q}\mathrm{n}_y + \left\{\frac{\partial u_h}{\partial y}\right\}_{\partial Q} + \theta_1 h\left[\frac{\partial^2 u_h}{\partial y^2}\right]_{\partial Q}\mathrm{n}_y, \quad \partial Q \in \mathscr{E}_h^0, \\
&\int_{\mathscr{E}_h^{\partial\Omega}}\left(\widehat{\frac{\partial u_h}{\partial x}} v_h \mathrm{n}_x + \widehat{\frac{\partial u_h}{\partial y}} v_h \mathrm{n}_y\right)\mathrm{d}s = -\lambda\int_{\mathscr{E}_h^{\partial\Omega}} u_h v_h \mathrm{d}s - \varsigma\int_{\mathscr{E}_h^{\partial\Omega}} \nabla_{\partial\Omega} u_h \nabla_{\partial\Omega} v_h \mathrm{d}s \\
&\qquad\qquad - \mu\int_{\mathscr{E}_h^{\partial\Omega}} {}_0^C D_t^\alpha u_h v_h \mathrm{d}s + \int_{\mathscr{E}_h^{\partial\Omega}} f_{\partial\Omega} v_h \mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where θ_0 and θ_1 are user-chosen positive constants. Summing (3.2) for each $Q \in \mathscr{Q}_h$, and then we can obtain

$$\begin{aligned}
&\left({}_0^C D_t^\alpha u_h, v_h\right)_\Omega + \mu\left({}_0^C D_t^\alpha u_h, v_h\right)_{\partial\Omega} + \mathbf{a_h}(u_h, v_h) = (f, v_h)_\Omega + (f_{\partial\Omega}, v_h)_{\partial\Omega}, \\
&(u_0, v_h) = (u_{h0}, v_h),
\end{aligned} \tag{3.3}$$

where

$$\mathbf{a_h}(u_h, v_h) = (\nabla u_h, \nabla v_h)_\Omega + \varsigma\,(\nabla_{\partial\Omega} u_h, \nabla_{\partial\Omega} v_h)_{\partial\Omega} + \lambda(u_h, v_h)_{\partial\Omega} + \sum_Q \int_{\partial Q} \widehat{\nabla u_h} \cdot [v_h] \cdot \mathbf{n} \mathrm{d}s, \quad \partial Q \in \mathscr{E}_h^0.$$

This finishes the description of the DDG method.

Remark 3.1. An appropriate choice of (θ_0, θ_1) is to ensure the stability of the method. Generally speaking, when $\theta_1 = 0$, θ_0 needs to be large enough to stabilize the scheme. In our calculations, we take $\theta_0 = 2$, $\theta_1 = \frac{1}{12}$ and the specific explanation can be seen in [25].

3.2. Time discretization

In order to deal with the initial singularity of time fractional diffusion problems, we use a graded mesh $t_j := T(j/N)^\varpi$ for $0 \le j \le N$, $N \in \mathbb{N}_+$ and $\varpi \ge 1$. For such nonuniform time levels, let $\Delta t_j = t_j - t_{j-1}$ for $1 \le j \le N$, and $\Delta t = \max_{1 \le j \le N} \Delta t_j$. We define a fractional time level $t_{j-\psi} = \psi t_{j-1} + (1-\psi) t_j$ for $\psi \in (0,1)$. In what follows, we let $\psi = \frac{\alpha}{2}$. In addition, the local step-size ratios are denote by

$$\phi_j = \frac{\Delta t_j}{\Delta t_{j+1}}, \quad 1 \le j \le N-1$$

with $\phi := \max_{1 \le j \le N-1} \phi_j$.

Then we introduce the following assumption to carry out the theoretical analyses later.

M. For $\varpi \ge 1$, there is a constant $C_\varpi > 0$ which satisfies

$$\Delta t_j \le C_\varpi \Delta t \min\left\{1, t_j^{1-1/\varpi}\right\} \quad \text{for} \quad 1 \le j \le N$$

with

$$t_j \le C_\varpi t_{j-1} \quad \text{and} \quad \frac{\Delta t_j}{t_j} \le C_\varpi \frac{\Delta t_{j-1}}{t_{j-1}} \quad \text{for} \quad 2 \le j \le N.$$

Obviously, the graded mesh satisfies **M**, and $\phi \le 1$. For any time sequence $(w^j)_{j=0}^N$, take the backward difference $\partial w^j = w^j - w^{j-1}$ and the interpolated value $w^{n-\psi} = \psi w^{n-1} + (1-\psi) w^n$. Let $\Pi_{1,j} w$ be the linear interpolant of w for t_{j-1} and t_j, and $\Pi_{2,j} w$ be the quadratic with t_{j-1}, t_j and t_{j+1}. Then, similar to [29], Alikhanov type scheme to $\left({}_0^C D_t^\alpha w\right)(t_{n-\psi})$ is as follows:

$$\begin{aligned}
(\mathscr{D}^\alpha w)^{n-\psi} &= \int_{t_{n-1}}^{t_{n-\psi}} \kappa_n'(s) \left(\Pi_{1,n} w\right)'(s) \mathrm{d}s + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \kappa_n'(s) \left(\Pi_{2,j} w\right)'(s) \mathrm{d}s \\
&= c_0^{(n)} \partial w^n + \sum_{j=1}^{n-1} \left(c_{n-j}^{(n)} \partial w^j + \phi_j g_{n-j}^{(n)} \partial w^{j+1} - g_{n-j}^{(n)} \partial w^j \right),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
c_{n-j}^{(n)} &= \frac{1}{\Delta t_j} \int_{t_{j-1}}^{\min\{t_j, t_{n-\psi}\}} \kappa_n'(s) \mathrm{d}s, & 1 \le j \le n, \\
g_{n-j}^{(n)} &= \frac{2}{\Delta t_j (\Delta t_j + \Delta t_{j+1})} \int_{t_{j-1}}^{t_j} (s - t_{j-\frac{1}{2}}) \kappa_n'(s) \mathrm{d}s, & 1 \le j \le n-1
\end{aligned}$$

with $\kappa_n'(t) = \omega_{1-\alpha}(t_{n-\psi} - t)$ and $\omega_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$.

From (3.4), we get the compact form

$$(\mathscr{D}^\alpha w_h)^{n-\psi} = \sum_{j=1}^{n} Z_{n-j}^{(n)} \partial w_h^j,$$

namely

$$(\mathscr{D}^\alpha w_h)^{n-\psi} = Z_0^{(n)} w_h^n - \sum_{j=1}^{n-1} \left(Z_{n-j-1}^{(n)} - Z_{n-j}^{(n)} \right) w_h^j - Z_{n-1}^{(n)} w_h^0,$$

and the local truncation error denote by

$$\Upsilon_t^{n-\psi} = \left({}_0^C D_t^\alpha w_h \right)(t_{n-\psi}) - (\mathscr{D}^\alpha w_h)^{n-\psi}, \quad 1 \le n \le N. \tag{3.5}$$

Furthermore, $Z_{n-j}^{(n)}$ are defined as follows. When $n=1$, $Z_0^{(1)} = c_0^{(1)}$. When $n \ge 2$,

$$Z_{n-j}^{(n)} = \begin{cases} c_0^{(n)} + \phi_{n-1} g_1^{(n)}, & j = n, \\ c_{n-j}^{(n)} + \phi_{j-1} g_{n-j+1}^{(n)} + g_{n-j}^{(n)}, & 2 \le j \le n-1, \\ c_{n-1}^{(n)} - g_{n-1}^{(n)}, & j = 1. \end{cases}$$

In addition, we introduce the following hypotheses:

(I) There is a constant $\pi_Z > 0$ to yield

$$Z_{n-j}^{(n)} \ge \frac{1}{\pi_Z \Delta t_j} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_n - s) \mathrm{d}s, \quad 1 \le j \le n \le N.$$

(II) $Z_{n-j}^{(n)}$ are monotone, that is $Z_{j-2}^{(n)} \ge Z_{j-1}^{(n)} > 0$ for $2 \le j \le n \le N$.

Furthermore, define the global consistency error

$$\mathscr{E}_{\text{glob}}^n := \sum_{j=1}^{n} \Xi_{n-j}^{(n)} \left| \Upsilon_t^{j-\psi} \right|, \quad 1 \le j \le n \le N,$$

where $\Xi_{n-j}^{(n)}$ are chosen to satisfy

$$\sum_{i=j}^{n} \Xi_{n-j}^{(n)} Z_{i-j}^{(i)} \equiv 1, \quad 1 \le j \le n \le N.$$

Thus, by (3.3), we obtain the fully-discrete numerical scheme: require that $u_h \in \mathcal{V}_h$ to satisfy

$$\begin{aligned} &\left((\mathscr{D}^\alpha u_h)^{n-\psi}, v_h \right)_\Omega + \mu \left((\mathscr{D}^\alpha u_h)^{n-\psi}, v_h \right)_{\partial\Omega} + \mathbf{a}_h(u_h^{n-\psi}, v_h) \\ =& \left(f^{n-\psi}, v_h \right)_\Omega + \left(f_{\partial\Omega}^{n-\psi}, v_h \right)_{\partial\Omega} \end{aligned} \tag{3.6}$$

for all $v_h \in \mathcal{V}_h$ and $1 \le n \le N$.

Remark 3.2. In our simulation, $[0,T]$ is divided into $[0,\widetilde{T}]$ and $[\widetilde{T},T]$ with N subintervals. In $[0,\widetilde{T}]$, we apply the graded mesh $t_n = \widetilde{T}(n/\widetilde{N})^\varpi$ for $n \in [0,\widetilde{N}]$. Furthermore, we utilize a uniform mesh in $[\widetilde{T},T]$ with step size Δt. Refer to [27], we take $\Delta t = \frac{T-\widetilde{T}}{N-\widetilde{N}}$, where $\widetilde{T} = 2^{-\varpi}$ and $\widetilde{N} = \lceil \frac{\varpi N}{2^\varpi - 1 + \varpi} \rceil$.

4. Stability

To establish the stability for the full scheme (3.6), we first introduce some lemmas.

Lemma 4.1. *For $1 \le n \le N$, we have*

$$\left((\mathscr{D}^\alpha w)^{n-\psi}, w^{n-\psi}\right)_\Omega + \mu\left((\mathscr{D}^\alpha w)^{n-\psi}, w^{n-\psi}\right)_{\partial\Omega} \ge \frac{1}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_{L^2_\mu(\Omega,\partial\Omega)}\Big).$$

Proof. Define $\psi^{(1)} := \frac{1}{2}$ and $\psi^{(n)} := \frac{Z_0^{(n)} - Z_1^{(n)}}{2Z_0^{(n)} - Z_1^{(n)}}$ for $n \ge 2$. It is easy to get $\psi^{(1)} \ge \psi$. Furthermore, [35, Lemma 3.1] gives that $\psi^{(n)} \ge \psi$ for $n \ge 2$ along with (II) holds. Consequently, by [36, Lemma 4.1], we arrive at

$$\begin{aligned}
\left((\mathscr{D}^\alpha w)^{n-\psi}, w^{n-\psi}\right)_\Omega &\ge \frac{1}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_\Omega\Big),\\
\left((\mathscr{D}^\alpha w)^{n-\psi}, w^{n-\psi}\right)_{\partial\Omega} &\ge \frac{1}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_{\partial\Omega}\Big).
\end{aligned}$$

Thus for $1 \le n \le N$, we get

$$\begin{aligned}
&\left((\mathscr{D}^\alpha w)^{n-\psi}, w^{n-\psi}\right)_\Omega + \mu\left((\mathscr{D}^\alpha w)^{n-\psi}, w^{n-\psi}\right)_{\partial\Omega}\\
\ge\ & \frac{1}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_\Omega\Big) + \frac{\mu}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_{\partial\Omega}\Big)\\
=\ & \frac{1}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_\Omega + \mu\|w^j\|^2_{\partial\Omega}\Big)\\
=\ & \frac{1}{2}\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial\Big(\|w^j\|^2_{L^2_\mu(\Omega,\partial\Omega)}\Big).
\end{aligned}$$

The proof is complete. $\square$

Lemma 4.2 (see [36]). *Suppose that the conditions* (I) *and* (II) *hold, and $\psi \in [0,1)$. Let $c_0 > 0$ be a constant independent of*

$$\Delta t \le \frac{1}{\sqrt[\alpha]{2\Gamma(2-\alpha)\pi_Z c_0}}.$$

If we find $(\xi^j)_{j=1}^N \ge 0$ and $(w^j)_{j=1}^N \ge 0$ such that

$$\sum_{j=1}^{n} Z_{n-j}^{(n)}\partial(w^j)^2 \le c_0(w^{n-\psi})^2 + w^{n-\psi}\xi^n, \quad 1 \le n \le N,$$

then

$$\begin{aligned}
w^n &\le 2E_\alpha\left(2\max(1,\phi)\pi_Z c_0 t_n^\alpha\right)\left(w^0 + \max_{1\le j\le n}\sum_{i=1}^{j}\Xi_{j-i}^{(j)}\xi^i\right)\\
&\le 2E_\alpha\left(2\max(1,\phi)\pi_Z c_0 t_n^\alpha\right)\left(w^0 + \pi_Z\Gamma(1-\alpha)\max_{1\le i\le n}\left\{t_i^\alpha \xi^i\right\}\right), \quad 1 \le n \le N,
\end{aligned}$$

and from $\phi \le 1$, it follows that

$$w^n \le 2E_\alpha\left(2\pi_Z c_0 t_n^\alpha\right)\left(w^0 + \max_{1\le j\le n}\sum_{i=1}^{j}\Xi_{j-i}^{(j)}\xi^i\right)$$
$$\le 2E_\alpha\left(2\pi_Z c_0 t_n^\alpha\right)\left(w^0 + \pi_Z\Gamma(1-\alpha)\max_{1\le i\le n}\left\{t_i^\alpha\xi^i\right\}\right),\quad 1\le n\le N.$$

Theorem 4.1. *Suppose that $\Delta t \le 1/\sqrt[\alpha]{11\Gamma(2-\alpha)}$, and there is a constant $r^* \in (0,1)$ to satisfy $1 - C^*\frac{(r^*)^2}{2\theta_0} \ge 0$. Let u_h^n be the solution of (3.6). Then the full scheme (3.6) is stable, and we have for $1 \le n \le N$*

$$\|u_h^n\|_{L^2_\mu(\Omega,\partial\Omega)} \tag{4.1}$$
$$\le 2E_\alpha\left(11t_n^\alpha\right)\left(\|u_{h0}\|_{L^2_\mu(\Omega,\partial\Omega)} + \frac{11\sqrt{2}}{2}\Gamma(1-\alpha)C_u\max_{1\le i\le n}\left\{t_i^\alpha\left(\|f^{i-\psi}\|_\Omega + \|f_{\partial\Omega}^{i-\psi}\|_{\partial\Omega}\right)\right\}\right),$$

where the constants C^* and C_u only depend on u.

Proof. Taking $v_h = u_h^{n-\psi}$ in (3.6), we arrive at

$$\left((\mathscr{D}^\alpha u_h)^{n-\psi}, u_h^{n-\psi}\right)_\Omega + \mu\left((\mathscr{D}^\alpha u_h)^{n-\psi}, u_h^{n-\psi}\right)_{\partial\Omega} + \mathbf{a}_h(u_h^{n-\psi}, u_h^{n-\psi})$$
$$= \left(f^{n-\psi}, u_h^{n-\psi}\right)_\Omega + \left(f_{\partial\Omega}^{n-\psi}, u_h^{n-\psi}\right)_{\partial\Omega},$$

and

$$\mathbf{a}_h(u_h^{n-\psi}, u_h^{n-\psi}) = (\nabla u_h^{n-\psi}, \nabla u_h^{n-\psi})_\Omega + \varsigma\left(\nabla_{\partial\Omega}u_h^{n-\psi}, \nabla_{\partial\Omega}u_h^{n-\psi}\right)_{\partial\Omega}$$
$$+ \lambda(u_h^{n-\psi}, u_h^{n-\psi})_{\partial\Omega} + \sum_Q\int_{\partial Q}\widehat{\nabla u_h^{n-\psi}}\cdot\left[u_h^{n-\psi}\right]\cdot\mathbf{n}\mathrm{d}s.$$

From Young's inequality, there exists $r^* \in (0,1)$ such that

$$\sum_Q\int_{\partial Q}\widehat{\nabla u_h^{n-\psi}}\left[u_h^{n-\psi}\right]\mathbf{n}\mathrm{d}s$$
$$= \sum_Q\int_{\partial Q}\left(\theta_0 h^{-1}\left[u_h^{n-\psi}\right]\left[u_h^{n-\psi}\right] + \left\{\nabla u_h^{n-\psi}\right\}\left[u_h^{n-\psi}\right]\mathbf{n}\right)\mathrm{d}s$$
$$+ \sum_Q\int_{\partial Q}\left(\theta_1 h\left[\frac{\partial^2 u_h^{n-\psi}}{\partial x^2}\right]\mathrm{n}_x + \theta_1 h\left[\frac{\partial^2 u_h^{n-\psi}}{\partial y^2}\right]\mathrm{n}_y\right)\left[u_h^{n-\psi}\right]\mathrm{d}s$$
$$\ge \theta_0 h^{-1}\sum_Q\int_{\partial Q}\left[u_h^{n-\psi}\right]^2\mathrm{d}s - r^*\sum_Q\int_{\partial Q}\left|\nabla u_h^{n-\psi}\right||\mathbf{n}|\left|\left[u_h^{n-\psi}\right]\right|\mathrm{d}s$$
$$\ge \theta_0\sum_Q\int_{\partial Q}h^{-1}\left[u_h^{n-\psi}\right]^2\mathrm{d}s - r^*\sum_Q h^{\frac{1}{2}}\left\|\nabla u_h^{n-\psi}\right\|_{\partial Q}h^{-\frac{1}{2}}\left\|\left[u_h^{n-\psi}\right]\right\|_{\partial Q}$$
$$\ge \frac{\theta_0}{2}\sum_Q\frac{1}{h}\left\|\left[u_h^{n-\psi}\right]\right\|_{\partial Q}^2 - \frac{(r^*)^2}{2\theta_0}\sum_Q h\left\|\nabla u_h^{n-\psi}\right\|_{\partial Q}^2.$$

Using the trace inequality, there is a constant $C^* > 0$ to satisfy

$$\sum_Q h\left\|\nabla u_h^{n-\psi}\right\|_{\partial Q}^2 \le C^*\sum_Q\left\|\nabla u_h^{n-\psi}\right\|_Q^2,$$

which gives

$$
\begin{aligned}
&\left\|\nabla u_h^{n-\psi}\right\|^2 + \sum_Q \int_{\partial Q} \widehat{\nabla u_h^{n-\psi}} \left[u_h^{n-\psi}\right] \mathbf{n} \mathrm{d}s \\
\geq & \sum_Q \left\|\nabla u_h^{n-\psi}\right\|_Q^2 + \frac{\theta_0}{2} \sum_Q h^{-1} \left\|\left[u_h^{n-\psi}\right]\right\|_{\partial Q}^2 - \frac{(r^*)^2}{2\theta_0} \sum_Q h \left\|\nabla u_h^{n-\psi}\right\|_{\partial Q}^2 \\
\geq & \left(1 - C^* \frac{(r^*)^2}{2\theta_0}\right) \sum_Q \left\|\nabla u_h^{n-\psi}\right\|_Q^2 + \frac{\theta_0}{2} \sum_Q h^{-1} \left\|\left[u_h^{n-\psi}\right]\right\|_{\partial Q}^2 .
\end{aligned}
$$

Thus, $\mathbf{a}_h(u_h^{n-\psi}, u_h^{n-\psi}) \geq 0$. Due to Lemma 4.1, we have the following inequality:

$$
\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{n} Z_{n-j}^{(n)} \partial \left(\left\|u_h^j\right\|_\Omega^2 + \mu \left\|u_h^j\right\|_{\partial\Omega}^2\right) \\
\leq & \left\|u_h^{n-\psi}\right\|_\Omega^2 + \mu \left\|u_h^{n-\psi}\right\|_{\partial\Omega}^2 + \left\|f^{n-\psi}\right\|_\Omega^2 + \left\|f_{\partial\Omega}^{n-\psi}\right\|_{\partial\Omega}^2 \\
\leq & \left\|u_h^{n-\psi}\right\|_\Omega^2 + \mu \left\|u_h^{n-\psi}\right\|_{\partial\Omega}^2 + C_u \left(\left\|u_h^{n-\psi}\right\|_\Omega + \mu \left\|u_h^{n-\psi}\right\|_{\partial\Omega}\right) \left(\left\|f^{n-\psi}\right\|_\Omega + \left\|f_{\partial\Omega}^{n-\psi}\right\|_{\partial\Omega}\right).
\end{aligned}
$$

Then, there is a constant $C_u > 0$ to get

$$
\begin{aligned}
&\sum_{j=1}^{n} Z_{n-j}^{(n)} \partial \left\|u_h^j\right\|_{L_\mu^2(\Omega,\partial\Omega)}^2 \\
\leq & 2\left(\left\|u_h^{n-\psi}\right\|_{L_\mu^2(\Omega,\partial\Omega)}^2 + C_u \left(\left\|u_h^{n-\psi}\right\|_\Omega + \mu \left\|u_h^{n-\psi}\right\|_{\partial\Omega}\right) \left(\left\|f^{n-\psi}\right\|_\Omega + \left\|f_{\partial\Omega}^{n-\psi}\right\|_{\partial\Omega}\right)\right) \\
\leq & 2 \left\|u_h^{n-\psi}\right\|_{L_\mu^2(\Omega,\partial\Omega)}^2 + 2\sqrt{2} C_u \left\|u_h^{n-\psi}\right\|_{L_\mu^2(\Omega,\partial\Omega)} \left(\left\|f^{n-\psi}\right\|_\Omega + \left\|f_{\partial\Omega}^{n-\psi}\right\|_{\partial\Omega}\right).
\end{aligned}
$$

By Lemma 4.2 (with $\pi_Z = 11/4$), it is easy to get the desired result (4.1). □

5. Convergence

Similar to Lemma 2.2 in [22], we assume the solution u of (1.1) satisfies the following regular conditions:

$$
\|u(\boldsymbol{x}, t)\|_{H_\mu^{p+1}(\Omega,\partial\Omega)} \leq C, \tag{5.1a}
$$

$$
\|D_t^\alpha u(\boldsymbol{x}, t)\|_{H_\mu^{p+1}(\Omega,\partial\Omega)} \leq C, \tag{5.1b}
$$

$$
\left\|\frac{\partial^i u}{\partial t^i}(\boldsymbol{x}, t)\right\|_{L_\mu^2(\Omega,\partial\Omega)} \lesssim \left(1 + t^{\sigma - i}\right), \quad i = 1, 2, 3 \tag{5.1c}
$$

with $\sigma \in (0, 1)$.

Before providing the error analysis, for a given piecewise smooth function q, we define a projection $\mathit{\Lambda}$: $H_\mu^{p+1}(\Omega, \partial\Omega) \to \mathcal{V}_h$ as follows:

$$
\int_{\bar{Q}} (\mathit{\Lambda} q - q)\, w \mathrm{d}x \mathrm{d}y = 0, \quad \forall w \in \mathbb{P}^{p-2}(\bar{Q}),
$$

$$\begin{aligned}
&\frac{\widehat{\partial(\varLambda^{(x)}q)}}{\partial x} := \theta_0 h^{-1}\big[\varLambda^{(x)}q\big]\mathrm{n}_x + \left\{\frac{\partial(\varLambda^{(x)}q)}{\partial x}\right\} + \theta_1 h\left[\frac{\partial^2(\varLambda^{(x)}q)}{\partial x^2}\right]\mathrm{n}_x\Big|_{\partial Q} = \frac{\partial q}{\partial x}\Big|_{\partial Q}, \quad \partial Q \in \mathscr{E}_h^0,\\
&\frac{\widehat{\partial(\varLambda^{(y)}q)}}{\partial y} := \theta_0 h^{-1}\big[\varLambda^{(y)}q\big]\mathrm{n}_y + \left\{\frac{\partial(\varLambda^{(y)}q)}{\partial y}\right\} + \theta_1 h\left[\frac{\partial^2(\varLambda^{(y)}q)}{\partial y^2}\right]\mathrm{n}_y\Big|_{\partial Q} = \frac{\partial q}{\partial y}\Big|_{\partial Q}, \quad \partial Q \in \mathscr{E}_h^0,\\
&(\varLambda q)\big|_{\partial\Omega} = q\big|_{\partial\Omega}, \quad \{\varLambda q\}\big|_{\partial Q} = q\big|_{\partial Q}, \qquad\qquad \partial Q \in \mathscr{E}_h^0,
\end{aligned} \tag{5.2}$$

where $\varLambda$ is the tensor product of $\varLambda^{(x)}$ and $\varLambda^{(y)}$, and the superscripts indicate the application of the one-dimensional operator $\varLambda^{(x)}$ or $\varLambda^{(y)}$ with respect to the corresponding variable x or y.

Lemma 5.1. *Assume that $q \in H_\mu^{p+1}(\Omega, \partial\Omega)$, then*

$$\sum_Q \big\|\varLambda q - q\big\|^2_{L^2_\mu(Q,\partial Q)} \lesssim h^{2p+2} \sum_Q |q|^2_{H_\mu^{p+1}(Q,\partial Q)}. \tag{5.3}$$

Proof. By the definition of the tensor product, we see that

$$\varLambda q = \varLambda^{(x)}\big(\varLambda^{(y)}q\big) = \varLambda^{(y)}\big(\varLambda^{(x)}q\big).$$

Using the triangular inequality, we get

$$\|\varLambda q - q\|_{L^2_\mu(Q,\partial Q)} \le \big\|\varLambda^{(x)}\big(\varLambda^{(y)}q\big) - \varLambda^{(x)}q\big\|_{L^2_\mu(Q,\partial Q)} + \big\|\varLambda^{(x)}q - q\big\|_{L^2_\mu(Q,\partial Q)}.$$

According to [37, Corollary 7.2], we obtain

$$\begin{aligned}
&\sum_Q \big\|\varLambda^{(x)}q - q\big\|^2_{L^2_\mu(Q,\partial Q)} \lesssim h^{2(p+1)} \sum_Q \int_{\bar{Q}} \big|\partial_x^{p+1}q\big|^2 \mathrm{d}x\mathrm{d}y,\\
&\sum_Q \big\|\varLambda^{(x)}q\big\|^2_{L^2_\mu(Q,\partial Q)} \lesssim \sum_{j=0}^{\min\{p,2\}} h^{2j} \sum_Q \int_{\bar{Q}} \big|\partial_x^j q\big|^2 \,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Thus, for any $0 \le m \le p - j$, we arrive at

$$\begin{aligned}
&\sum_Q \big\|\varLambda^{(x)}\big(\varLambda^{(y)}q\big) - \varLambda^{(x)}q\big\|^2_{L^2_\mu(Q,\partial Q)}\\
\lesssim\ & \sum_{j=0}^{\min\{p,2\}} h^{2j} \sum_Q \int_{\bar{Q}} \big|\partial_x^j\big(\varLambda^{(y)}q - q\big)\big|^2 \mathrm{d}x\mathrm{d}y\\
\lesssim\ & \sum_{j=0}^{\min\{p,2\}} h^{2j} \sum_Q \int_{\bar{Q}} \big|\varLambda^{(y)}\partial_x^j q - \partial_x^j q\big|^2 \mathrm{d}y\mathrm{d}x\\
\lesssim\ & \sum_{j=0}^{\min\{p,2\}} h^{2j} h^{2(m+1)} \sum_Q \int_{\bar{Q}} \big|\partial_y^{m+1}\partial_x^j q\big|^2 \,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Let $m = p - j$, we derive

$$\sum_Q \big\|\varLambda^{(x)}\big(\varLambda^{(y)}q\big) - \varLambda^{(x)}q\big\|^2_{L^2_\mu(Q,\partial Q)} \lesssim h^{2(p+1)} \sum_{j=0}^{\min\{p,2\}} \sum \int_{\bar{Q}} \big|\partial_x^j \partial_y^{p-j+1} q\big|^2 \,\mathrm{d}x\mathrm{d}y.$$

The proof is complete. $\square$

Lemma 5.2 (see [35]). *Assume that the condition* $\mathbf{M}$ *holds, and the function* $w \in C^3(0,T]$ *admits an initial singularity,* $|w'''(t)| \lesssim 1 + t^{\sigma-3}$ *for* $t \in (0,T]$. *Here,* $\sigma \in (0,1)$ *is a real parameter. Then for* $1 \le n \le N$*, the global consistency error can be bounded*

$$\sum_{i=1}^{n} \Xi_{n-i}^{(n)} \big|\Upsilon_t^i[w]\big|_{L^2_\mu(\Omega,\partial\Omega)} \lesssim \frac{1}{\sigma(1-\alpha)} \Delta t^{\min\{\varpi\sigma,2\}}.$$

Lemma 5.3. *For any function* $w(t) \in C^2(0,T]$ *and* $n = 1,\dots,N-1$*, we have*

$$\big|w(t_{n-\psi}) - \big(\psi w(t_{n-1}) + (1-\psi) w(t_n)\big)\big| \lesssim \Delta t_n^2 \max_{t_{n-1}\le t\le t_n} |w''(t)|.$$

Proof. Using Taylor's theorem, for $\eta_1, \eta_2 \in [t_{n-1}, t_n]$, we obtain

$$w(t_n) = w(t_{n-\psi}) + w'(t_{n-\psi})(\psi \Delta t_n) + \frac{\psi^2 w''(\eta_1)}{2} \Delta t_n^2,$$

and

$$w(t_{n-1}) = w(t_{n-\psi}) + w'(t_{n-\psi})(\psi - 1)\Delta t_n + \frac{(\psi-1)^2 w''(\eta_2)}{2} \Delta t_n^2.$$

Furthermore

$$\begin{aligned}
&\big|w(t_{n-\psi}) - \big(\psi w(t_{n-1}) + (1-\psi) w(t_n)\big)\big| \\
=&\left|\frac{(1-\psi)\psi^2 w''(\eta_1)}{2}\Delta t_n^2 + \frac{\psi(\psi-1)^2 w''(\eta_2)}{2}\Delta t_n^2\right| \\
\lesssim& \Delta t_n^2 \max_{t_{n-1}\le t\le t_n} |w''(t)|.
\end{aligned}$$

The proof is complete. □

Theorem 5.1. *Under the regular assumption* (5.1) *and the assumption* $\mathbf{M}$*, if the conditions of Theorem* 4.1 *hold, then the solution* u_h^n *of* (3.6) *is convergent and for* $1 \le n \le N$*, we have*

$$\|u(t_n) - u_h^n\|_{L^2_\mu(\Omega,\partial\Omega)} \lesssim \frac{1}{\sigma(1-\alpha)} \Delta t^{\min\{\varpi\sigma,2\}} + h^{p+1},$$

where $u(t_n) = u(\cdot, t_n)$.

Proof. Let

$$u^{n-\psi} - u_h^{n-\psi} = \big(\Lambda u^{n-\psi} - u_h^{n-\psi}\big) - \big(\Lambda u^{n-\psi} - u^{n-\psi}\big) = \varepsilon^{n-\psi} - \epsilon^{n-\psi}.$$

Then, we can obtain the following expression directly from (3.3):

$$\begin{aligned}
&\Big((\mathscr{D}^\alpha u)^{n-\psi}, v_h\Big)_\Omega + \mu\Big((\mathscr{D}^\alpha u)^{n-\psi}, v_h\Big)_{\partial\Omega} + \mathbf{a}_h\Big(u\big(t_{n-\psi}\big), v_h\Big) \\
&= \big(f(t_{n-\psi}), v_h\big)_\Omega - \Big(\Upsilon_t^{n-\psi}, v_h\Big)_\Omega - \mu\Big(\Upsilon_t^{n-\psi}, v_h\Big)_{\partial\Omega}.
\end{aligned} \tag{5.4}$$

Subtracting (3.6) from (5.4) respectively, we get

$$\begin{aligned}
&\Big((\mathscr{D}^\alpha (u-u_h))^{n-\psi}, v_h\Big)_\Omega + \mu\Big((\mathscr{D}^\alpha (u-u_h))^{n-\psi}, v_h\Big)_{\partial\Omega} + \mathbf{a}_h\Big(u\big(t_{n-\psi}\big) - u_h^{n-\psi}, v_h\Big) \\
&= -\Big(\Upsilon_t^{n-\psi}, v_h\Big)_\Omega - \mu\Big(\Upsilon_t^{n-\psi}, v_h\Big)_{\partial\Omega}.
\end{aligned} \tag{5.5}$$

Furthermore, we derive

$$\begin{aligned}
&\left((\mathscr{D}^{\alpha}\varepsilon)^{n-\psi},v_h\right)_{\Omega}+\mu\left((\mathscr{D}^{\alpha}\varepsilon)^{n-\psi},v_h\right)_{\partial\Omega}\\
&\quad+\mathbf{a}_h\Big(u\big(t_{n-\psi}\big)-u^{n-\psi},v_h\Big)+\mathbf{a}_h(\varepsilon^{n-\psi},v_h)\\
=&\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi},v_h\right)_{\Omega}+\mu\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi},v_h\right)_{\partial\Omega}\\
&\quad+\mathbf{a}_h(\epsilon^{n-\psi},v_h)-\left(\Upsilon_t^{n-\psi},v_h\right)_{\Omega}-\mu\left(\Upsilon_t^{n-\psi},v_h\right)_{\partial\Omega}.
\end{aligned}\tag{5.6}$$

Let $v_h=\varepsilon^{n-\psi}$ in (5.6), we obtain

$$\begin{aligned}
&\left((\mathscr{D}^{\alpha}\varepsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\Omega}+\mu\left((\mathscr{D}^{\alpha}\varepsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}\\
&\quad+\mathbf{a}_h\Big(u\big(t_{n-\psi}\big)-u^{n-\psi},\varepsilon^{n-\psi}\Big)+\mathbf{a}_h\Big(\varepsilon^{n-\psi},\varepsilon^{n-\psi}\Big)\\
=&\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\Omega}+\mu\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}\\
&\quad+\mathbf{a}_h\Big(\epsilon^{n-\psi},\varepsilon^{n-\psi}\Big)-\left(\Upsilon_t^{n-\psi},\varepsilon^{n-\psi}\right)_{\Omega}-\mu\left(\Upsilon_t^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega},
\end{aligned}\tag{5.7}$$

and

$$\begin{aligned}
\mathbf{a}_h\Big(\epsilon^{n-\psi},\varepsilon^{n-\psi}\Big)=&\left(\nabla\epsilon^{n-\psi},\nabla\varepsilon^{n-\psi}\right)_{\Omega}+\varsigma\left(\nabla_{\partial\Omega}\epsilon^{n-\psi},\nabla_{\partial\Omega}\varepsilon^{n-\psi}\right)_{\partial\Omega}\\
&+\lambda\left(\epsilon^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}-\sum_{Q}\int_{\partial Q}\widehat{\nabla\epsilon^{n-\psi}}\cdot\left[\varepsilon^{n-\psi}\right]\cdot\mathbf{n}ds\\
=&-\left(\epsilon^{n-\psi},\Delta\varepsilon^{n-\psi}\right)_{\Omega}+\left(\epsilon^{n-\psi},\frac{\partial\varepsilon^{n-\psi}}{\partial\mathbf{n}}\right)_{\partial\Omega}+\varsigma\left(\nabla_{\partial\Omega}\epsilon^{n-\psi},\nabla_{\partial\Omega}\varepsilon^{n-\psi}\right)_{\partial\Omega}\\
&+\lambda\left(\epsilon^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}-\sum_{Q}\int_{\partial Q}\widehat{\nabla\epsilon^{n-\psi}}\cdot\left[\varepsilon^{n-\psi}\right]\cdot\mathbf{n}ds.
\end{aligned}$$

According to the definitions of projection operators (5.2), we arrive at $\mathbf{a}_h(\epsilon^{n-\psi},\varepsilon^{n-\psi})=0$. Thus, (5.7) can be simplified to

$$\begin{aligned}
&\left((\mathscr{D}^{\alpha}\varepsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\Omega}+\mu\left((\mathscr{D}^{\alpha}\varepsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}\\
&\quad+\mathbf{a}_h\Big(u\big(t_{n-\psi}\big)-u^{n-\psi},\varepsilon^{n-\psi}\Big)+\mathbf{a}_h\Big(\varepsilon^{n-\psi},\varepsilon^{n-\psi}\Big)\\
=&\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\Omega}+\mu\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}+\left(\Upsilon_t^{n-\psi},\varepsilon^{n-\psi}\right)_{\Omega}+\mu\left(\Upsilon_t^{n-\psi},\varepsilon^{n-\psi}\right)_{\partial\Omega}\\
=&\left((\mathscr{D}^{\alpha}\epsilon)^{n-\psi}\big|_{\Omega}+\mu\,(\mathscr{D}^{\alpha}\epsilon)^{n-\psi}\big|_{\partial\Omega}-\Upsilon_t^{n-\psi}\big|_{\Omega}-\mu\Upsilon_t^{n-\psi}\big|_{\partial\Omega},\varepsilon^{n-\psi}\right).
\end{aligned}\tag{5.8}$$

Let

$$\mathcal{R}^{n-\psi}=(\mathscr{D}^{\alpha}\epsilon)\big|_{\Omega}^{n-\psi}+\mu\,(\mathscr{D}^{\alpha}\epsilon)^{n-\psi}\big|_{\partial\Omega}-\Upsilon_t^{n-\psi}\big|_{\Omega}-\mu\Upsilon_t^{n-\psi}\big|_{\partial\Omega}.$$

So, (3.5), (5.1b) and (5.3) give that

$$\begin{aligned}
&\left\|\mathcal{R}^{n-\psi}\right\|\le\left\|(\mathscr{D}^{\alpha}\epsilon)\big|_{\Omega}^{n-\psi}+\mu\,(\mathscr{D}^{\alpha}\epsilon)^{n-\psi}\big|_{\partial\Omega}\right\|+\left\|\Upsilon_t^{n-\psi}\big|_{\Omega}+\mu\Upsilon_t^{n-\psi}\big|_{\partial\Omega}\right\|\\
=&\left\|(D_t^{\alpha}\epsilon)\,(t_{n-\psi})\big|_{\Omega}-\Upsilon_t^{n-\psi}\big|_{\Omega}+\mu\,(D_t^{\alpha}\epsilon)\,(t_{n-\psi})\big|_{\partial\Omega}-\mu\Upsilon_t^{n-\psi}\big|_{\partial\Omega}\right\|+\left\|\Upsilon_t^{n-\psi}\big|_{\Omega}+\mu\Upsilon_t^{n-\psi}\big|_{\partial\Omega}\right\|\\
\le&\left\|(D_t^{\alpha}(\Lambda u-u))\,(t_{n-\psi})\big|_{\Omega}+\mu\,(D_t^{\alpha}(\Lambda u-u))\,(t_{n-\psi})\big|_{\partial\Omega}\right\|+2\left\|\Upsilon_t^{n-\psi}\big|_{\Omega}+\mu\Upsilon_t^{n-\psi}\big|_{\partial\Omega}\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\|\left(\Lambda\left(D_t^\alpha u\right)-D_t^\alpha u\right)\left(t_{n-\psi}\right)\right\|_\Omega+\mu\left\|\left(\Lambda\left(D_t^\alpha u\right)-D_t^\alpha u\right)\left(t_{n-\psi}\right)\right\|_{\partial\Omega}+2\left\|\Upsilon_t^{n-\psi}\right|_\Omega+\mu\Upsilon_t^{n-\psi}\left|_{\partial\Omega}\right\| \\
&\lesssim h^{p+1}\left\|D_t^\alpha u\left(t_{n-\psi}\right)\right\|_{H_\mu^{p+1}(\Omega,\partial\Omega)}+\left\|\Upsilon_t^{n-\psi}\right|_\Omega+\mu\Upsilon_t^{n-\psi}\left|_{\partial\Omega}\right\| \\
&\lesssim h^{p+1}+\left\|\Upsilon_t^{n-\psi}\right\|_\Omega+\left\|\Upsilon_t^{n-\psi}\right\|_{\partial\Omega}.
\end{aligned}
$$

In addition

$$
\begin{aligned}
&\left\|\mathbf{a}_h\Big(u\big(t_{n-\psi}\big)-u^{n-\psi},\varepsilon^{n-\psi}\Big)\right\| \\
=&\bigg\|\left(\nabla\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right),\nabla\varepsilon^{n-\psi}\right)_\Omega+\varsigma\left(\nabla_{\partial\Omega}\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right),\nabla_{\partial\Omega}\varepsilon^{n-\psi}\right)_{\partial\Omega} \\
&\quad+\lambda\left(\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right),\varepsilon^{n-\psi}\right)_{\partial\Omega}-\sum_Q\int_{\partial Q}\nabla\left(\widehat{u\big(t_{n-\psi}\big)}-\widehat{u^{n-\psi}}\right)\cdot\left[\varepsilon^{n-\psi}\right]\cdot\mathbf{n}\mathrm{d}s\bigg\| \\
=&\Bigg\|-\left(\Delta\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right),\varepsilon^{n-\psi}\right)_\Omega+\left(\frac{\partial\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right)}{\partial\mathbf{n}},\varepsilon^{n-\psi}\right)_{\partial\Omega} \\
&\quad-\varsigma\left(\Delta\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right),\varepsilon^{n-\psi}\right)_{\partial\Omega}+\lambda\left(\left(u\big(t_{n-\psi}\big)-u^{n-\psi}\right),\varepsilon^{n-\psi}\right)_{\partial\Omega} \\
&\quad-\sum_Q\int_{\partial Q}\nabla\left(\widehat{u\big(t_{n-\psi}\big)}-\widehat{u^{n-\psi}}\right)\cdot\left[\varepsilon^{n-\psi}\right]\cdot\mathbf{n}\mathrm{d}s\Bigg\|.
\end{aligned}
$$

Then, by using Lemma 5.3, we have

$$
\left\|\mathbf{a}_h\Big(u\big(t_{n-\psi}\big)-u^{n-\psi},\varepsilon^{n-\psi}\Big)\right\|\lesssim\Delta t_n^2\max_{t_{n-1}\le t\le t_n}\left\|u''(t)\right\|_{L_\mu^2(\Omega,\partial\Omega)}\left\|\varepsilon^{n-\psi}\right\|_{L_\mu^2(\Omega,\partial\Omega)}.
$$

Consequently, from Lemma 4.1 and (5.8), we obtain

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^n Z_{n-j}^{(n)}\partial\left\|\varepsilon^j\right\|_{L_\mu^2(\Omega,\partial\Omega)}^2 &\lesssim\left\|\mathcal{R}^{n-\psi}\right\|\left\|\varepsilon^{n-\psi}\right\|_{L_\mu^2(\Omega,\partial\Omega)}+\left\|\mathbf{a}_h\big(u\big(t_{n-\psi}\big)-u^{n-\psi},\varepsilon^{n-\psi}\big)\right\| \\
&\lesssim\left(\left\|\mathcal{R}^{n-\psi}\right\|+\Delta t_n^2\max_{t_{n-1}\le t\le t_n}\left\|u''(t)\right\|_{L_\mu^2(\Omega,\partial\Omega)}\right)\left\|\varepsilon^{n-\psi}\right\|_{L_\mu^2(\Omega,\partial\Omega)}.
\end{aligned}
$$

Proceeding as the proof of Theorem 4.1, we get the desired result, where Lemmas 5.2 and 5.3 together with (5.1a) and (5.1c) are used. □

Remark 5.1. For the proof of Theorem 5.1, we use $u_{h0}=\Lambda u_0$. In fact, we can also choose the L^2-projection of u_0 and still maintain the same accuracy in space. In addition, the present method can reach the optimal convergence order $\mathcal{O}(\Delta t^2)$ in time when $\varpi\ge\max\{1,2/\sigma\}$.

Remark 5.2. In the derivation of error equation (5.5), we utilize $f^{n-\psi}$ and $f_{\partial\Omega}^{n-\psi}$ instead of $f(t_{n-\psi})$ and $f_{\partial\Omega}(t_{n-\psi})$, which leads to additional error terms $f(t_{n-\psi})-f^{n-\psi}$ and $f_{\partial\Omega}(t_{n-\psi})-f_{\partial\Omega}^{n-\psi}$ actually. However, this will not affect the convergence results only if f and $f_{\partial\Omega}$ satisfies the regularity properties in Lemma 5.3.

6. Numerical Experiments

To testify the results in Theorem 5.1, we choose the computational domain $\Omega=(0,1)\times(0,1)$. We obtain the L^2-norm errors, together with spatial convergence order on uniform $M\times M$ ($M\in\mathbb{N}_+$) rectangular meshes and temporal convergence order on graded mesh at $T=0.5$.

Example 6.1. Consider time fractional diffusion problem

$$ {}_0^C D_t^\alpha u(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T] $$

with initial condition

$$ u_0 = (x-0.5)^2(x^2+1)(y-0.5)^2(y^2+1), \quad \boldsymbol{x} \in \bar{\Omega} $$

and boundary condition

$$ {}_0^C D_t^\alpha u(\boldsymbol{x},t) = -2u(\boldsymbol{x},t) + \Delta_{\partial\Omega} u(\boldsymbol{x},t) - \partial_{\mathbf{n}} u(\boldsymbol{x},t) + f_{\partial\Omega}(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T]. $$

The functions f and $f_{\partial\Omega}$ are selected to make the exact solution $u = (1+t^\sigma)(x-0.5)^2(x^2+1)(y-0.5)^2(y^2+1)$. In Tables 6.1 and 6.2, the time step size Δt is small enough to guarantee

Table 6.1: The L^2-norm errors and convergence orders in spatial direction when $p = 1$ for Example 6.1.

α	M	$\Vert u(t_n) - u_h^n\Vert_\Omega$	order	$\Vert u(t_n) - u_h^n\Vert_{\partial\Omega}$	order
0.2	8	9.0620e−02		9.7308e−02	
	16	2.4188e−02	1.91	2.6599e−02	1.87
	32	6.2705e−03	1.95	6.9620e−03	1.93
	64	1.6211e−03	1.95	1.7596e−03	1.98
	128	4.1324e−04	1.97	4.4973e−04	1.97
0.5	8	9.4933e−02		9.2671e−02	
	16	2.6167e−02	1.86	2.5040e−02	1.89
	32	6.9305e−03	1.92	6.4376e−03	1.96
	64	1.7690e−03	1.97	1.5942e−03	2.01
	128	4.4209e−04	2.00	4.0319e−04	1.98
0.8	8	9.2209e−02		8.9514e−02	
	16	2.5108e−02	1.88	2.3980e−02	1.90
	32	6.5942e−03	1.93	6.2358e−03	1.94
	64	1.6885e−03	1.97	1.5800e−03	1.98
	128	4.3514e−04	1.96	4.0399e−04	1.97

Table 6.2: The L^2-norm errors and convergence orders in spatial direction when $p = 2$ for Example 6.1.

α	M	$\Vert u(t_n) - u_h^n\Vert_\Omega$	order	$\Vert u(t_n) - u_h^n\Vert_{\partial\Omega}$	order
0.2	8	5.9201e−03		6.0337e−03	
	16	7.8732e−04	2.91	7.8692e−04	2.94
	32	9.9740e−05	2.98	1.0109e−04	2.96
	64	1.2793e−05	2.96	1.2946e−05	2.97
	128	1.6374e−06	2.97	1.6248e−06	2.99
0.5	8	5.7728e−03		5.2115e−03	
	16	7.5054e−04	2.94	6.9732e−04	2.90
	32	9.6710e−05	2.96	9.0112e−05	2.95
	64	1.2432e−05	2.96	1.1427e−05	2.98
	128	1.5794e−06	2.98	1.4450e−06	2.98
0.8	8	6.6530e−03		6.1702e−03	
	16	9.0692e−04	2.88	8.1981e−04	2.91
	32	1.1726e−04	2.95	1.0458e−04	2.97
	64	1.4618e−05	3.00	1.3205e−05	2.99
	128	1.8490e−06	2.98	1.6773e−06	2.98

Table 6.3: The L^2-norm errors and convergence orders in temporal direction when $\alpha = 0.4, \sigma = 0.8$ for Example 6.1.

	ϖ	N	$\Vert u(t_n) - u_h^n \Vert_\Omega$	order	$\Vert u(t_n) - u_h^n \Vert_{\partial\Omega}$	order
		32	2.8027e−02		3.6682e−02	
		64	7.4332e−03	1.91	9.9487e−03	1.88
	2	128	2.0369e−03	1.87	2.6895e−03	1.89
		256	6.0016e−04	1.76	7.8371e−04	1.78
$\min\{\varpi\sigma, 2\}$				1.60		1.60
		32	4.1608e−02		3.8211e−02	
		64	9.7181e−03	2.10	9.3710e−03	2.03
	$5/2$ ($\varpi_{\rm opt}$)	128	2.3863e−03	2.03	2.3572e−03	1.99
		256	5.8285e−04	2.03	5.9863e−04	1.98
$\min\{\varpi\sigma, 2\}$				2.00		2.00
		32	3.3905e−02		2.9972e−02	
		64	8.0124e−03	2.08	7.2189e−03	2.05
	3	128	1.9186e−03	2.06	1.7968e−03	2.01
		256	4.7629e−04	2.01	4.4057e−04	2.03
$\min\{\varpi\sigma, 2\}$				2.00		2.00

the spatial error is dominant. For $\alpha = 0.2$, 0.5 and 0.8, we report the computational errors and convergence orders for $p = 1$ and $p = 2$ in space respectively, which imply the spatial convergence order is $\mathcal{O}(h^{p+1})$. In Table 6.3, it is noticed that the convergence order is $\mathcal{O}(\Delta t^{\min\{\varpi\sigma,2\}})$ in time, and the optimal second-order accuracy is gained if $\varpi \geq \varpi_{\rm opt}$. Here, $\varpi_{\rm opt}$ is the optimal mesh parameter, and sufficiently refined spatial meshes are used. The numerical results are compatible with the conclusion of Theorem 5.1.

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