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A NEWTON-TYPE GLOBALLY CONVERGENT INTERIOR-POINT METHOD TO SOLVE MULTI-OBJECTIVE OPTIMIZATION PROBLEMS*

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Abstract

This paper proposes an interior-point technique for detecting the nondominated points of multi-objective optimization problems using the direction-based cone method. Cone method decomposes the multi-objective optimization problems into a set of single-objective optimization problems. For this set of problems, parametric perturbed KKT conditions are derived. Subsequently, an interior point technique is developed to solve the parametric perturbed KKT conditions. A differentiable merit function is also proposed whose stationary point satisfies the KKT conditions. Under some mild assumptions, the proposed algorithm is shown to be globally convergent. Numerical results of unconstrained and constrained multi-objective optimization test problems are presented. Also, three performance metrics (modified generational distance, hypervolume, inverted generational distance) are used on some test problems to investigate the efficiency of the proposed algorithm. We also compare the results of the proposed algorithm with the results of some other existing popular methods.

Mathematics subject classification: 65N06, 65B99.
Key words: Cone method, Interior point method, Merit function, Newton method, Global convergence.

1. Introduction

The vast majority of practical optimization problems [2,16,31,33] consist of multi-objective problems. Applications of multi-objective optimization problems can be found in a various fields, including engineering design [2], optimal control systems [31], chemical engineering [33], machine learning [16], etc. Therefore, identification and characterization of the solutions to MOPs have become a very important task. MOPs consider to optimize several conflicting objectives simultaneously. Therefore, most often, a single solution that performs well for each objective functions does not exist. In solving MOP problems, sometimes decision makers come up with a compromise solution by analyzing a set of points that are representative of the entire Pareto set [29]. A feasible point is called Pareto optimal (non-dominated point) if no objective

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can be improved without sacrificing at least one other objective. When solving an MOP, the goal is to identify all possible Pareto optimal solutions.

MOPs have been solved through several techniques [23] over the last few years. The reputed classical methods such as weighted sum [19, 24], ϵ-constraint [18], physical programming [26], normal boundary intersection [5], etc., are known to find the Pareto optimal solutions. However, these methods either not able to yield a complete Pareto front or require some prior information regarding its location. Recently, a cone method [12] has been established that can generate all Pareto solutions, and no knowledge about the position of the Pareto front is required.

The formulation of cone method is found to be similar to the Pascoletti-Serafini [30] technique for vector optimization. Cone method [12] has the ability to generate both convex and nonconvex parts of the Pareto front. The formulation of the cone method (see Section 2) transforms the multi-objective optimization problem into a set of direction-based-parametric single objective problem. Although the formulation of cone method is detailed in [12], how to effectively solve the formulated direction-based parametric subproblems is not given therein. In this paper, we concern towards this direction and attempt to apply interior-point method to solve the subproblems.

In 1955, foundation of the interior-point approach was laid introduced by Frisk [11]. Subsequently, Fiacco and McCormick [10] reformulate the problem $\min\{f(x) : c(x) = 0, x \geq 0, c(x) \in \mathbb{R}^m, x \in \mathbb{R}^n\}$ as an unconstrained minimization problem and proved the global convergence of interior-point method. In 1960's, one type of interior point methods (classical log-barrier method) was used broadly. In 1970's, it was proven [27] that the Hessians for barrier methods are ill-conditioned near the optima. Therefore, despite fair finding of the log-barrier method, other methods became primary topics for research.

In 1984, Karmarkar [17] published an algorithm that solves linear programming problems in polynomial time. This was a huge improvement over existing simplex method, which solved linear programming problems in the worst-case by exponential time. In a very short time, it was found that Karmarkar's algorithm was equivalent to the log-barrier method [13]. Thereafter, the interest in interior-point methods resurged.

In 1989, Megiddo [25] first presented an interior-point method, which simultaneously solves the primal and dual problems and describes the properties of primal-dual central path for linear programming. Thereafter, a primal-dual barrier method to solve linear programming problems was implemented in [20]. To solve linear programming problems and quadratic programming problems, widely used method was barrier methods [32, 41].

As a result of the popularity of interior-point methods for linear and quadratic programming studies on their use for nonlinear optimization continue till today. The proposed method exploits the efficiencies of the cone method [12] and interior-point method. In this work, we introduce a novel differentiable merit function that helps to decide the convergence of the proposed algorithm towards the solution. The stationary points of this merit function satisfy the perturbed KKT conditions. Further, a Newton-type method is applied to solve KKT conditions. We also present the global convergence results of the proposed method.

This paper is structured as follows. In Section 2, we provide the required terminologies and notations, and briefly explain the cone method. In Section 3, we formulate an interior-point method for a nonlinear problem, which is formulated in Section 2, and find the search direction formulas. In Section 4, a merit function and its properties are presented. In Section 5, we show that the proposed algorithm is globally convergent. Section 5, refers to numerical results of the proposed method. Finally, Section 6 ends with a few concluding remarks.

2. Preliminaries and Terminologies

The following is a multi-objective optimization problem (MOP):

$$\text{(MOP)}\begin{cases}\text{minimize} & F(x)\\ \text{subject to} & x \in \mathcal{X},\end{cases} \tag{2.1}$$

where $F(x) = (f_1(x), \ldots, f_p(x))^\top$, $p \geq 2$ is a vector-valued function (multiobjective function), and $\mathcal{X} = \{x \in \mathbb{R}^n : \theta_i(x) \geq 0,\ x_j \geq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$. Each component function of $F(x)$ and each $\theta_i(x)$, $i = 1, \ldots, m$, are twice continuously differentiable.

We call the vector $x = (x_1, \ldots, x_n)^\top$ as the vector of decision variables and the set $\mathcal{X}$ as the decision feasible region. We denote the image of the decision feasible region $\mathcal{X}$ under the vector-valued objective function F by $\mathcal{Y}$, i.e., $\mathcal{Y} = F(\mathcal{X}) = \{(f_1(x), \ldots, f_p(x))^\top : x \in \mathcal{X}\}$. The set $\mathcal{Y}$ is referred to the objective feasible region. Note that $\mathcal{X}$ is a subset of $\mathbb{R}^n$ and $\mathcal{Y}$ is a subset of $\mathbb{R}^p$.

Due to the conflicting nature of the objective functions $f_1, f_2, \ldots, f_p$, and the nonexistence of a linear ordering in $\mathbb{R}^p$, the optimality concept for an MOP gets differed from that of conventional single objective optimization problems. The notion of optimality for an MOP is Pareto optimality [29]. The definition of Pareto optimality is based on a dominance structure on $\mathbb{R}^p$. For the required dominance relation for Pareto optimality, we use the following notations:

- $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p : y \geqq 0\}$ is referred to the non-negative orthant of $\mathbb{R}^p$, where for a $y = (y_1, y_2, \ldots, y_p)^\top \in \mathbb{R}^p$, $y \geqq 0$ represents $y_i \geq 0$ for all $i = 1, \ldots, p$.
- $\mathbb{R}^p_{\geq} = \{y \in \mathbb{R}^p : y \geq 0\}$, where $y \geq 0$ denotes $y \geqq 0$ but $y \neq 0$.
- $\mathbb{R}^p_{>} = \{y \in \mathbb{R}^p : y > 0\}$ represents the interior of $\mathbb{R}^p_{\geqq}$, where $y > 0$ indicates that $y_i > 0$ for all $i = 1, \ldots, p$.
- The relations $\leqq, \leq$ and $<$ can also be defined in a similar way.
- For two vectors $y_1, y_2 \in \mathbb{R}^p$, we say that the vector y_1 dominates y_2 (in the sense of minimization) if $y_1 \leq y_2$.

Throughout the paper, we also use the following notations. For an $x \in \mathbb{R}^n$,

- $\|x\| = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$.
- Given a vector $x \in \mathbb{R}^n$, we write $X = \operatorname{diag}(x)$, e represents a column vector whose all components are one, and denote $x^{-1} = X^{-1}e$.
- $d(x, P) = \min\{\|x - y\| : y \in P\}$.

Definition 2.1 (Pareto optimality [7]). *A point $\hat{x} \in \mathcal{X}$ is called efficient or Pareto optimal if there is no other $x \in \mathcal{X}$ such that $F(x) \leq F(\hat{x})$. If $\hat{x}$ is efficient, $F(\hat{x})$ is called a nondominated point.*

The collection of all efficient points of the MOP (2.1) is denoted by $\mathcal{X}_E$. The set of all nondominated points, also known as Pareto front is represented by $\mathcal{Y}_N$. Evidently, $\mathcal{Y}_N = F(\mathcal{X}_E)$. Generally, Pareto front contains a large number of points. Therefore, it is estimated by a sufficiently large and diverse set of points on the Pareto front.

The next section briefly describes the idea of cone method [12] to generate the Pareto optimal points of an MOP.

2.1. Description of the Cone method

As described in [12], firstly, it determines the ideal point $F^* = (f_1^*, \ldots, f_p^*)^\top$, where $f_i^* = \min\{f_i(x) : x \in \mathcal{X}\}$. Then, by solving the following minimization problem corresponding to a particular $\hat{\beta} \in \mathbb{S}^{k-1}_{\geqq} = \mathbb{S}^{k-1} \cap \mathbb{R}^k_{\geqq}$ (where $\mathbb{S}^{k-1}$ represents the unit sphere in $\mathbb{R}^k$), the solution of the MOP (2.1) can be obtained

$$\mathrm{CM}(\hat{\beta}) \quad \left\{\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & t\hat{\beta} \geqq F(x) - F^*, \\ & \theta_i(x) \geq 0,\ i = 1, \ldots, m, \\ & x \geqq 0,\ t \geq 0. \end{array}\right\} \tag{2.2}$$

Note that the problem (2.2) is a single-objective parametric problem with parameter $\hat{\beta}$. To generate the nondominated set, one needs to solve the problem (2.2) for several values of $\hat{\beta}$ in $\mathbb{S}^{p-1}_{\geqq}$. For generating a set of spreaded nondominated points, [12] suggested to take the expression $\hat{\beta}$ as follows:

$$\left(\cos\phi_1, \cos\phi_2 \sin\phi_1, \cos\phi_3 \sin\phi_2 \sin\phi_1, \ldots, \cos\phi_{p-1} \prod_{i=1}^{p-2} \sin\phi_i, \prod_{i=1}^{p-1} \sin\phi_i\right), \tag{2.3}$$

where $0 \leq \phi_i \leq \frac{\pi}{2}, i = 1, \ldots, p-1$.

The next section formulates the Newton scheme for an interior-point method (IPM) to solve the parametric problem $\mathrm{CM}(\hat{\beta})$.

3. Interior-Point Method

In this section, an IPM is discussed to solve $\mathrm{CM}(\hat{\beta})$ for each given $\beta \in \mathbb{S}^{k-1}$. In the sequel, $\mathrm{CM}(\hat{\beta})$ is formulated into barrier problem and then Karush-Kuhn-Tucker (KKT) conditions are derived. Thereafter, IPM takes the advantage of Newton method to solve the system of KKT.

Introducing the vectors $\mathrm{x} = (x_1, \ldots, x_n, t)^\top$, $c = (0, \ldots, 0, 1)^\top$, $\hat{\beta} = (\beta_1, \ldots, \beta_p)^\top$, and denoting $f_j(\mathrm{x}) = f_j(x)$, $j = 1, \ldots, p$ and $f(\mathrm{x}) = F(\mathrm{x}) - F^*$, $\mathrm{CM}(\hat{\beta})$ reduces to the following problem:

$$\left\{\begin{array}{ll} \text{minimize} & c^\top \mathrm{x} \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - f(\mathrm{x}) - v = 0, \\ & \theta(\mathrm{x}) - w = 0, \\ & \mathrm{x} \geqq 0,\ v \geqq 0,\ w \geqq 0, \end{array}\right. \tag{3.1}$$

where $v = (v_1, \ldots, v_p)^\top$, $w = (w_1, \ldots, w_m)^\top$ and $\theta(\mathrm{x}) = (\theta_1(x), \ldots, \theta_m(x))^\top$.

In problem (3.1), the exclusion of the non-negative vectors x, v and w are achieved by setting them within a barrier function as follows:

$$\begin{cases} \text{minimize} & b\,(\mathrm{x}, v, w, \mu) \\ \text{subject to} & \hat{\beta}c^\top \mathrm{x} - f(\mathrm{x}) - v = 0, \\ & \theta(\mathrm{x}) - w = 0, \end{cases} \tag{3.2}$$

where

$$b\,(\mathrm{x}, v, w, \mu) = c^\top \mathrm{x} - \mu \left(\sum_{j=1}^{n+1} \log(\mathrm{x}_j) + \sum_{i=1}^{p} \log(v_i) + \sum_{l=1}^{m} \log(w_l) \right)$$

and $\mu > 0$ is the barrier parameter.

In this paper, our main focus will be on solving the following first order perturbed KKT conditions. For $\mu > 0$ and $(\mathrm{x}, v, w, s, y, z) > 0$

$$\begin{bmatrix} c - s - \nabla_{\mathrm{x}} \left(\hat{\beta}c^\top \mathrm{x} - f(\mathrm{x}) \right)^\top y - (\nabla_{\mathrm{x}}\theta(\mathrm{x}))^\top z \\ -\mu e + VYe \\ -\mu e + WZe \\ -\mu e + S\mathrm{X}e \\ -f(\mathrm{x}) + \hat{\beta}c^\top \mathrm{x} - v \\ \theta(\mathrm{x}) - w \end{bmatrix} = 0, \tag{3.3}$$

where $s = \mu \mathrm{X}^{-1}e,\ \mathrm{X} = \mathrm{diag}(\mathrm{x}_1, \ldots, \mathrm{x}_n, t),\ V = \mathrm{diag}(v_1, \ldots, v_p),\ W = \mathrm{diag}(w_1, \ldots, w_m),\ Y = \mathrm{diag}(y_1, \ldots, y_p),\ Z = \mathrm{diag}(z_1, \ldots, z_m)$ and $S = \mathrm{diag}(s_1, \ldots, s_n, s_{n+1})$. For the rest of the paper, we denote the matrix in the left-hand side of (6) by $\mathcal{D}_\mu(\mathrm{x}, v, w, s, y, z)$.

The KKT system (3.3) need not an ill-conditioned system of equations [9], and also the vector $(\mathrm{x}, v, w, s, y, z)$ keep away from the zero at every iteration. For any $\mu > 0$, we call $\Lambda_\mu = (\mathrm{x}_\mu, v_\mu, w_\mu, s_\mu, y_\mu, z_\mu)$ a perturbed KKT point if it satisfies the perturbed KKT conditions (3.3). Clearly, for $\mu = 0$, the perturbed KKT system is the KKT system corresponding to the problem (3.1).

Definition 3.1. *(i) A point $\Lambda = (\mathrm{x}, v, w, s, y, z)$ is said to be an interior point for the barrier problem* (3.2) *if*

$$(\mathrm{x}, v, w, s, y, z) > 0.$$

(ii) A point $\Lambda = (\mathrm{x}, v, w, s, y, z)$ is said to be a quasi-central point for the problem (3.2) *if it satisfies the following conditions for any $\mu > 0$:*

$$\begin{cases} -\mu e + VYe = 0, \\ -\mu e + WZe = 0, \\ -\mu e + S\mathrm{X}e = 0, \\ -f(\mathrm{x}) + \hat{\beta}c^\top \mathrm{x} - v = 0, \\ \ \theta(\mathrm{x}) - w = 0. \end{cases} \tag{3.4}$$

The set of all points that satisfy (3.4) is called quasi-central path.

We simplified the expressions below by using the following notations:

$$A_{\hat{\beta}}(\mathrm{x}) = \nabla_{\mathrm{x}}\big(\hat{\beta}c^\top \mathrm{x} - f(\mathrm{x})\big), \quad B(\mathrm{x}) = \nabla_{\mathrm{x}}\theta(\mathrm{x}),$$

$$H(\mathrm{x},y,z)=\sum_{j=1}^{s} y_j\nabla^2 f_j(\mathrm{x})-\sum_{i=1}^{m+1} z_i\nabla^2\theta_i(\mathrm{x}),\quad y\geqq 0,\quad z\geqq 0. \tag{3.5}$$

For a fixed $\mu>0$, the Newton step $\Delta\Lambda=(\Delta\mathrm{x},\Delta v,\Delta w,\Delta s,\ \Delta y,\Delta z)$ at the interior-point $\Lambda=(\mathrm{x},v,w,s,y,z)$ is obtained by solving the following system:

$$\bar{\mathfrak{D}}_\mu(\Lambda)\Delta\Lambda=-q_{\hat\beta}(\Lambda,\mu), \tag{3.6}$$

where

$$\bar{\mathfrak{D}}_\mu(\Lambda)=\begin{bmatrix} H(\mathrm{x},y,z) & 0 & 0 & -I & -(A_{\hat\beta}(\mathrm{x}))^\top & -(B(\mathrm{x}))^\top\\ 0 & Y & 0 & 0 & V & 0\\ 0 & 0 & Z & 0 & 0 & W\\ S & 0 & 0 & X & 0 & 0\\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & 0 & 0 & 0\\ B(\mathrm{x}) & 0 & -I & 0 & 0 & 0\end{bmatrix},$$

$$q_{\hat\beta}(\Lambda,\mu)=\begin{bmatrix} c-s-(A_{\hat\beta}(\mathrm{x}))^\top y-(B(\mathrm{x}))^\top z\\ -\mu e+VYe\\ -\mu e+WZe\\ -\mu e+S\mathrm{X}e\\ \hat\beta c^\top\mathrm{x}-f(\mathrm{x})-v\\ \theta(\mathrm{x})-w\end{bmatrix}.$$

The matrix $\bar{\mathfrak{D}}_\mu(\Lambda)$ is not symmetric. However, it can be made symmetric by multiplying the first row by -1, the second row by $-V^{-1}$, the third row by $-W^{-1}$ and the forth row by S^{-1}. Accordingly, we get

$$\begin{bmatrix} -H(\mathrm{x},y,z) & 0 & 0 & I & (A_{\hat\beta}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top\\ 0 & -V^{-1}Y & 0 & 0 & -I & 0\\ 0 & 0 & -W^{-1}Z & 0 & 0 & -I\\ I & 0 & 0 & S^{-1}X & 0 & 0\\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & 0 & 0 & 0\\ B(\mathrm{x}) & 0 & -I & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\\ \Delta s\\ \Delta y\\ \Delta z\end{bmatrix}=\begin{bmatrix}\sigma_{\hat\beta}\\ -\gamma_1\\ -\gamma_2\\ \gamma_3\\ \varrho_{\hat\beta}\\ \rho\end{bmatrix}, \tag{3.7}$$

where

$$\begin{aligned}
&\sigma_{\hat\beta}(\mathrm{x},s,y,z)=c-s-(A_{\hat\beta}(\mathrm{x}))^\top y-(B(\mathrm{x}))^\top z,\\
&\gamma_1(v,y)=\mu V^{-1}e-y,\\
&\gamma_2(w,z)=\mu W^{-1}e-z,\\
&\gamma_3(x,s)=\mu S^{-1}e-\mathrm{x},\\
&\varrho_{\hat\beta}(\mathrm{x},v)=f(\mathrm{x})+v-\hat\beta c^\top\mathrm{x},\\
&\rho(\mathrm{x},w)=w-\theta(\mathrm{x}).
\end{aligned} \tag{3.8}$$

Note that $\varrho_{\hat\beta}$ and ρ together find *primal infeasibility* and $\sigma_{\hat\beta}$ gives *dual infeasibility.* If $\rho_{\hat\beta}$ and ρ vanish at a point, then the point is primal feasible. Moreover, let

$$\nu(\Lambda;\mu)=\max\{\|\rho\|,\|\rho_{\hat\beta}\|,\|\sigma_{\hat\beta}\|,\|\gamma_1\|,\|\gamma_2\|,\|\gamma_3\|\}. \tag{3.9}$$

For a given $\epsilon > 0$, we define the approximated perturbed KKT point as an interior point Λ that satisfies $\nu(\Lambda;\mu) \le \epsilon$. Note that $\nu(\Lambda;\mu) = 0$ and Λ is interior point if and only if Λ is a perturbed KKT point. Also, $\nu(\Lambda;0) = 0$ and $\Lambda \ge 0$ if and only if Λ is a KKT point of the problem (3.1).

We note that second and third equations of (3.7) can be used to eliminate Δv and Δw without producing any off-diagonal fill-in in the remaining system with the help of the following equations:

$$\begin{cases} \Delta v = VY^{-1}(\gamma_1 - \Delta y), \\ \Delta w = WZ^{-1}(\gamma_2 - \Delta z), \\ \Delta s = S\mathrm{X}^{-1}\left(\gamma_3 - \Delta \mathrm{x}\right). \end{cases} \tag{3.10}$$

Accordingly, from (3.7), the resulting *reduced KKT system* is given by

$$\begin{bmatrix} -\left(H(\mathrm{x},y,z) + S\mathrm{X}^{-1}\right) & (A_{\hat\beta}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ A_{\hat\beta}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix} \begin{bmatrix} \Delta \mathrm{x} \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat\beta} - S\mathrm{X}^{-1}\gamma_3 \\ \varrho_{\hat\beta} + VY^{-1}\gamma_1 \\ \rho + WZ^{-1}\gamma_2 \end{bmatrix}. \tag{3.11}$$

The system (3.11) has a unique solution, provided the matrix $H(\mathrm{x},y,z) + S\mathrm{X}^{-1}$ is nonsingular (see [40]). The solution of the system (3.11) provides $\Delta\mathrm{x}, \Delta y$ and Δz. Then, by using (3.10), one can obtain $\Delta v, \Delta w$ and Δs. The following theorem gives explicit formulas of the solution to system (3.11).

Theorem 3.1. *We denote*

$$N_{\hat\beta}(\Lambda) = H(\mathrm{x},y,z) + S\mathrm{X}^{-1} + (A_{\hat\beta}(\mathrm{x}))^\top V^{-1}YA_{\hat\beta}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1}ZB(\mathrm{x}).$$

If at a point $\Lambda = (\mathrm{x}, v, w, s, y, z)$, $N_{\hat\beta}(\Lambda)$ is nonsingular, then the system (3.7) has a unique solution. In particular,

$$\begin{aligned} \Delta\mathrm{x} &= N_{\hat\beta}^{-1}\left(-\sigma_{\hat\beta} + S\mathrm{X}^{-1}\gamma_3 + (A_{\hat\beta}(\mathrm{x}))^\top(\gamma_1 + V^{-1}Y\varrho_{\hat\beta}) + (B(\mathrm{x}))^\top(\gamma_2 + W^{-1}Z\rho)\right), \\ \Delta v &= A_{\hat\beta}(\mathrm{x})\Delta\mathrm{x} - \varrho_{\hat\beta}, \\ \Delta w &= B(\mathrm{x})\Delta\mathrm{x} - \rho. \end{aligned} \tag{3.12}$$

Proof. By solving the second and the third equations of (3.11) for Δy and Δz, we obtain

$$\begin{aligned} \Delta y &= V^{-1}Y\varrho_{\hat\beta} + \gamma_1 - V^{-1}YA_{\hat\beta}(\mathrm{x})\Delta\mathrm{x}, \\ \Delta z &= W^{-1}Z\rho + \gamma_2 - W^{-1}ZB(\mathrm{x})\Delta\mathrm{x}. \end{aligned}$$

Eliminating Δy and Δz from the first block of the system (3.11), we get

$$\Delta\mathrm{x} = N_{\hat\beta}^{-1}\left(-\sigma_{\hat\beta} + S\mathrm{X}^{-1}\gamma_3 + (A_{\hat\beta}(\mathrm{x}))^\top(\gamma_1 + V^{-1}Y\varrho_{\hat\beta}) + (B(\mathrm{x}))^\top(\gamma_2 + W^{-1}Z\rho)\right).$$

We notice that the square matrix of order $(n+m+p+1)\times(n+m+p+1)$ in the left side of the system (3.11) is quasi-definite, and therefore is nonsingular in nature (see [40])). Hence, we can compute Δv and Δw uniquely as follows:

$$\Delta v = A_{\hat{\beta}}(\mathrm{x})\Delta \mathrm{x} - \varrho_{\hat{\beta}}, \quad \Delta w = B(\mathrm{x})\Delta \mathrm{x} - \rho.$$

The proof is complete. □

Note: Let $\Lambda = (\mathrm{x}, v, w, s, y, z)$ be the current point of iteration and the matrix $N_{\hat{\beta}}(\Lambda)$ is nonsingular. Then, Theorem 3.1 provides the solution of the system (3.11). If at any iteration the matrix $N_{\hat{\beta}}(\Lambda)$ is singular, we replace the matrix $H(\mathrm{x}, y, z)$ by $\hat{H}(\mathrm{x}, y, z)$ to make the matrix $N_{\hat{\beta}}(\Lambda)$ nonsingular, where

$$\hat{H}(\mathrm{x}, y, z) = H(\mathrm{x}, y, z) + \lambda I$$

and $\lambda > 0$ is chosen such that the matrix $\hat{H}(\mathrm{x}, y, z)$ is positive definite, I being the identity matrix of the order of $\hat{H}$.

To find a solution of (3.3), the algorithm that we propose below proceeds from an initial point $(\mathrm{x}^{(0)}, v^{(0)}, w^{(0)}, s^{(0)}, y^{(0)}, z^{(0)})$; then, at the k-th iteration, it determines a search direction $(\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)}, \Delta s^{(k)}, \Delta y^{(k)}, \Delta z^{(k)})$ with the help of Theorem 3.1 at $(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, s^{(k)}, y^{(k)}, z^{(k)})$; lastly, it chooses a step length $\alpha^{(k)}$, and then finds the next iterate by

$$\begin{cases} \mathrm{x}^{(k+1)} = \mathrm{x}^{(k)} + \alpha^{(k)}\Delta\mathrm{x}^{(k)}, \\ v^{(k+1)} = v^{(k)} + \alpha^{(k)}\Delta v^{(k)}, \\ w^{(k+1)} = w^{(k)} + \alpha^{(k)}\Delta w^{(k)}, \\ s^{(k+1)} = s^{(k)} + \alpha^{(k)}\Delta s^{(k)}, \\ y^{(k+1)} = y^{(k)} + \alpha^{(k)}\Delta y^{(k)}, \\ z^{(k+1)} = z^{(k)} + \alpha^{(k)}\Delta z^{(k)}, \end{cases} \tag{3.13}$$

where the choice of $\alpha^{(k)}$ is detailed in the next section.

As described in (3.13), to move from current to the next iterate, the proposed algorithm first determines the search direction and then the step length. To make the barrier function in (3.2) well defined across the iterates, the step length is suitably chosen. Also, the identification of every iteration towards the solution of the KKT system (3.3) is discussed in the upcoming section.

4. Computation of Step Size

In this section, we discuss the computation of step length to be taken along the search directions that are determined in Theorem 3.1. The proposed algorithm updates the iteration point by (3.13). To guarantee that the successive points $\mathrm{x}_+ = \mathrm{x} + \alpha\Delta\mathrm{x},\ v_+ = v + \alpha\Delta v,\ w_+ = w + \alpha\Delta w,\ s_+ = s + \alpha\Delta s,\ y_+ = y + \alpha\Delta y$ are interior points, we choose the step length α by the following standard ratio formula (see [3]):

$$\alpha = \min\left\{\delta\left(\max_{i,j,l}\left\{-\frac{\Delta\mathrm{x}_l}{\mathrm{x}_l}, -\frac{\Delta v_j}{v_j}, -\frac{\Delta y_j}{y_j}, -\frac{\Delta w_i}{w_i}, -\frac{\Delta s_l}{s_l}, -\frac{\Delta z_i}{z_i}\right\}\right)^{-1}, 1\right\} \tag{4.1}$$

for some $0 < \delta \leq 1$.

Although the step length calculated by (4.1) ensures that the vectors x, v, w and s remain in the interior, there is no guarantee of the reduction in the objective function and convergence of the generated sequence to a minimum point. This can be seen by taking the unconstrained optimization problem $f(x) = (1 + x^2)^{1/2}$ with an initial $|x_0| > 1$. Merit functions shorten the interval $[0, \alpha]$ such that an appropriate reduction towards optimality can be made along the search direction.

4.1. Merit function

To solve nonlinear constrained optimization problems, IPMs simultaneously minimizes both of objective function and an infeasibility measure. Therefore, the progress in the proposed IPM towards optimality is measured by a merit function that incorporates both the objective function and the infeasibility terms. In this article, we introduce a merit function that is corresponding to the barrier problem (3.2).

For any $\mu > 0$, consider the merit function $\mathcal{M}_{\eta,\mu} : \mathbb{R}^{(n+1)+p+m+(n+1)+p+m} \to \mathbb{R}$, defined by

$$\mathcal{M}_{\eta,\mu}(\Lambda) = c^\top \mathrm{x} + \varrho_{\hat{\beta}}^\top y + \rho^\top z - s^\top \mathrm{x} + \eta \Psi_\mu(\Lambda), \tag{4.2}$$

where η is the nonnegative penalty parameter, and

$$\Psi_\mu(\Lambda) = \frac{1}{2}\left(\varrho_{\hat{\beta}}^\top \rho_{\hat{\beta}} + \rho^\top \rho\right) + \mathrm{x}^\top s + v^\top y + w^\top z - \mu \left(\sum_{i=1}^{n+1} \log(\mathrm{x}_i s_i) + \sum_{j=1}^{p} \log(v_j y_j) + \sum_{l=1}^{m} \log(w_l z_l) \right)$$

is the penalty term of the merit function $\mathcal{M}_{\eta,\mu}$. Clearly, the penalty term is well-defined for $\Lambda > 0$. If the penalty parameter $\eta > 0$ is big and we attempt to minimize the merit function $\mathcal{M}_{\eta,\mu}$, then a lot of computaional effort will be concentrated on making the penalty term Ψ_μ towards zero.

The next two theorems address the global property of the merit function.

Theorem 4.1. *Consider the barrier problem* (3.2)*. Let the interior point* $\Lambda_\mu^* = (\mathrm{x}_\mu^*, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$ *be such that* $\mathcal{D}_\mu(\Lambda_\mu^*) = 0$ *for* $\mu > 0$*. Then, for any* $\eta > 0$*, the point* x_μ^* *is the stationary point of* $\mathcal{M}_{\eta,\mu}(\mathrm{x}, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$*.*

Proof. The gradient of the merit function $\mathcal{M}_{\eta,\mu}$ at point $(\mathrm{x}, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$ is

$$\begin{aligned}
&\nabla_{\mathrm{x}} \mathcal{M}_{\eta,\mu}(\mathrm{x}, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)\\
&= c + \left(\nabla_{\mathrm{x}} \varrho_{\hat{\beta}}(\mathrm{x}, v_\mu^*)\right)^\top y_\mu^* + \left(\nabla_{\mathrm{x}} \rho(\mathrm{x}, w_\mu^*)\right)^\top z_\mu^* - s_\mu^*\\
&\quad + \eta \left[\left(\varrho_{\hat{\beta}}(\mathrm{x}, v_\mu^*)\right)^\top \nabla_{\mathrm{x}} \varrho_{\hat{\beta}}(\mathrm{x}, v_\mu^*) + \left(\rho(\mathrm{x}, w_\mu^*)\right)^\top \nabla_{\mathrm{x}} \rho(\mathrm{x}, w_\mu^*) + s_\mu^* - \mu \mathrm{X}^{-1} e \right].
\end{aligned}$$

For any $\mu > 0$, at the interior point Λ_μ^*, the primal infeasibilities $\varrho_{\hat{\beta}}(\mathrm{x}_\mu^*, v_\mu^*)$ and $\rho(\mathrm{x}_\mu^*, w_\mu^*)$ vanish. Therefore, after replacing x by x_μ^* in the last expression, we get

$$\begin{aligned}
\nabla_{\mathrm{x}} \mathcal{M}_{\eta,\mu}(\Lambda_\mu^*) &= c - s_\mu^* - \left(A_{\hat{\beta}}(\mathrm{x}_\mu^*)\right)^\top y_\mu^* - \left(B(\mathrm{x}_\mu^*)\right)^\top z_\mu^* + \eta \left[s_\mu^* - \mu (\mathrm{X}_\mu^*)^{-1} e\right]\\
&= \sigma_{\hat{\beta}}(\mathrm{x}_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*) + \eta \left[s_\mu^* - \mu (\mathrm{X}_\mu^*)^{-1} e\right] = 0,
\end{aligned}$$

because the dual infeasibility $\sigma_{\hat{\beta}}$ and the complementarity terms vanish at Λ_μ^* (see (3.3)). $\square$

Theorem 4.2. *Consider the barrier problem* (3.2). *Let the interior point* $\Lambda_\mu^* = (\mathrm{x}_\mu^*, v_\mu^*, w_\mu^*, y_\mu^*, z_\mu^*)$ *be such that* $\mathcal{D}_\mu(\Lambda_\mu^*) = 0$ *for* $\mu > 0$. *Then, there exists* $\bar{\eta} > 0$ *such that for* $\eta \geq \bar{\eta}$, *the Hessian of the merit function* $\nabla_x^2 \mathcal{M}_{\eta,\mu}(\Lambda_\mu^*)$ *is positive definite.*

Proof. The Hessian of the merit function $\mathcal{M}_{\eta,\mu}$ is

$$\begin{aligned}
\nabla_{\mathrm{x}}^2 \mathcal{M}_{\eta,\mu}(\Lambda_\mu^*) &= H(\mathrm{x}_\mu^*, y_\mu^*, z_\mu^*) \\
&\quad + \eta \left[\left(\nabla_x \varrho_{\hat{\beta}}(\mathrm{x}_\mu^*, v_\mu^*)\right)^\top \nabla_x \varrho_{\hat{\beta}}(\mathrm{x}_\mu^*, v_\mu^*) \left(\nabla_{\mathrm{x}} \rho(\mathrm{x}_\mu^*, w_\mu^*)\right)^\top \nabla_{\mathrm{x}} \rho(\mathrm{x}_\mu^*, w_\mu^*) + \mu (\mathrm{X}_\mu^*)^{-2} \right] \\
&= H(\mathrm{x}_\mu^*, y_\mu^*, z_\mu^*) + \eta S_\mu^* \left(\mathrm{X}_\mu^*\right)^{-1} \\
&\quad + \eta \left[\left(\nabla_x \varrho_{\hat{\beta}}(\mathrm{x}_\mu^*, v_\mu^*)\right)^\top \nabla_x \varrho_{\hat{\beta}}(\mathrm{x}_\mu^*, v_\mu^*) + \left(\nabla_{\mathrm{x}} \rho(\mathrm{x}_\mu^*, w_\mu^*)\right)^\top \nabla_{\mathrm{x}} \rho(\mathrm{x}_\mu^*, w_\mu^*) \right].
\end{aligned}$$

Choosing $\tilde{\eta} = \max\{|\lambda_H| : \lambda_H$ is an eigenvalue of $H(\mathrm{x}_\mu^*, y_\mu^*, z_\mu^*)\}$, we notice that $\tilde{\eta} > 0$ and the matrix $H(\mathrm{x}_\mu^*, y_\mu^*, z_\mu^*) + \tilde{\eta} S_\mu^* (\mathrm{X}_\mu^*)^{-1}$ is positive definite. Hence, for all $\eta \geq \tilde{\eta}$, $\nabla_x^2 \mathcal{M}_{\eta,\mu}(\Lambda^*)$ is positive definite. □

Note: By Theorems 4.1 and 4.2, for any $\mu > 0$, if Λ_μ^* satisfies $\mathcal{D}_\mu(\Lambda_\mu^*) = 0$, then there exists a $\bar{\eta} > 0$ such that

$$\mathrm{x}_\mu^* = \arg\min \mathcal{M}_{\eta,\mu}(\mathrm{x}, v_\mu^*, w_\mu^*, y_\mu^*, z_\mu^*) \quad \text{for all} \quad \eta \geq \bar{\eta}.$$

Theorem 4.3. *For any* $\mu > 0$, *the penalty term* Ψ_μ *has a unique minimum value* $3\mu(1 - \log \mu)$.

Proof. The result can be obtained by computing the minimum of the following function:

$$\varphi(\xi, \varsigma, \omega) = \xi + \varsigma + \omega - \mu \left(\log(\xi) + \log(\varsigma) + \log(\omega)\right).$$

The function φ has the minimizer $\xi = \mu, \varsigma = \mu$, and $\omega = \mu$. Therefore, the minimum value of the function φ is $3\mu\left(1 - \log \mu\right)$. Also, the function φ is convex. Hence, $\xi = \mu$, $\varsigma = \mu$, and $\omega = \mu$ is the unique minimizer of the function φ. □

4.2. Descent direction

This section ensures that the Newton direction (3.12) is descent for the penalty function Ψ_μ and the merit function $\mathcal{M}_{\eta,\mu}$.

Theorem 4.4. *Consider the barrier problem* (3.2). *Suppose that the point* $\Lambda = (\mathrm{x}, v, w, s, y, z)$ *is an interior point. Then, for any* $\mu > 0$, *the following results hold:*

(i) The Newton direction $\Delta\Lambda = (\Delta \mathrm{x}, \Delta v, \Delta w, \Delta s, \Delta v, \Delta w)$ *is descent at* Λ *for the penalty term* Ψ_μ *if and only if* Λ *is not quasi-central point.*

(ii) The Newton direction $\Delta\Lambda = (\Delta \mathrm{x}, \Delta v, \Delta w, \Delta s, \Delta y, \Delta z)$ *is descent at* Λ *for the merit function* $\mathcal{M}_{\eta,\mu}$ *if and only if* Λ *is not quasi-central point.*

Proof. (i) We can easily compute the following gradients:

$$\begin{aligned}
&\nabla_{\mathrm{x}} \Psi_\mu = -\left(A_{\hat{\beta}}(\mathrm{x})\right)^\top \varrho_{\hat{\beta}} - \left(B(\mathrm{x})\right)^\top \rho + s - \mu x^{-1}, \quad \nabla_v \Psi_\mu = \varrho_{\hat{\beta}} + y - \mu v^{-1}, \\
&\nabla_w \Psi_\mu = \rho + z - \mu w^{-1}, \quad \nabla_s \Psi_\mu = \mathrm{x} - \mu s^{-1},
\end{aligned}$$

$$\nabla_y \Psi_\mu = v - \mu y^{-1}, \quad \nabla_z \Psi_\mu = w - \mu z^{-1}.$$

The directional derivative of Ψ_μ in the direction $\Delta\Lambda$ is

$$\begin{aligned}
(\nabla \Psi_\mu)^\top \Delta\Lambda &= (\nabla_{\mathrm{x}} \Psi_\mu)^\top \Delta \mathrm{x} + (\nabla_v \Psi_\mu)^\top \Delta v + (\nabla_w \Psi_\mu)^\top \Delta w + (\nabla_s \Psi_\mu)^\top \Delta s \\
&\quad + (\nabla_y \Psi_\mu)^\top \Delta y + (\nabla_z \Psi_\mu)^\top \Delta z \\
&= \Big(-\big(A_{\hat\beta}(\mathrm{x})\big)^\top \varrho_{\hat\beta} - \big(B(\mathrm{x})\big)^\top \rho\Big)^\top \Delta \mathrm{x} + s^\top \Delta \mathrm{x} - \mu(\mathrm{x}^{-1})^\top \Delta \mathrm{x} \\
&\quad + \varrho_{\hat\beta}^\top \Delta v + \rho^\top \Delta w + \mathrm{x}^\top \Delta s - \mu (s^{-1})^\top \Delta s - \mu \left(v^{-1}\right)^\top \Delta v \\
&\quad - \mu \left(w^{-1}\right)^\top \Delta w - \mu \left(y^{-1}\right)^\top \Delta y - \mu \left(z^{-1}\right)^\top \Delta z \\
&= -\varrho_{\hat\beta}^\top (A_{\hat\beta}(\mathrm{x}) \Delta \mathrm{x} - \Delta v) - \rho^\top (B(\mathrm{x}) - \Delta w) + \mathrm{x}^\top \Delta s + s^\top \Delta \mathrm{x} \\
&\quad - \mu \left((\mathrm{x}^{-1})^\top \Delta \mathrm{x} + (s^{-1})^\top \Delta s\right) - \mu \left((v^{-1})^\top \Delta v + (y^{-1})^\top \Delta y\right) \\
&\quad - \mu \left((w^{-1})^\top \Delta w + (z^{-1})^\top \Delta z\right) \\
&= -\varrho_{\hat\beta}^\top \varrho_{\hat\beta} - \rho^\top \rho + 2(n+1)\mu - \mu \left((\mathrm{x}^{-1})^\top \Delta \mathrm{x} + (s^{-1})^\top \Delta s\right) \\
&\quad - \mu \left((v^{-1})^\top \Delta v + (y^{-1})^\top \Delta y\right) - \mu \left((w^{-1})^\top \Delta w + (z^{-1})^\top \Delta z\right),
\end{aligned}$$

where the last equality is obtained by the last two equations of the system (3.7) and the complementarity conditions.

Now, if we set $\Theta = (\mathrm{X}S)^{1/2} e,\ \Upsilon = (VY)^{1/2} e$ and $\Phi = (WZ)^{1/2} e$ then we obtain

$$(\nabla \Psi_\mu)^\top \Delta\Lambda = -\left(\|\varrho_{\hat\beta}\|^2 + \|\rho\|^2 + \|\Theta - \mu\Theta^{-1}\|^2 + \|\Upsilon - \mu\Upsilon^{-1}\|^2 + \|\Phi - \mu\Phi^{-1}\|^2\right) < 0. \quad (4.3)$$

(ii) We have

$$\begin{aligned}
(\nabla \mathcal{M}_{\eta,\mu})^\top \Delta\Lambda &= (\nabla_{\mathrm{x}} \mathcal{M}_{\eta,\mu})^\top \Delta \mathrm{x} + (\nabla_v \mathcal{M}_{\eta,\mu})^\top \Delta v + (\nabla_w \mathcal{M}_{\eta,\mu})^\top \Delta w + (\nabla_s \mathcal{M}_{\eta,\mu})^\top \Delta s \\
&\quad + (\nabla_y \mathcal{M}_{\eta,\mu})^\top \Delta y + (\nabla_z \mathcal{M}_{\eta,\mu})^\top \Delta z \\
&= \sigma_{\hat\beta}^\top \Delta \mathrm{x} - \mathrm{x}^\top \Delta s + y^\top \Delta v + z^\top \Delta w + \varrho_{\hat\beta}^\top \Delta y + \rho^\top \Delta z + \eta \left(\nabla \Psi_\mu\right)^\top \Delta\Lambda.
\end{aligned}$$

Since $(\nabla \Psi_\mu)^\top \Delta\Im < 0$, the least value of penalty parameter η to make the Newton direction as descent for the merit function $\mathcal{M}_{\eta,\mu}$ is

$$\tilde\eta = \frac{\sigma_{\hat\beta}^\top \Delta \mathrm{x} - \mathrm{x}^\top \Delta s + y^\top \Delta v + z^\top \Delta w + \varrho_{\hat\beta}^\top \Delta y + \rho^\top \Delta z}{|\left(\nabla \Psi_\mu\right)^\top \Delta\Lambda|}. \quad (4.4)$$

Hence, for all $\eta > \tilde\eta$, the Newton direction $\Delta\Lambda$ is descent for the merit function if and only if Λ is not quasi-central point. □

4.3. Sufficient decrease

If the penalty parameter η is such that $\eta > \tilde\eta$, then, we can write $\eta = \tilde\eta + \delta$, where $\delta > 0$. Since

$$\left(\nabla \mathcal{M}_{\eta,\mu}(\Lambda)\right)^\top \Delta\Lambda = \delta \left(\nabla \Psi_\mu\right)^\top \Delta\Lambda \quad (4.5)$$

and $(\nabla\Psi_\mu)^\top\Delta\Lambda < 0$, then for any $\kappa \in (0,1)$, there exists a positive number $\bar{\alpha}$ such that for any $\alpha \in (0,\bar{\alpha})$, the following condition is satisfied:

$$\Psi_\mu(\Lambda + \alpha\Delta\Lambda) \leq \Psi_\mu(\Lambda) + \alpha\kappa \left(\nabla\Psi_\mu\right)^\top \Delta\Lambda. \tag{4.6}$$

As

$$\left(\nabla\mathcal{M}_{\eta,\mu}\right)^\top \Delta\Lambda < 0 \quad \text{for} \quad \eta > \tilde{\eta}$$

for some $\alpha' \in (0,\bar{\alpha})$, we get

$$\mathcal{M}_{\eta,\mu}(\Lambda + \alpha'\Delta\Lambda) \leq \mathcal{M}_{\eta,\mu}(\Lambda) + \delta\alpha'\kappa \left(\nabla\mathcal{M}_{\eta,\mu}\right)^\top \Delta\Lambda. \tag{4.7}$$

Definition 4.1. *The following set of points represents the γ-neighbourhood of the quasi-central path corresponding to μ :*

$$\begin{aligned}\mathcal{N}_\gamma(\Lambda;\mu) = \big\{\Lambda : \mathrm{x} > 0, s > 0, v > 0, w > 0, y > 0, z > 0,\ \|\varrho_{\hat{\beta}}\|^2 + \|\rho\|^2 \\ + \|\Theta - \mu\Theta^{-1}\|^2 + \|\Upsilon - \mu\Upsilon^{-1}\|^2 + \|\Phi - \mu\Phi^{-1}\|^2 \leq \gamma\mu\big\},\end{aligned} \tag{4.8}$$

where $(\gamma,\mu) > 0,\ \Theta = (\mathrm{X}S)^{1/2}e,\ \Upsilon = (VY)^{1/2}e,$ and $\Phi = (WZ)^{1/2}e.$

By this definition, we are able to calculate the distance of an interior point from the KKT point corresponding to a $\mu > 0$.

4.4. Update of the penalty parameter

The penalty parameter η is chosen such that the Newton direction is a descent for the merit function. Corresponding to a μ and a given $\tilde{\delta}$, the current penalty parameter η_{new} is updated as follows:

$$\eta_{\text{new}} = \begin{cases}\tilde{\eta} + \tilde{\delta}, & \text{if} \quad \tilde{\eta} + \tilde{\delta} > \eta, \\ \tilde{\eta} + \delta, & \text{otherwise},\end{cases} \tag{4.9}$$

where $\tilde{\eta}$ is calculated by (4.4).

4.5. Description of the initial point and the barrier parameter

In Algorithm 4.1, the initial point is chosen such that all the components of the vectors $\mathrm{x}^{(0)}$, $s^{(0)}$, $v^{(0)}$, $w^{(0)}$, $y^{(0)}$ and $z^{(0)}$ are positive. If an MOP has bound constraints with negative values, then we can easily formulate the problem in the form of (2.1). Consequently, we can always initialize x such that $x_i > 0$ for all $i \in \{1,\ldots,n\}$.

Algorithm 4.1 is based on the strategy of the quasi-central path (3.4). It solves the KKT system (3.3) for decreasing values of $\mu > 0$ until $\nu(\Lambda;\mu)$ is not less than a given precision parameter ϵ (see (3.9)). In the proposed algorithm, we initialize the value of the barrier parameter μ_0 as (see [39])

$$\mu_0 = \frac{\mathrm{x}^\top s + v^\top y + w^\top z}{p+m}. \tag{4.10}$$

Then, we solve the KKT system (3.3) for μ_0 until it satisfies the (μ,γ)-neighbourhood condition (4.1). Suppose an iterate $\Lambda = (\mathrm{x}, v, w, s, y, z)$ lies in (μ,γ)-neighbourhood (see (4.1)) but fails the stopping criteria condition $\nu(\Lambda;\mu) \leq \epsilon$ for the outer loop, then we decrease the barrier parameter at the $(k+1)$-th iteration by the value

$$\mu^{(k+1)} = \wp\Big(\|\varrho_{\hat{\beta}}^{(k)}\|^2 + \|\rho^{(k)}\|^2 + \|\Theta^{(k)} - \mu^{(k)}(\Theta^{(k)})^{-1}\|^2 + \|\Upsilon^{(k)} - \mu(\Upsilon^{(k)})^{-1}\|^2 + \|\Phi^{(k)} - \mu(\Phi^{(k)})^{-1}\|^2\Big), \tag{4.11}$$

where $\wp \in (0,1)$, $\Theta^{(k)} = (\mathrm{X}^{(k)}S^{(k)})^{1/2}e$, $\Upsilon^{(k)} = (V^{(k)}Y^{(k)})^{1/2}e$ and $\Phi^{(k)} = (W^{(k)}Z^{(k)})^{1/2}e$. In Algorithm 4.1, we describe a step-by-step procedure for finding Pareto optimal points for a given optimization problem with the help of the above process.

Algorithm 4.1. Ideal Cone-IPM (IC-IPM) for MOPs

Inputs:
(a) Given MOP:

$$\begin{cases} \text{minimize} & F(x) \\ \text{subject to} & x \in \mathcal{X}. \end{cases}$$

(b) Provide the number of subproblems to be solved, N

Finding the Ideal Point: Find $f_i^* = \min\{f_i(x) : x \in \mathcal{X}\}$ for each $i \in \{1, \ldots, p\}$ using IPM [3], and then set ideal point $F^* = (f_1^*, f_2^*, \ldots, f_p^*)^\top$

Initialization:
Set Pareto set $\leftarrow \emptyset$
Give an initial point such that $\Lambda^{(k)} = (\mathrm{X}^{(k)}, s^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}) > 0$
Choose the values of the parameters $\tilde{\tilde{\delta}} = 2$ and $\kappa = 0.85$
Give a value of the precision parameter $\epsilon > 0$ for the optimum solutions to (3.2)
Set $k \leftarrow 0$

for $i = 1 : 1 : N$ **do**
  Choose randomly a direction $\hat{\beta}$ from (2.3)
  whhile $\nu(\Lambda^{(k)}) \geq \epsilon$ **do**
    Choose $\mu^{(k)}$ by (4.10)
    while $\Lambda^{(k)} \notin \mathcal{N}_{\mu^{(k)}}(\gamma)$ **do** (inner loop)
      Calculate the direction $(\Delta\mathrm{X}^{(k)}, \Delta s, \Delta v^{(k)}, \Delta w^{(k)})$ by using (3.12)
      Choose step length α by the formula (4.1)
      Calculate $\tilde{\eta}$ by the expression (4.4)
      Calculate $\eta^{(k)}$ and $\delta^{(k)}$ from to ensure that the Newton direction is descent for $\mathcal{M}_{\eta,\mu}$
      Find $\alpha^{(k)} \in (0, \alpha)$ such that the following Armijo condition is satisfied

$$\mathcal{M}_{\eta,\mu^{(k)}}(\Lambda^{(k)} + \alpha^{(k)}\Delta\Lambda^{(k)}) \leq \mathcal{M}_{\eta,\mu^{(k)}}(\Lambda^{(k)}) + \delta^{(k)}\alpha^{(k)}\kappa\left(\nabla\mathcal{M}_{\eta,\mu^{(k)}}\right)^\top \Delta\Lambda^{(k)}$$

      Set $\Lambda^{(k+1)} = \Lambda^{(k)} + \alpha\Delta\Lambda^{(k)}$
    end while
  end while
  Calculate $F(\mathrm{X}^{(k)}) = \hat{\beta}c^T\mathrm{X}^{(k)} - v^{(k)}$
  Update Pareto set $\leftarrow$ Pareto set $\bigcup\{F(\mathrm{X}^{(k)})\}$
end for
return Pareto set

5. Global Convergence Results

The following section discusses the global convergence theory for the Algorithm 4.1. The process of global convergence analysis begins with an additional set of assumptions and lemmas.

5.1. Assumptions

Assumptions for developing the convergence theory of Algorithm 4.1 are as follows:

(A1) For $\mathrm{x} \geq 0$, the set $\{\nabla\theta_1(\mathrm{x}), \nabla\theta_2(\mathrm{x}), \ldots, \nabla\theta_m(\mathrm{x})\}$ is linearly independent.

(A2) The iteration sequences $\mathrm{x}^{(k)}$, $v^{(k)}$, $w^{(k)}$, $y^{(k)}$ and $z^{(k)}$ generated by the Algorithm 4.1 is bounded above.

(A3) The matrix $H(\mathrm{x}, y, z) + S\mathrm{X}^{-1}$ is positive definite.

Lemma 5.1. *Assume* $\Lambda = (\mathrm{x}, v, w, s, y, z)$ *is an interior point. Then, for any* $\mu > 0$ *and under the assumption* (A3)*, the matrix* $D_{\hat{\beta}}(\Lambda)$ *is nonsingular.*

Proof. See [9]. □

Lemma 5.2. *For any* $\mu > 0$*, the sequence of interior points* $\{\Lambda^{(k)} : \Lambda^{(k)} > 0\}$*, produced by the inner loop of Algorithm* 4.1 *is bounded whenever the conditions (A2) and (A3) follows. Moreover, the components* $\mathrm{x}_i^{(k)}$, $s_i^{(k)}$, $v_j^{(k)}$, $y_j^{(k)}$, $w_l^{(k)}$ *and* $z_l^{(k)}$ *are bounded away from zero.*

Proof. By assumption (A2), the sequences $\{\mathrm{x}^{(k)}\}$, $\{v^{(k)}\}$ and $\{w^{(k)}\}$ are bounded above. Let $\Psi_\mu^{(0)}$ be the value of penalty term at the initial point. Note that the inequality

$$3\mu\,(1 - \log\mu) \leq \Psi_\mu(\Lambda^{(k)} + \alpha^{(k)}\Delta\Lambda^{(k)}) \leq \Psi_\mu^{(0)} \tag{5.1}$$

holds by Theorems 4.3 and 4.4.

Further, observe that

$$\begin{aligned}
&\mathrm{x}_j^{(k)}s_j^{(k)} - \mu\log\left(\mathrm{x}_j^{(k)}s_j^{(k)}\right) \to \infty \quad \text{if either} \quad \mathrm{x}_j^{(k)}s_j^{(k)} \to 0 \quad \text{or} \quad \mathrm{x}_j^{(k)}s_j^{(k)} \to \infty,\\
&v_i^{(k)}y_i^{(k)} - \mu\log\left(v_i^{(k)}y_i^{(k)}\right) \to \infty \quad \text{if either} \quad v_i^{(k)}y_i^{(k)} \to 0 \quad \text{or} \quad v_i^{(k)}y_i^{(k)} \to \infty,\\
&w_l^{(k)}z_l^{(k)} - \mu\log\left(w_l^{(k)}z_l^{(k)}\right) \to \infty \quad \text{if either} \quad w_l^{(k)}z_l^{(k)} \to 0 \quad \text{or} \quad w_l^{(k)}z_l^{(k)} \to \infty.
\end{aligned}$$

However, by (5.1), we see that the products $\mathrm{x}_j^{(k)}s_j^{(k)}$, $v_i^{(k)}y_i^{(k)}$ and $w_l^{(k)}z_l^{(k)}$ are bounded above and bounded away from zero. Therefore, the sequences $s^{(k)}$, $y^{(k)}$ and $z^{(k)}$ are bounded above and the components $\mathrm{x}_j^{(k)}$, $s_j^{(k)}$, $v_i^{(k)}$, $y_i^{(k)}$, $w_l^{(k)}$ and $z_l^{(k)}$ are bounded away from zero. □

Lemma 5.3. *Under the assumption (A1), (A2) and (A3), the merit function* $\mathcal{M}_{\eta,\mu}$ *is bounded below.*

Proof. By Theorem 4.3, the penalty term is bounded below and the term

$$c^\top\mathrm{x} + \varrho_{\hat{\beta}}^\top y + \rho^\top z - s^\top\mathrm{x}$$

is bounded below due to Assumptions (A1) and (A2), and Lemma 5.2. Hence, the merit function $\mathcal{M}_{\eta,\mu}$ is bounded below. $\square$

Theorem 5.1. *(i) Consider $\mu > 0$. Assume that the sequence $\{\Lambda^{(k)} = (\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, s^{(k)}, y^{(k)}, z^{(k)})\}$ is created by the inner loop of Algorithm 4.1 and its limit point is $\Lambda^*_\mu = (\mathrm{x}^*_\mu, v^*_\mu, w^*_\mu, s^*_\mu, y^*_\mu, z^*_\mu)$. If $\bar{\mathcal{D}}_{\hat{\beta}}$ is continuous at Λ^* and $\bar{\mathcal{D}}_{\hat{\beta}}(\Lambda^*)$ is nonsingular then, Λ^*_μ lies on the quasi-central path* (3.4)*.*

(ii) Suppose the problem (3.2) *has an optimal point in $\mathcal{N}_\gamma(\Lambda;\mu)$ for a fixed μ and γ. Let the starting point $\Lambda^{(0)}$ is an interior point. Then Algorithm* 4.1 *will terminate in at most*

$$\frac{\log\left(\frac{\gamma\mu}{\Xi(\Lambda^{(0)})}\right)}{\log(1-\tau)}$$

iterations, where

$$\Xi(\Lambda^{(0)}) = \Big(\|\varrho^{(0)}_{\hat{\beta}}\|^2 + \|\rho^{(0)}\|^2 + \|\Theta^{(0)} - \mu(\Theta^{(0)})^{-1}\|^2 + \|\Upsilon^{(0)} - \mu(\Upsilon^{(0)})^{-1}\|^2 + \|\Phi^{(0)} - \mu(\Phi^{(0)})^{-1}\|^2\Big).$$

Proof. (i) Since the limit point of the sequence $\{\Lambda^{(k)}\}$ is Λ^*_μ then, there exists a convergent subsequence $\Lambda^{(k_l)}$ such that $\Lambda^{(k_l)} \to \Lambda^*_\mu$.

We need to prove three properties regarding the subsequence $\Lambda^{(k_l)}$:

(a) The limit point Λ^*_μ is an interior point.

(b) The sequence of search direction $\{\Delta\Lambda^{(k_l)}\}$ is bounded.

(c) The sequence of steplengths $\{\alpha^{(k_l)}\}$ is bounded away from zero.

The first property can be obtained by Lemma 5.2, stating that the components $\mathrm{x}^{(k_l)}_j$ and $s^{(k_l)}_j$ are bounded away from zero. Therefore,

$$\mathrm{x}^{(k_l)} \to \mathrm{x}^*_\mu > 0 \quad \text{and} \quad s^{(k_l)} \to s^*_\mu > 0.$$

The property (b) holds because the set $E = \{\Lambda^{(k_l)}, \Lambda^*_\mu\}$ is compact and $\mathcal{D}$, $(\bar{\mathcal{D}})^{-1}$ are continuous functions on E.

To prove the property (c), if possible let the steplength sequence $\{\alpha^{k_l}\}$ generated by the Algorithm 4.1 is not bounded away from zero. Then,

$$\lim_{l\to\infty} \frac{\mathrm{x}^{(k_l)}_j}{|\Delta\mathrm{x}_j|^{(k_l)}} = 0 \quad \text{holds for at least one } j \in \{1, 2, \ldots\}.$$

Since $\mathrm{x}^{(k_l)}_j$ is bounded away from zero, $|\Delta\mathrm{x}_j|^{(k_l)}$ tends to infinity. However, this goes against the fact that $\{\Delta\Lambda^{(k)}\}$ is bounded. Hence, (c) holds.

Now, we are able to proof for the key part of the theorem. Consider the iterative sequence

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \alpha^{(k)}\Delta\Lambda^{(k)},$$

where $\alpha^{(k)}$ is computed by (4.1) and $\Delta\Lambda^{(k)}$ is determined by solving (3.11). At k-th iteration, let $\eta^{(k)} > 0$ and $\mu^{(k)} > 0$ stand for the value of the penalty parameter and barrier parameter, respectively. Also, $\Lambda^{(k)}$ satisfies (see Theorem 4.4)

$$\left(\nabla \mathcal{M}_{\eta^{(k)},\mu^{(k)}}(\Lambda^{(k)})\right)^{\top} \Delta\Lambda^{(k)} < 0.$$

Since the step length $\alpha^{(k)}$ is bounded away from zero, the sequence $\{\Lambda^{(k)}\}$ satisfies (see [14])

$$\left(\nabla \mathcal{M}_{\eta^{(k)},\mu^{(k)}}(\Lambda^{(k)})\right)^{\top} \frac{\Delta\Lambda^{(k)}}{\|\Delta\Lambda^{(k)}\|} \to 0.$$

However, the sequence $\{\Delta\Lambda^{(k)}\}$ is bounded and by (4.5), we obtain

$$\left(\nabla \mathcal{M}_{\eta^{(k)},\mu^{(k)}}(\Lambda^{(k)})\right)^{\top} \Delta\Lambda^{(k)} = \delta^{(k)} \left(\nabla \Psi_{\mu}(\Lambda^{(k)})\right)^{\top} \Delta\Lambda^{(k)} \to 0.$$

Since $\delta^{(k)} > 0$ and by (4.1), we obtain

$$\begin{aligned}\left(\nabla \Psi_{\mu}(\Lambda^{(k)})\right)^{\top} \Delta\Lambda^{(k)} = -\Big(&\|\varrho_{\hat{\beta}}^{(k)}\|^2 + \|\rho^{(k)}\|^2 + \|\Theta^{(k)} - \mu(\Theta^{(k)})^{-1}\|^2 \\ &+ \|\Upsilon^{(k)} - \mu(\Upsilon^{(k)})^{-1}\|^2 + \|\Phi^{(k)} - \mu(\Phi^{(k)})^{-1}\|^2\Big) \to 0.\end{aligned} \tag{5.2}$$

This directly implies that

$$\begin{aligned}&\lim_{k\to\infty} \varrho_{\hat{\beta}}^{(k)} = 0, \quad \lim_{k\to\infty} \rho^{(k)} = 0, \quad \lim_{k\to\infty} \mathrm{X}^{(k)} S^{(k)} e = \mu e, \\ &\lim_{k\to\infty} V^{(k)} Y^{(k)} e = \mu e, \quad \lim_{k\to\infty} W^{(k)}, \\ &Z^{(k)} e = \mu e.\end{aligned}$$

(ii) From (5.2), we can conclude that there exists $\tau_1, \tau_2, \tau_3, \tau_4,$ and τ_5 such that

$$\begin{aligned}&\|\varrho_{\hat{\beta}}^{(k+1)}\|^2 \le (1-\tau_1)\|\varrho_{\hat{\beta}}^{(k)}\|^2, \quad \|\rho^{(k+1)}\|_2^2 \le (1-\tau_2)\|\rho^{(k)}\|^2, \\ &\|\Theta^{(k+1)} - \mu(\Theta^{(k+1)})^{-1}\|^2 \le (1-\tau_3)\|\Theta^{(k)} - \mu(\Theta^{(k)})^{-1}\|^2, \\ &\|\Upsilon^{(k+1)} - \mu(\Upsilon^{(k+1)})^{-1}\|^2 \le (1-\tau_4)\|\Upsilon^{(k)} - \mu(\Upsilon^{(k)})^{-1}\|^2, \\ &\|\Phi^{(k+1)} - \mu(\Phi^{(k+1)})^{-1}\|^2 \le (1-\tau_5)\|\Phi^{(k)} - \mu(\Phi^{(k)})^{-1}\|^2.\end{aligned}$$

Denoting

$$\begin{aligned}\Xi(\Lambda^{(k)}) = \Big(&\|\varrho_{\hat{\beta}}^{(k)}\|^2 + \|\rho^{(k)}\|^2 + \|\Theta^{(k)} - \mu(\Theta^{(k)})^{-1}\|^2 \\ &+ \|\Upsilon^{(k)} - \mu t(\Upsilon^{(k)})^{-1}\|_2^2 + \|\Phi^{(k)} - \mu(\Phi^{(k)})^{-1}\|^2\Big)\end{aligned}$$

and $\tau = \min\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, we obtain

$$\Xi(\Lambda^{(k)}) \le (1-\tau)\Xi(\Lambda^{(k-1)}) \le (1-\tau)^2\Xi(\Lambda^{(k-1)}) \le \cdots \le (1-\tau)^k\Xi(\Lambda^{(0)}), \tag{5.3}$$

suppose that at the k-th iteration, $\Lambda^{(k)}$ lies in the γ-neighbourhood of the quasi-central path (4.1) corresponding to μ. Therefore,

$$\Xi(\Lambda^{(k)}) \leq (1-\tau)^k \Xi(\Lambda^{(0)}) \leq \gamma\mu.$$

Hence, the number of iterations to reach γ-neighbourhood of the quasi-central path is

$$\frac{\log\big(\gamma\mu/\Xi(\Lambda^{(0)})\big)}{\log(1-\tau)}.$$

The proof is complete. □

We now show the global convergence of Algorithm 4.1.

Theorem 5.2. *Assume that the assumptions (A1), (A2) and (A3) hold true. Then, the sequence* $\{\Lambda^{(k)} = (\mathrm{x}^{(k)}, v^{(k)}, w^{(k)},\ s^{(k)}, y^{(k)}, z^{(k)})\}$ *generated by Algorithm* 4.1 *has a limit point* $\Lambda^* = (\mathrm{x}^*, v^*, w^*, s^*, y^*, z^*)$*. Moreover, the limit point* Λ^* *satisfies* $\mathcal{D}(\Lambda^*) = 0$.

Proof. Since Λ^* is the limit point of the sequence $\{\Lambda^{(k)}\}$, there exists a convergent subsequence $\{\Lambda^{(k_l)}\}$ such that $\Lambda^{(k_l)} \to \Lambda^*$, where $\Lambda^{(k_l)} \in \mathcal{N}_{\mu_k}(\gamma)$ and $\mu_k \to 0$. As the step length is bounded away from zero (by Lemma 5.2), $\big(\Delta\mathrm{x}^{(k_l)}, \Delta v^{(k_l)}, \Delta w^{(k_l)}, \Delta s^{(k_l)}, \Delta y^{(k_l)}, \Delta z^{(k_l)}\big) \to 0$. From the first equation of (3.6), we have

$$\begin{aligned} c - s^{(k_l)} - \Delta s^{(k_l)} - \big(A_{\hat\beta}(\mathrm{x}^{(k_l)})\big)^\top \big(y^{(k_l)} + \Delta y^{(k_l)}\big) \\ - \big(B(\mathrm{x}^{(k_l)})\big)^\top \big(z^{(k_l)} + \Delta z^{(k_l)}\big) \\ = -H\big(\mathrm{x}^{(k_l)}, y^{(k_l)}, z^{(k_l)}\big)\Delta\mathrm{x}^{(k_l)}. \end{aligned}$$

Taking $k_l \to \infty$, and since $\{H(\mathrm{x}^{(k_l)}, y^{(k_l)}, z^{(k_l)})\}$ is bounded, we obtain

$$c - s^* - \big(A_{\hat\beta}(\mathrm{x}^*)\big)^\top y^* - \big(B(\mathrm{x}^*)\big)^\top z^* = 0. \tag{5.4}$$

Since $\Lambda^{(k_l)} \in \mathcal{N}_{\mu_{k_l}}(\gamma)$,

$$\begin{aligned} \big\|\varrho_{\hat\beta}(\mathrm{x}^{(k_l)}, v^{(k_l)})\big\|^2 + \big\|\rho(\mathrm{x}^{(k_l)}, w^{(k_l)})\big\|^2 + \big\|\Theta^{(k_l)} - \mu^{(k_l)}(\Theta^{(k_l)}t)^{-1}\big\|^2 \\ + \big\|\Upsilon^{(k_l)} - \mu^{(k_l)}(\Upsilon^{(k_l)})^{-1}\big\|^2 + \big\|\Phi^{(k_l)} - \mu^{(k_l)}(\Phi^{(k_l)})^{-1}\big\|^2 \leq \gamma\mu^{k_l}. \end{aligned}$$

Again taking $k_l \to \infty$ then $\mu^{k_l} \to 0$ and

$$\begin{cases} \varrho_{\hat\beta}(\mathrm{x}^*, v^*) = 0, \\ \rho(\mathrm{x}^*, w^*) = 0, \\ \mathrm{X}^* S^* e = 0, \\ V^* Y^* e = 0, \\ W^* Z^* e = 0. \end{cases} \tag{5.5}$$

Hence, from (5.5), $\mathcal{D}(\Lambda^*) = 0$. Therefore, x^* is a KKT point. □

6. Numerical Experiments

This subsection reports the outcomes of several types of test problems found in the literature. The performance of the Algorithm 4.1 is tested on constrained multiobjective as well as box constrained multiobjective problems. The test have been carried out on a PC with Intel Core i7-4770U 3.40 GHz CPU and 4GB RAM in MATLAB 2020a. We take some widely used multiobjective test problems (BNH, SRN, TNK, CONSTR, Kita, SWG) to test the performance of Algorithm 4.1. The test problem Kita is a maximization problem and remaining are the minimization problems. The details of these five constrained test problems are shown in Table 6.1. The Pareto points of these problems are shown in blue (see Figs. 6.1-6.3).

Table 6.1: Constrained test problems used in this study.

Problem	Source	n	p	Number of subproblems	Accuracy
BNH	[4]	2	2	300	1.0E-6
SRN	[36]	2	2	75	1.0E-6
TNK	[37]	2	2	100	1.0E-6
CONSTR	[6]	2	3	400	1.0E-6
Kita	[6]	2	3	150	1.0E-6
SGW	[34]	2	2	300	1.0E-6

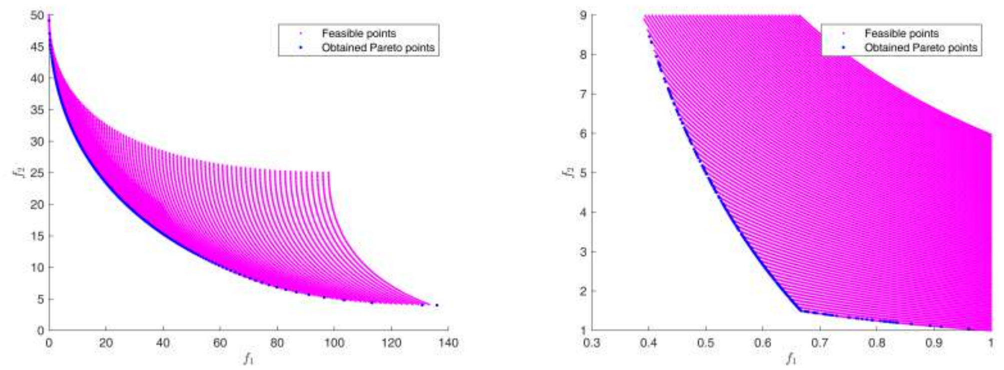

Fig. 6.1. Obtained Pareto points of BNH and CONSTR problems by Algorithm 4.1.

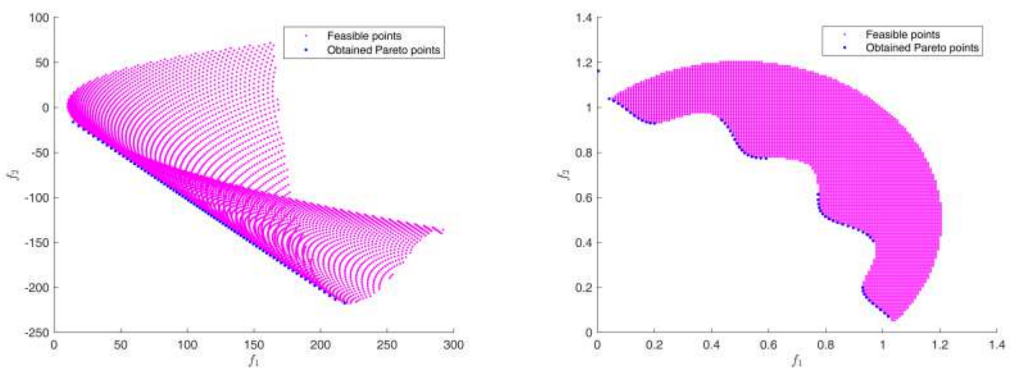

Fig. 6.2. Obtained Pareto points of SRN and TNK problems by Algorithm 4.1.

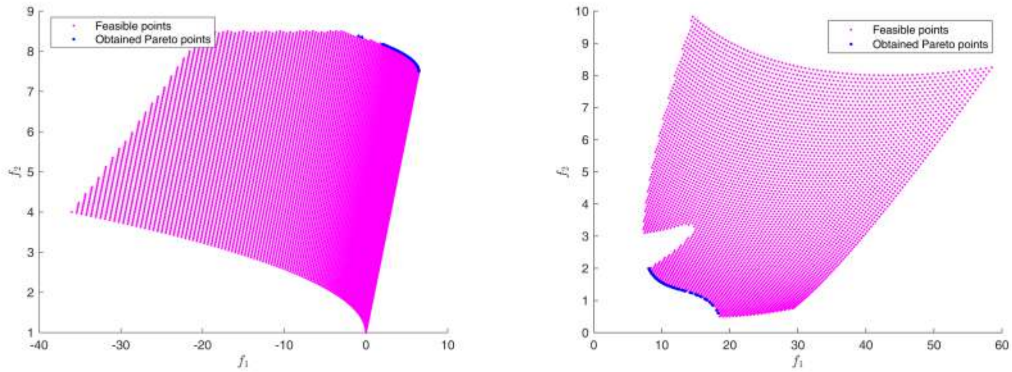

Fig. 6.3. Obtained Pareto points of Kita and SGW problem by Algorithm 4.1.

6.1. Performance metrics

To measure the performance of Algorithm 4.1, it is necessary to ascertain how close obtained solution is to the actual solution. For this propose, the literature contains various evaluation criteria (see [21]). In this article, we used three main performance measures such as GD+ (modified generational distance) [15], HV (hyper volume) [22] and IGD (inverted generational distance) [44].

Let x^* be an efficient point that is obtained by the proposed algorithm, and $f^* = (f_1^*, f_2^*, \ldots, f_p^*)$ be an ideal point. Then, the modified generational distance (GD+) is calculated by the following formula

$$\text{GD+} = \sqrt{\sum_{i=1}^{p} \big(\max\{f_i(x^*) - f_i^*, 0\}\big)^2}. \tag{6.1}$$

Apart from GD+, HV is calculated [22]. HV is the value obtained by finding the area enclosed by the generated nondominated solution points and and a reference point, say $R(x_{(0)}) = (f_1(x_{(0)}), f_2(x_{(0)}), \ldots, f_p(x_{(0)}))$, which must be dominated by all $F(x^i)$, for $i \in \{1, \ldots, m\}$. Suppose that the solution set is sorted by increasing order with respect to f_1. Then, HV is calculated by

$$\text{HV} = \sum_{i=0}^{m-1} \prod_{j=1}^{p} \Big(f_j\big(x^{(0)} - f_j(x_{i+1})\big)\Big). \tag{6.2}$$

A higher HV value indicates better algorithm performance.

Another commonly used performance measure is IGD. IGD is calculated as follows:

$$\text{IGD}(G, G^*) = \frac{\sum_{x \in G^*} d(x, G)}{|G^*|}, \tag{6.3}$$

where the set G consists the approximation of the Pareto front and the set G^* contains uniformly distributed known nondominated points.

6.2. Performance of the Algorithm 4.1 on some test problems

We take the following optimization problem (FON [38]) to test the performance measure of Algorithm 4.1:

$$\begin{cases} \text{minimize} & \left(1-\exp\left(-\sum_{i=1}^{n}\left(x_i-\frac{1}{\sqrt{n}}\right)^2\right), 1-\exp\left(-\sum_{i=1}^{n}\left(x_i+\frac{1}{\sqrt{n}}\right)^2\right)\right)^\top \\ \text{subject to} & x\in[-4,4]. \end{cases} \tag{6.4}$$

The efficient set is

$$\left\{x\in\mathbb{R}^n:\ x_i\in\left[-\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right],\ i=1,\ldots,n\right\}.$$

The performance measures (GD+, HV, IGD) of Algorithm 4.1 for the problem (6.4) are calculated by taking different values of n (see Table 6.2). Also, the obtained Pareto points comparison with true Pareto front are shown in Fig. 6.4.

Table 6.2: Performance measures of test problem FON.

Number of decision variables (n)	Number of subproblems	GD+	HV	IGD
4	300	0.1760E-4	0.34047	6.2512E-4
9	200	1.4097E-4	0.33941	8.8491E-4
16	150	0.1171E-4	0.33880	8.8334E-4
25	100	8.1465E-4	0.33716	8.1986E-4

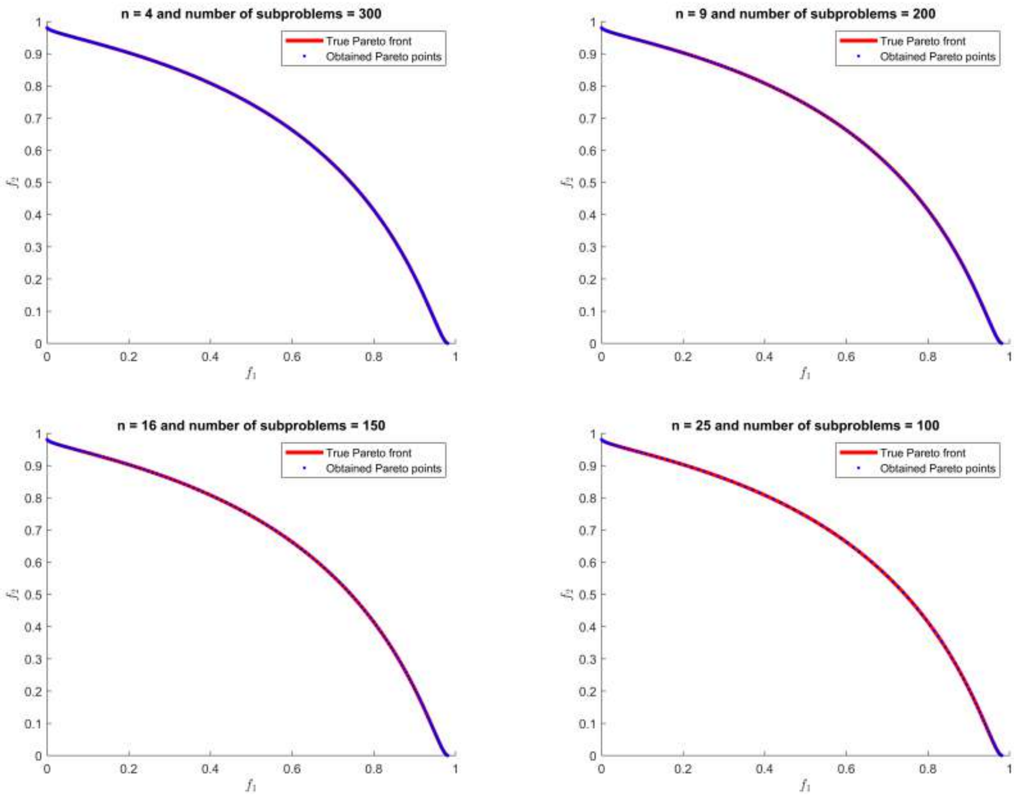

Fig. 6.4. Obtained Pareto points of test problem FON by Algorithm 4.1.

• CEC09 and Zitzler, Deb and Thiele (ZDT) test suit

So far, we have seen the performance of IC-IPM on constrained MOPs and an unconstrained MOP with bound constraints (FON). The efficiency of IC-IPM is demonstrated by the good approximation of the Pareto fronts in all the above problems. Below, we use CEC09 test problems and ZDT test suite to further test the algorithm's capabilities. We evaluate the problems from CEC09 test suite [42] that are smooth. In comparison to the previous test problems, these are more complex problems. Table 6.3 shows the results of CEC09, obtained by IC-IPM and other well-known techniques, such as MOEA [43], ENS-MOEA [43], etc. The comparison Table 6.3 shows that the performance (based on IGD values) of Algorithm 4.1 (IC-IPM) is better compared to other algorithms on test suite CEC09. The obtained Pareto points, along with the true Pareto front, are depicted below in Fig. 6.5.

Table 6.3: Comparison of IGD scores for CEC09.

Problem	FRD [42]	FD [42]	RD [42]	OD [42]	MOEA/D [43]	ENS-MOEA/D [43]	Cultural MOQPSO [1]	MOWOATS [1]	IC-IPM
UF1	9.61E-3	6.40E-3	2.52E-3	2.78E-3	2.01E-3	1.64E-3	1.11E-2	2.32E-3	1.04E-4
UF2	8.41E-3	7.20E-3	9.43E-3	9.80E-3	4.82E-3	4.03E-3	2.15E-2	2.21E-3	1.12E-4
UF3	4.72E-3	3.11E-3	9.30E-3	1.05E-2	1.06E-2	2.66E-3	3.75E-2	9.77E-3	1.43E-4
UF4	5.92E-2	7.88E-2	8.81E-2	8.58E-2	6.24E-2	4.21E-2	5.98E-2	1.83E-3	1.33E-4
UF7	5.62E-3	6.30E-3	5.41E-3	3.24E-3	1.80E-3	1.72E-3	1.13E-2	2.12E-3	2.35E-4
UF8	6.60E-2	6.11E-2	5.69E-2	5.62E-2	4.28E-2	3.10E-2	1.18E-2	3.61E-3	5.78E-3

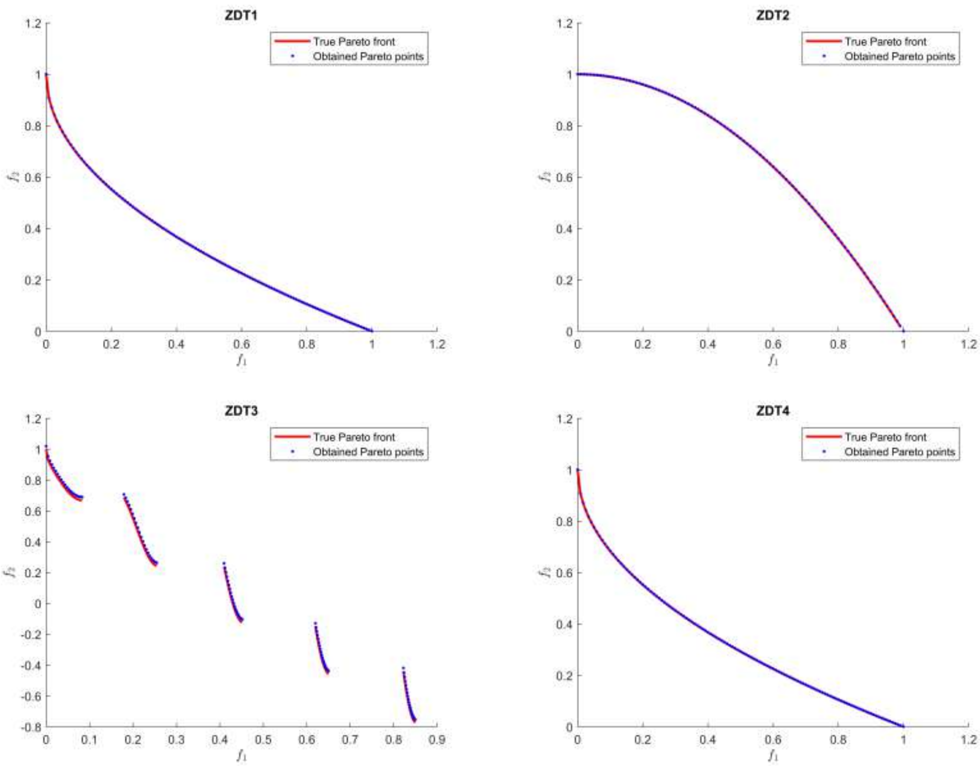

Fig. 6.5. Obtained Pareto points of test problem ZDT1, ZDT2, ZDT3 and ZDT4 by Algorithm 4.1.

For the ZDT test suite, the outcomes achieved by IC-IPM and the reputed methods such as MOEA/D PBI Approach, MOEA/D Tchebycheff (TE) Approach, MOEA/D Weighted Sum (WS) Approach, Pareto-adaptive weight vectors ($pa\lambda$) based MoEA/D approach, NSGA-II, Cultural MOQPSO and MOWOATS are presented in Table 6.4. The obtained Pareto points, along with the true Pareto front, are depicted below in Fig. 6.6.

Table 6.4: Comparison of IGD values for the ZDT benchmark suite.

Problem	WS [35]	TE [35]	PBI [35]	NSGA-II [35]	$pa\lambda$-MOEA/D [35]	Cultural MOQPSO [1]	MOWOATS [1]	IC-IPM
ZDT1	5.42E-4	6.48E-4	1.14E-4	7.94E-4	5.79E-4	6.13E-3	1.30E-3	1.05E-4
ZDT2	1.30E-2	5.84E-4	7.02E-4	8.15E-4	6.02E-4	4.87E-3	7.32E-4	1.27E-4
ZDT3	4.93E-3	2.01E-3	2.06E-3	1.19E-3	1.97E-3	1.89E-1	2.87E-3	1.01E-4
ZDT4	7.01E-3	6.65E-4	7.85E-4	8.14E-4	5.93E-4	5.26E-3	2.85E-4	1.45E-4

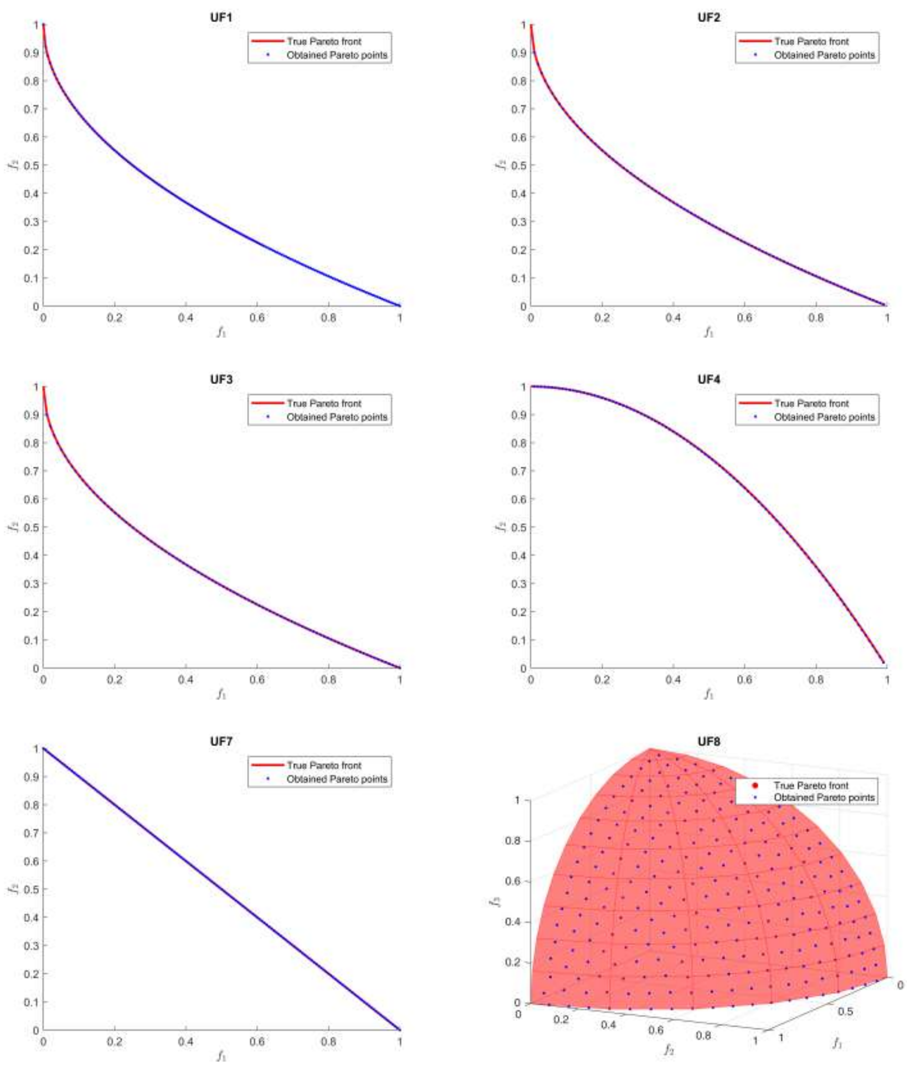

Fig. 6.6. Obtained Pareto points of UF1, UF2, UF3, UF4, UF7 and UF8 by Algorithm 4.1.

7. Conclusions

This work has introduced an interior-point approach, with the help of the cone method (IC-IPM), to find a subset of nondominated points of an MOPs. In proposed method, IC has been used to transform an MOP into a collection of single objective optimization problems. Each single objective optimization problem of the collection has been solved by IPM. To find the solution of each subproblem by IPM, a barrier problem and its KKT conditions have been derived. In order to the solve KKT conditions, the iteration started with the initial point and then calculated the direction by Newton method. Thereafter, step length has been chosen so that the nonnegative variables remain nonnegative. A new merit function has been also proposed with global properties (Theorems 4.1-4.3). Merit function has helped to take a suitable step length along the search direction.

Theorem 4.4 shows that the search direction calculated by Theorem 3.1 is descent for the merit function (4.2). Furthermore, we have proved the global convergence of the proposed algorithm under standard assumptions. We have demonstrated through numerical experiments that Algorithm 4.1 can solve constrained MOPs efficiently. We have used three performance measures (GD+, HV, IGD) to test the efficiency of the Algorithm 4.1 on some standard test suite. The performance of Algorithm 4.1 has compared with some existing popular algorithms. Tables 6.3 and 6.4 has shown that Algorithm 4.1 is comparatively efficient.

Throughout the paper, the objective and constraints functions are assumed to be twice continuously differentiable. As a result, IC-IPM method is not capable of dealing with non-smooth problems. Hence, future study will focus on expanding this technique to tackle the vast majority of problems (smooth and non-smooth).

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