

# UNIFORM ERROR BOUNDS OF A CONSERVATIVE COMPACT FINITE DIFFERENCE METHOD FOR THE QUANTUM ZAKHAROV SYSTEM IN THE SUBSONIC LIMIT REGIME\*

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## Abstract

In this paper, we consider a uniformly accurate compact finite difference method to solve the quantum Zakharov system (QZS) with a dimensionless parameter  $0 < \varepsilon \leq 1$ , which is inversely proportional to the acoustic speed. In the subsonic limit regime, i.e., when  $0 < \varepsilon \ll 1$ , the solution of QZS propagates rapidly oscillatory initial layers in time, and this brings significant difficulties in devising numerical algorithm and establishing their error estimates, especially as  $0 < \varepsilon \ll 1$ . The solvability, the mass and energy conservation laws of the scheme are also discussed. Based on the cut-off technique and energy method, we rigorously analyze two independent error estimates for the well-prepared and ill-prepared initial data, respectively, which are uniform in both time and space for  $\varepsilon \in (0, 1]$  and optimal at the fourth order in space. Numerical results are reported to verify the error behavior.

*Mathematics subject classification:* 35Q55, 65M06, 65M12, 65M15.

*Key words:* Quantum Zakharov system, Subsonic limit, Compact finite difference method, Uniformly accurate, Error estimate.

## 1. Introduction

Consider the quantum Zakharov system (QZS) for describing the nonlinear interaction between high-frequency quantum Langmuir and low-frequency quantum ion-acoustic waves [12, 17],

$$\begin{cases} iE_t^\varepsilon + \Delta E^\varepsilon - \lambda^2 \Delta^2 E^\varepsilon = N^\varepsilon E^\varepsilon, \\ \varepsilon^2 N_{tt}^\varepsilon - \Delta N^\varepsilon + \lambda^2 \Delta^2 N^\varepsilon = \Delta |E^\varepsilon|^2, \\ E^\varepsilon(x, 0) = E_0(x), \quad N^\varepsilon(x, 0) = N_0^\varepsilon(x), \quad \partial_t N^\varepsilon(x, 0) = N_1^\varepsilon(x), \end{cases} \quad x \in \mathbb{R}^d, \quad t > 0, \quad (1.1)$$

where  $i^2 = -1$ ,  $E^\varepsilon : \mathbb{R}^{d+1} \rightarrow \mathbb{C}$  denotes the slowly varying envelope of the rapidly oscillatory electric field,  $N^\varepsilon : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  represents the low-frequency variation of the density of the ions. The dimensionless parameter  $\varepsilon \in (0, 1]$  is inversely proportional to the speed of ion sound, the quantum effect  $\lambda > 0$  is the ratio of the ion plasma and the temperature of electrons, and

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$E_0(x)$ ,  $N_0^\varepsilon(x)$ , and  $N_1^\varepsilon(x)$  are given initial datum. The QZS are deduced from a multiple time-scale method applied to a set of quantum hydrodynamic (QHD) equations under quasineutral assumption [17]. It extends the classical Zakharov system ( $\lambda = 0$ ) [31] to the quantum realm. When either the ion-plasma frequency is high or the electrons temperature is low, the quantum effect is non-negligible and can be characterized by the fourth-order perturbation with a quantum parameter  $\lambda$ . It can be applied for quantum decay and four-wave instabilities with relevant changes of the classical dispersion [12], or for the enhancement of modulational instabilities due to combination of partial coherence and quantum corrections [20]. We refer to [12, 16, 20] for more background in physics.

For the case  $\varepsilon = \mathcal{O}(1)$ , in recent years, there are many results on the QZS (1.1) from the physical and mathematical points of view [8, 10, 15–18, 22–24, 30]. Particularly, Haas and Shukla [17] pointed out that the quantum corrections induced qualitative and quantitative changes, inhibiting singularities and allowing for oscillations of the width of the Langmuir envelope field. Misra *et al.* [22] revealed that the system is destabilized via a supercritical hopf-bifurcation, and periodic, chaotic, and hyperchaotic behaviors of the Fourier-mode amplitudes are identified by the analysis of Lyapunov exponent spectra and the power spectrum. For the well-posedness, the QZS is locally well-posed in  $L^2$  data for dimension up to eight and globally well-posed for  $1 \leq d \leq 5$  [8], which is different from the classical ZS, while the local and global well-posedness of the corresponding Cauchy problem is known only for  $1 \leq d \leq 3$  [11, 13]. Moreover, the QZS is globally well-posed for initial data  $(E_0, N_0^\varepsilon, N_1^\varepsilon) \in H^k \times H^{k-1} \times H^{k-3}$  with  $k \geq 2$  and  $d = 1, 2, 3$  without any size constraints on the initial data [15]. This suggests that including some more physical effects in the equations which results as a more complicated system may make the mathematical understanding much easier.

For the numerical part, there are few results for the QZS (1.1). Xiao *et al.* [29] developed a conservative finite difference scheme for the modified ZS with high-order space fractional quantum correction. Recently, Baumstark and Schratz [3] presented a new class of asymptotic preserving trigonometric integrators for QZS, and the scheme converges to the classical ZS in the limit  $\lambda \rightarrow 0$  uniformly in the time discretization parameter. In [32], we proposed and analysed a highly accurate conservative method for solving the QZS, which is fourth-order accurate in space and second-order accurate in time. Zhang [33] developed a fully explicit and efficient method by applying a time splitting technique and an exponential integrator for time integration combined with the Fourier pseudospectral method in space for the QZS. Some interesting dynamical phenomena was also included.

For the QZS (1.1) in the subsonic limit regime, i.e.,  $\varepsilon \rightarrow 0^+$ , Fang *et al.* [11] proved the solution of the corresponding fourth-order Schrödinger part converges to the solution of quantum modified nonlinear Schrödinger equation (QM-NLSE)

$$\begin{cases} i\partial_t E - (-\Delta + \lambda^2 \Delta^2)E + I_\lambda(|E|^2)E = 0, & x \in \mathbb{R}^d, \quad t > 0, \\ E(x, 0) = E_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

and  $E^\varepsilon(x, t) \rightarrow E(x, t)$ , where  $I_\lambda = (I - \lambda^2 \Delta)^{-1}$  and  $I$  is the identity operator. Convergence rates of the subsonic limit regime from the QZS (1.1) to the QM-NLSE (1.2) and initial layers, as well as the propagation of oscillatory waves, have been rigorously investigated in the literatures [6, 7, 9, 15]. Fang *et al.* [9] studied the existence and the stability of the standing waves of the QZS (1.1) for  $1 \leq d \leq 3$ . The low-regularity global well-posedness of the subsonic limit and its semi-classical limit ( $\lambda \rightarrow 0$ ) were studied in [6]. Particularly the solution of the QZS exhibits highly oscillatory initial layers caused by the incompatibility of the initial data. This high oscillation

in time brings significant difficulties in devising numerical algorithms and establishing their error estimates, especially when  $\varepsilon \ll 1$ .

Along the numerical part of QZS in the subsonic limit regime, extensive numerical studies have been carried out for the ZS, i.e., when the quantum effect is absent ( $\lambda = 0$ ). Cai and Yuan [4] developed a linearly-implicit conservative finite difference scheme to the classical ZS in the subsonic regime, and obtained the  $\varepsilon$  dependent error bounds. Bao and Su [1] designed a uniformly accurate finite difference method and established rigorously its uniform error bounds for the classical ZS with  $\varepsilon \in (0, 1]$ . In [2], they further proposed a time splitting combined with exponential wave integrator sine pseudospectral method for the classical ZS with  $\varepsilon \in (0, 1]$ .

The aim of this paper is devoted to developing and analysing a conservative linearly-implicit fourth-order compact difference scheme for solving the QZS (1.1), whose efficiency has been widely verified in solving a large number of equations [5, 19, 26, 28, 34]. The proposed method is efficient to implement and only two independent linear systems are solved at each time step, which is very efficient. Furthermore, the scheme preserves the mass and energy which is of vital importance for long-time dynamics and stability.

An outline of this paper is as follows. In Section 2, we present a linearly-implicit compact difference scheme for the QZS (1.1). The conservative laws of the scheme are given in Section 3. In Section 4, the detailed error analysis of the scheme are discussed. Numerical experiments are given in Section 5 to demonstrate the accuracy of the scheme. Finally, a brief conclusion is drawn in Section 6.

## 2. Numerical Methods

For the simplicity of notation, we only deal with the scheme and analysis in one space dimension, i.e.,  $d = 1$  and extensions to higher dimensions are straightforward. The original problem is truncated on a bounded interval  $\Omega = (a, b)$  with zero Dirichlet boundary condition

$$\begin{cases} i\partial_t E^\varepsilon(x, t) + \partial_{xx} E^\varepsilon(x, t) - \lambda^2 \partial_x^4 E^\varepsilon(x, t) - N^\varepsilon(x, t) E^\varepsilon(x, t) = 0, & x \in \Omega, \quad t > 0, \\ \varepsilon^2 \partial_{tt} N^\varepsilon(x, t) - \partial_{xx} N^\varepsilon(x, t) + \lambda^2 \partial_x^4 N^\varepsilon(x, t) - \partial_{xx} |E^\varepsilon(x, t)|^2 = 0, & x \in \Omega, \quad t > 0, \\ E^\varepsilon(x, 0) = E_0(x), \quad N^\varepsilon(x, 0) = N_0^\varepsilon(x), \quad \partial_t N^\varepsilon(x, 0) = N_1^\varepsilon(x), & x \in \Omega, \\ E^\varepsilon(a, t) = E^\varepsilon(b, t) = 0, \quad N^\varepsilon(a, t) = N^\varepsilon(b, t) = 0, \\ \partial_{xx} E^\varepsilon(a, t) = \partial_{xx} E^\varepsilon(b, t) = 0, \quad \partial_{xx} N^\varepsilon(a, t) = \partial_{xx} N^\varepsilon(b, t) = 0. \end{cases} \quad (2.1)$$

We suppose the problem (2.1) possesses a unique solution which is smooth enough and the initial data  $N_1^\varepsilon$  satisfies the compatibility condition [14]

$$\int_a^b N_1^\varepsilon(x) dx = 0, \quad \sum_{k=1}^M N_1^\varepsilon(a + kh) = 0 \quad \text{for } h > 0 \quad \text{with } Mh = b - a, \quad M \in \mathbb{N}. \quad (2.2)$$

It is well-known that the QZS (2.1) conserves the mass

$$\|E^\varepsilon(\cdot, t)\|_{L^2(\Omega)} := \int_\Omega |E^\varepsilon(x, t)|^2 dx = \int_\Omega |E_0(x)|^2 dx = \|E^\varepsilon(\cdot, 0)\|_{L^2(\Omega)}, \quad (2.3)$$

and the energy

$$\mathcal{E}(t) := \int_\Omega \left( |\partial_x E^\varepsilon(x, t)|^2 + N^\varepsilon(x, t) |E^\varepsilon(x, t)|^2 + \frac{1}{2} \left( \varepsilon^2 |\partial_x U^\varepsilon(x, t)|^2 + |N^\varepsilon(x, t)|^2 \right) \right)$$

$$+\lambda^2|\partial_{xx}E^\varepsilon(x,t)|^2 + \frac{\lambda^2}{2}|\partial_x N^\varepsilon(x,t)|^2) dx \equiv \mathcal{E}(0) \tag{2.4}$$

with the potential function  $U^\varepsilon := U^\varepsilon(x,t) \in \mathbb{R}$  is defined by

$$-\partial_{xx}U^\varepsilon = N_t^\varepsilon, \quad U^\varepsilon|_{\partial\Omega} = 0. \tag{2.5}$$

In the subsonic limit regime, i.e.,  $\varepsilon \rightarrow 0^+$ , the QZS (2.1) reduces to the following QM-NLSE:

$$\begin{cases} i\partial_t E(x,t) + \Delta_\lambda E(x,t) + (I_\lambda |E(x,t)|^2) E(x,t) = 0, & x \in \Omega, \quad t > 0, \\ E(x,0) = E_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

where  $\Delta_\lambda = \partial_{xx} - \lambda^2 \partial_x^4$  and  $I_\lambda = (I - \lambda^2 \partial_{xx})^{-1}$ . Suppose that the initial data  $N_0^\varepsilon$  and  $N_1^\varepsilon$  satisfy

$$\begin{cases} N_0^\varepsilon(x) = -I_\lambda |E_0(x)|^2 + \varepsilon^\alpha w_0(x), & \alpha \geq 0, \quad x \in \Omega, \\ N_1^\varepsilon(x) = 2I_\lambda \left( \text{Im} \left( \Delta_\lambda E_0(x) \overline{E_0(x)} \right) \right) + \varepsilon^{\beta-1} w_1(x), & \beta \geq 0, \quad x \in \Omega, \end{cases} \tag{2.7}$$

where  $\alpha, \beta \geq 0$  are non-negative parameters describing the consistency between the initial data of QZS (2.1) and the initial value of QM-NLSE (2.6). Moreover, the initial data are usually classified into well-prepared initial data ( $\alpha \geq 2, \beta \geq 2$ ), less-ill prepared initial data ( $1 \leq \alpha, \beta < 2$ ) and ill-prepared initial data ( $0 \leq \alpha, \beta < 1$ ).

To achieve the difference scheme, choose time step  $\tau := \frac{T}{J}$  and mesh size  $h := \frac{b-a}{M}$  with  $J, M$  two positive integers. Denote the time steps and grid points as

$$t_k := k\tau, \quad k = 0, 1, \dots, J, \quad x_j := a + jh, \quad j = 0, 1, \dots, M.$$

Let  $E_j^k, E_j^{\varepsilon,k}$  and  $N_j^{\varepsilon,k}$  be the numerical approximations of the exact solutions  $E(x_j, t_k), E^\varepsilon(x_j, t_k)$  and  $N^\varepsilon(x_j, t_k)$ , respectively. Denote  $E^k, E^{\varepsilon,k}$  and  $N^{\varepsilon,k}$  by the corresponding numerical solution vectors at time  $t = t_k$ . As usual, we introduce the following finite difference operators:

$$\begin{aligned} \delta_t^+ v_j^k &= \frac{1}{\tau} (v_j^{k+1} - v_j^k), \quad \delta_t^- v_j^k = \delta_t^+ v_{j-1}^k, \quad \delta_t v_j^k = \frac{v_j^{k+1} - v_j^{k-1}}{2\tau}, \quad \delta_t^2 v_j^k = \delta_t^+ \delta_t^- v_j^k, \\ \delta_x^+ v_j^k &= \frac{1}{h} (v_{j+1}^k - v_j^k), \quad \delta_x^- v_j^k = \delta_x^+ v_{j-1}^k, \quad \delta_x v_j^k = \frac{v_{j+1}^k - v_{j-1}^k}{2h}, \quad \delta_x^2 v_j^k = \delta_x^+ \delta_x^- v_j^k, \\ \delta_x^4 v_j^k &= \delta_x^2 (\delta_x^2 v_j^k), \quad v_j^{k-\frac{1}{2}} = \frac{1}{2} (v_j^k + v_j^{k-1}), \quad v_j^{\bar{k}} = \frac{1}{2} (v_j^{k+1} + v_j^{k-1}). \end{aligned}$$

Define  $\mathbb{V}_h := \{v = \{v_j\} \mid 0 \leq j \leq M, v_0 = v_M = 0\}$ . Furthermore, we set  $\delta_x^2 v_0 = \delta_x^2 v_M = 0$  for  $v \in \mathbb{V}_h$  when involved in view of the boundary conditions in (2.1). For any grid functions  $u, v \in \mathbb{V}_h$ , denote the inner product and the norms as

$$\langle u, v \rangle = h \sum_{j=1}^{M-1} u_j \bar{v}_j, \quad \|v\| = \left( h \sum_{j=1}^{M-1} |v_j|^2 \right)^{\frac{1}{2}}, \quad \|v\|_\infty = \max_{1 \leq j \leq M-1} |v_j|,$$

where  $\bar{v}_j$  is the complex conjugate of  $v_j$ .

As shown in [19, 28, 32], the standard spatial fourth-order compact finite difference operator  $\mathcal{A}$  is defined as

$$\mathcal{A}v_j^k = \left( I + \frac{h^2}{12} \delta_x^2 \right) v_j^k = \frac{1}{12} (v_{j-1}^k + 10v_j^k + v_{j+1}^k).$$

One can see that  $\mathcal{A}v_{xx}(x_j, t_k) = \delta_x^2 v(x_j, t_k) + \mathcal{O}(h^4)$  for  $v(\cdot, t_k) \in C^6([a, b])$ . The corresponding matrix of the operator  $\mathcal{A}$  is  $A = \frac{1}{12}(10I + S)$ , where

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{(M-1) \times (M-1)}$$

and similarly  $H = A^{-1}$ ,  $D = \frac{1}{h^2}(S - 2I)$  correspond to the operator  $\mathcal{A}^{-1}$  and  $\delta_x^2$  defined on  $\mathbb{V}_h$ , respectively. The linearly-implicit compact finite difference scheme of QZS (2.1) reads for  $k \geq 1$  as

$$\begin{aligned} i\delta_t^- E_j^{\varepsilon,k} &= \left(-\mathcal{A}^{-1}\delta_x^2 + N_j^{\varepsilon,k-\frac{1}{2}}\right) E_j^{\varepsilon,k-\frac{1}{2}} + \lambda^2 (\mathcal{A}^{-1}\delta_x^2)^2 E_j^{\varepsilon,k-\frac{1}{2}}, & 1 \leq j \leq M-1, \\ \varepsilon^2 \delta_t^2 N_j^{\varepsilon,k} &= \mathcal{A}^{-1}\delta_x^2 N_j^{\varepsilon,\bar{k}} - \lambda^2 (\mathcal{A}^{-1}\delta_x^2)^2 N_j^{\varepsilon,\bar{k}} + \mathcal{A}^{-1}\delta_x^2 |E_j^{\varepsilon,k}|^2, & 1 \leq j \leq M-1, \\ E_0^{\varepsilon,k} = E_M^{\varepsilon,k} &= 0, \quad N_0^{\varepsilon,k} = N_M^{\varepsilon,k} = 0, \quad \delta_x^2 E_0^{\varepsilon,k} = \delta_x^2 E_M^{\varepsilon,k} = 0, \quad \delta_x^2 N_0^{\varepsilon,k} = \delta_x^2 N_M^{\varepsilon,k} = 0, \end{aligned} \tag{2.8}$$

i.e., we find  $N^{\varepsilon,k}, E^{\varepsilon,k}, p^{\varepsilon,k}, q^{\varepsilon,k} \in \mathbb{V}_h$  such that

$$i\delta_t^- E^{\varepsilon,k} = (-HD + \lambda^2 H^2 D^2 + N^{\varepsilon,k-\frac{1}{2}}) E^{\varepsilon,k-\frac{1}{2}}, \quad k \geq 1, \tag{2.9a}$$

$$\varepsilon^2 \delta_t^2 N^{\varepsilon,k} = (HD - \lambda^2 H^2 D^2) N^{\varepsilon,\bar{k}} + HD |E^{\varepsilon,k}|^2, \quad k \geq 1, \tag{2.9b}$$

where we have used the property that  $HD = DH$ . From now on, we use the notation  $Xv = X(v_1, \dots, v_{M-1})^T$  for  $X \in \mathbb{C}^{(M-1) \times (M-1)}$  and  $v \in \mathbb{V}_h$ . Furthermore, we set the initial conditions

$$E_j^{\varepsilon,0} = E_0(x_j), \quad N_j^{\varepsilon,0} = N_0^\varepsilon(x_j) = -I_\lambda(|E_0|^2)(x_j) + \varepsilon^\alpha w_0(x_j). \tag{2.10}$$

To implement the three-level scheme (2.9), we need to fix the value  $N^{\varepsilon,1}$  at the first step. According to Taylor expansion, we get

$$\begin{aligned} N^\varepsilon(x_j, \tau) &\approx N_0^\varepsilon(x_j) + \tau N_t^\varepsilon(x_j, 0) + \frac{\tau^2}{2} N_{tt}^\varepsilon(x_j, 0) \\ &= N_0^\varepsilon(x_j) + \tau N_1^\varepsilon(x_j) + \frac{\tau^2}{2} N_{tt}^\varepsilon(x_j, 0), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} N_1^\varepsilon(x_j) &= 2I_\lambda (\text{Im}(\Delta_\lambda E_0 \overline{E_0})) (x_j) + \varepsilon^{\beta-1} w_1(x_j), \\ \partial_{tt} N^\varepsilon(x_j, 0) &= \frac{1}{\varepsilon^2} \partial_{xx} \left( (I - \lambda^2 \partial_{xx}) N^\varepsilon(x_j, 0) + |E^\varepsilon(x_j, 0)|^2 \right) = \varepsilon^{\alpha-2} \Delta_\lambda w_0(x_j). \end{aligned}$$

This approximation suggests that  $|N_j^{\varepsilon,1}| \sim \tau \varepsilon^{\beta-1} + \tau^2 \varepsilon^{\alpha-2}$ , which requires very small  $\tau$  to bound the value of  $N^{\varepsilon,1}$  when  $\varepsilon \ll 1$  and  $0 \leq \beta < 1$  or  $0 \leq \alpha < 2$ . To remedy this, replacing  $\tau/\varepsilon$  in (2.11) by the trigonometric functions  $\sin(\tau/\varepsilon)$ , which is uniformly bounded for all  $\tau$  and  $\varepsilon \in (0, 1]$ , we get a modified approximation for  $N^\varepsilon(x_j, \tau)$  by  $N_j^{\varepsilon,1}$

$$\begin{aligned} N_j^{\varepsilon,1} &= N_j^{\varepsilon,0} + 2\tau I_\lambda (\text{Im}(\Delta_\lambda E_0 \overline{E_0})) (x_j) + \varepsilon^\beta \sin\left(\frac{\tau}{\varepsilon}\right) w_1(x_j) \\ &\quad + 2\varepsilon^\alpha \sin^2\left(\frac{\tau}{2\varepsilon}\right) \Delta_\lambda w_0(x_j). \end{aligned} \tag{2.12}$$

Noticing that the scheme (2.9) is efficient to implement, and it suffices to solve two linear systems at each time step. Moreover, it is easy to derive the uniqueness of the solution by standard analysis [14].

**Remark 2.1.** To get  $v = I_\lambda(u)$  in (2.10) and (2.12), we can solve the linear equation  $v - \lambda^2 v_{xx} = u$  with homogeneous boundary condition by sine pseudospectral method, which results as an approximation with spectral accuracy, or replace the operator  $I_\lambda$  by a fourth-order approximation  $\delta_I = (I - \lambda^2 HD)^{-1}$ .

### 3. Conservative Properties

In this section, we present the conservation properties for the difference scheme (2.9). We start with the following lemmas.

**Lemma 3.1.** *For any grid function  $u, v \in \mathbb{V}_h$ , it holds that*

$$\langle \delta_x^2 u, v \rangle = \langle Du, v \rangle = -[\delta_x^+ u, \delta_x^+ v],$$

where  $[p, q] = h \sum_{j=0}^{M-1} p_j \overline{q_j}$ .

**Lemma 3.2 ([5]).** *The matrix  $-HD$  is symmetric positive definite and for any grid function  $u, v \in \mathbb{V}_h$ , we have*

$$-\langle \mathcal{A}^{-1} \delta_x^2 u, v \rangle = -\langle HDu, v \rangle = \langle Ru, Rv \rangle, \quad \langle (\mathcal{A}^{-1} \delta_x^2)^2 u, v \rangle = \langle \mathcal{A}^{-1} \delta_x^2 u, \mathcal{A}^{-1} \delta_x^2 v \rangle,$$

where  $R$  is obtained by Cholesky decomposition for  $-HD$ , denoted as  $-HD = R^T R$ .

**Lemma 3.3.** *For any  $v \in \mathbb{V}_h$ , we have*

$$\|\delta_x^+ v\|^2 \leq -\langle \mathcal{A}^{-1} \delta_x^2 v, v \rangle = \|Rv\|^2 \leq 2\|\delta_x^+ v\|^2, \quad \|v\| \leq \|\mathcal{A}^{-1} v\| = \|Hv\| \leq \frac{3}{2}\|v\|, \quad (3.1)$$

where  $\|\delta_x^+ v\|^2 = [\delta_x^+ v, \delta_x^+ v] = h \sum_{j=0}^{M-1} |\delta_x^+ v_j|^2$ .

*Proof.* Denote the eigenvalues and the corresponding eigenvectors of  $S$  by  $\lambda_S^j$  and  $y^j$ , respectively. It is well known that  $\lambda_S^j = 2 \cos(j\pi/M)$  for  $j = 1, \dots, M-1$ . Hence, by definition of  $D$  and  $H$ , we have the eigenvalues of  $D$  and  $H$  as

$$\begin{aligned} \lambda_D^j &= \frac{\lambda_S^j - 2}{h^2} = -\frac{4}{h^2} \sin^2\left(\frac{j\pi}{2M}\right), \\ \lambda_H^j &= \frac{12}{10 + \lambda_S^j} = \frac{6}{5 + \cos(j\pi/M)}, \\ \lambda_{HD}^j &= \lambda_H^j \lambda_D^j = -\frac{24 \sin^2(j\pi/2M)}{h^2(5 + \cos(j\pi/M))} \end{aligned}$$

with the same eigenvectors  $y^j$ . Noticing that

$$2\lambda_D^j \leq \lambda_{HD}^j \leq \lambda_D^j,$$

$D$  and  $HD$  share the same eigenvectors, we get  $D - HD$  and  $HD - 2D$  are both symmetric positive definite matrices. Thus for any  $v \in \mathbb{V}_h$ , we have

$$\langle Dv, v \rangle \geq \langle HDv, v \rangle \geq \langle 2Dv, v \rangle,$$

which leads to the first inequality of (3.1) by applying Lemmas 3.1 and 3.2. The second inequality can be easily derived by noticing that  $1 \leq \lambda_H^j \leq \frac{3}{2}$  and the fact that  $H$  is symmetric positive definite.  $\square$

Denote  $U_j^{\varepsilon, k+\frac{1}{2}}$  ( $1 \leq j \leq M-1, k \geq 0$ ) by the solution of

$$-\mathcal{A}^{-1} \delta_x^2 U_j^{\varepsilon, k+\frac{1}{2}} = \delta_t^+ N_j^{\varepsilon, k}, \quad \text{i.e.,} \quad -HDU^{\varepsilon, k+\frac{1}{2}} = \delta_t^+ N^{\varepsilon, k} \quad (3.2)$$

with boundary condition  $U_0^{\varepsilon, k+\frac{1}{2}} = U_M^{\varepsilon, k+\frac{1}{2}} = 0$ . Based on the discrete Sobolev inequality, it follows that

$$\|U^{\varepsilon, k+\frac{1}{2}}\| \lesssim \|\delta_x^+ U^{\varepsilon, k+\frac{1}{2}}\| \lesssim \|\delta_t^+ N^{\varepsilon, k}\|. \quad (3.3)$$

**Theorem 3.1.** *The difference scheme (2.9) conserves the mass*

$$\|E^{\varepsilon, k}\| \equiv \|E^{\varepsilon, 0}\|, \quad k \geq 1, \quad (3.4)$$

and energy

$$\begin{aligned} \mathcal{E}^k &= \|RE^{\varepsilon, k+1}\|^2 + \|RE^{\varepsilon, k}\|^2 + \lambda^2 \left( \|\mathcal{A}^{-1} \delta_x^2 E^{\varepsilon, k+1}\|^2 + \|\mathcal{A}^{-1} \delta_x^2 E^{\varepsilon, k}\|^2 \right) + \varepsilon^2 \|RU^{\varepsilon, k+\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2} \left( \|N^{\varepsilon, k+1}\|^2 + \|N^{\varepsilon, k}\|^2 \right) + \frac{\lambda^2}{2} \left( \|RN^{\varepsilon, k+1}\|^2 + \|RN^{\varepsilon, k}\|^2 \right) \\ &\quad + \left\langle N^{\varepsilon, k+\frac{1}{2}}, |E^{\varepsilon, k+1}|^2 + |E^{\varepsilon, k}|^2 \right\rangle \equiv \mathcal{E}^0, \quad k \geq 1. \end{aligned} \quad (3.5)$$

*Proof.* Computing the inner product of (2.9a) with  $E^{\varepsilon, k-\frac{1}{2}}$ , and taking the imaginary parts, we get

$$\|E^{\varepsilon, k}\| = \|E^{\varepsilon, k-1}\| = \dots = \|E^{\varepsilon, 0}\|.$$

Computing the inner product of (2.9a) with  $E^{\varepsilon, k} - E^{\varepsilon, k-1}$ , applying Lemma 3.2 and taking the real parts yields

$$\begin{aligned} &\frac{1}{2} \|RE^{\varepsilon, k}\|^2 - \frac{1}{2} \|RE^{\varepsilon, k-1}\|^2 + \frac{\lambda^2}{2} \|\mathcal{A}^{-1} \delta_x^2 E^{\varepsilon, k}\|^2 - \frac{\lambda^2}{2} \|\mathcal{A}^{-1} \delta_x^2 E^{\varepsilon, k-1}\|^2 \\ &\quad + \frac{1}{2} \left\langle N^{\varepsilon, k-\frac{1}{2}}, |E^{\varepsilon, k}|^2 - |E^{\varepsilon, k-1}|^2 \right\rangle = 0, \end{aligned} \quad (3.6)$$

which indicates

$$\begin{aligned} &\frac{1}{2} \|RE^{\varepsilon, k+1}\|^2 - \frac{1}{2} \|RE^{\varepsilon, k-1}\|^2 + \frac{\lambda^2}{2} \|\mathcal{A}^{-1} \delta_x^2 E^{\varepsilon, k+1}\|^2 - \frac{\lambda^2}{2} \|\mathcal{A}^{-1} \delta_x^2 E^{\varepsilon, k-1}\|^2 \\ &\quad + \frac{1}{2} \left\langle N^{\varepsilon, k+\frac{1}{2}}, |E^{\varepsilon, k+1}|^2 - |E^{\varepsilon, k}|^2 \right\rangle + \frac{1}{2} \left\langle N^{\varepsilon, k-\frac{1}{2}}, |E^{\varepsilon, k}|^2 - |E^{\varepsilon, k-1}|^2 \right\rangle = 0. \end{aligned} \quad (3.7)$$

Taking the inner product of  $\tau(U^{\varepsilon, k-\frac{1}{2}} + U^{\varepsilon, k+\frac{1}{2}})$  with (2.9b), and making use of (3.2), we get

$$\begin{aligned} &\varepsilon^2 \|RU^{\varepsilon, k+\frac{1}{2}}\|^2 - \varepsilon^2 \|RU^{\varepsilon, k-\frac{1}{2}}\|^2 + \frac{1}{2} \|N^{\varepsilon, k+1}\|^2 - \frac{1}{2} \|N^{\varepsilon, k-1}\|^2 \\ &\quad + \frac{\lambda^2}{2} \|RN^{\varepsilon, k+1}\|^2 - \frac{\lambda^2}{2} \|RN^{\varepsilon, k-1}\|^2 + \left\langle |E^{\varepsilon, k}|^2, N^{\varepsilon, k+1} - N^{\varepsilon, k-1} \right\rangle = 0. \end{aligned} \quad (3.8)$$

Finally, (3.7)+ $\frac{1}{2}$  · (3.8) yields that

$$\begin{aligned} & \frac{1}{2} \|RE^{\varepsilon,k+1}\|^2 + \frac{\lambda^2}{2} \|\mathcal{A}^{-1}\delta_x^2 E^{\varepsilon,k+1}\|^2 + \frac{\varepsilon^2}{2} \|RU^{\varepsilon,k+\frac{1}{2}}\|^2 + \frac{1}{4} \|N^{\varepsilon,k+1}\|^2 \\ & + \frac{\lambda^2}{4} \|RN^{\varepsilon,k+1}\|^2 + \frac{1}{2} \left\langle N^{\varepsilon,k+\frac{1}{2}}, |E^{\varepsilon,k+1}|^2 + |E^{\varepsilon,k}|^2 \right\rangle \\ = & \frac{1}{2} \|RE^{\varepsilon,k-1}\|^2 + \frac{\lambda^2}{2} \|\mathcal{A}^{-1}\delta_x^2 E^{\varepsilon,k-1}\|^2 + \frac{\varepsilon^2}{2} \|RU^{\varepsilon,k-\frac{1}{2}}\|^2 + \frac{1}{4} \|N^{\varepsilon,k-1}\|^2 \\ & + \frac{\lambda^2}{4} \|RN^{\varepsilon,k-1}\|^2 + \frac{1}{2} \left\langle N^{\varepsilon,k-\frac{1}{2}}, |E^{\varepsilon,k}|^2 + |E^{\varepsilon,k-1}|^2 \right\rangle, \end{aligned}$$

i.e.,  $\mathcal{E}^k = \mathcal{E}^{k-1}$ . The proof is complete. □

### 4. Error Analysis

Let  $T^*$  be the maximum common existence time for the solution  $(E^\varepsilon(x, t), N^\varepsilon(x, t))$  to the QZS (2.1) and the solution  $E(x, t)$  to the QM-NLSE (2.6) with homogeneous Dirichlet boundary conditions. For  $0 < T < T^*$ , suppose that the exact solutions  $(E^\varepsilon(x, t), N^\varepsilon(x, t))$  and  $E(x, t)$  are smooth enough. More precisely, according to the results in [21, 25], we assume

$$\begin{aligned} & \|E^\varepsilon\|_{L^\infty([0,T];W^{10,\infty}(\Omega))} + \|E^\varepsilon\|_{W^{1,\infty}([0,T];W^{6,\infty}(\Omega))} \\ & + \varepsilon^{1-\alpha^*} \|E^\varepsilon\|_{W^{2,\infty}(0,T;W^{6,\infty}(\Omega))} + \varepsilon^{2-\alpha^\dagger} \|E^\varepsilon\|_{W^{3,\infty}([0,T];W^{2,\infty}(\Omega))} \lesssim 1, \\ (A) \quad & \|N^\varepsilon\|_{L^\infty([0,T];W^{9,\infty}(\Omega))} + \varepsilon^{1-\alpha^*} \|N^\varepsilon\|_{W^{1,\infty}([0,T];W^{8,\infty}(\Omega))} \\ & + \varepsilon^{2-\alpha^\dagger} \|N^\varepsilon\|_{W^{2,\infty}([0,T];W^{5,\infty}(\Omega))} + \varepsilon^{3-\alpha^\dagger} \|N^\varepsilon\|_{W^{3,\infty}([0,T];W^{4,\infty}(\Omega))} \\ & + \varepsilon^{4-\alpha^\dagger} \|N^\varepsilon\|_{W^{4,\infty}([0,T];W^{1,\infty}(\Omega))} + \varepsilon^{5-\alpha^\dagger} \|N^\varepsilon\|_{W^{5,\infty}([0,T];L^\infty(\Omega))} \lesssim 1 \end{aligned}$$

with the convergence

$$(B) \quad \|E^\varepsilon - E\|_{L^\infty([0,T];H^2(\Omega))} \lesssim \varepsilon^{1+\alpha^*}, \quad \|N^\varepsilon + I_\lambda |E|^2\|_{L^\infty([0,T];H^1(\Omega))} \lesssim \varepsilon^{\alpha^\dagger},$$

as well as the initial data

$$(C) \quad \|N_0\|_{W^{2,\infty}(\Omega)} + \|E_0\|_{W^{4,\infty}(\Omega)} + \|w_0\|_{W^{4,\infty}(\Omega)} + \|w_1\|_{L^\infty(\Omega)} \lesssim 1,$$

where

$$\alpha^* = \min\{1, \alpha, \beta\}, \quad \alpha^\dagger = \min\{\alpha, \beta, 2\}. \tag{4.1}$$

In view of (2.12), (3.6), Taylor expansion and assumption (C), we can easily prove that

$$\varepsilon \|\delta_x^+ U^{\varepsilon,\frac{1}{2}}\| \lesssim \varepsilon \|\delta_t^- N^{\varepsilon,1}\| \lesssim 1, \quad \|N^{\varepsilon,0}\| + \|N^{\varepsilon,1}\| \lesssim 1,$$

and

$$\begin{aligned} & \|RE^{\varepsilon,1}\|^2 + \lambda^2 \|\mathcal{A}^{-1}\delta_x^2 E^{\varepsilon,1}\|^2 + \left\langle N^{\varepsilon,\frac{1}{2}}, |E^{\varepsilon,1}|^2 \right\rangle \\ = & \|RE^{\varepsilon,0}\|^2 + \lambda^2 \|\mathcal{A}^{-1}\delta_x^2 E^{\varepsilon,0}\|^2 + \left\langle N^{\varepsilon,\frac{1}{2}}, |E^{\varepsilon,0}|^2 \right\rangle \lesssim 1, \end{aligned}$$

which suggests that  $\mathcal{E}^k = \mathcal{E}^0$  is bounded. Thus, in one space dimension, the *a priori* bounds for  $E^{\varepsilon,k}$  can be established by a standard argument [32],

$$\|E^{\varepsilon,k}\|_\infty \leq \sqrt{2\|E^{\varepsilon,k}\|\|\delta_x^+ E^{\varepsilon,k}\|} \leq \sqrt{2\|E^{\varepsilon,k}\|\|RE^{\varepsilon,k}\|} \leq C_a, \quad k \geq 0. \tag{4.2}$$

Denote the error functions

$$e_j^{\varepsilon,k} = E^\varepsilon(x_j, t_k) - E_j^{\varepsilon,k}, \quad n_j^{\varepsilon,k} = N^\varepsilon(x_j, t_k) - N_j^{\varepsilon,k}, \quad 1 \leq j \leq M. \tag{4.3}$$



#### 4.1. Main results

We present the corresponding error estimates of the difference scheme (2.9)-(2.12).

**Theorem 4.1.** *Under the assumptions (A), (B) and (C), there exist  $h_0 > 0$  and  $\tau_0 > 0$  independent of  $\varepsilon \in (0, 1]$  such that, when  $h \in (0, h_0]$  and  $\tau \in (0, \tau_0]$ , the difference scheme (2.9)-(2.12) with well-prepared and less-ill-prepared initial data ( $\alpha, \beta \geq 1$ ) satisfy the following two error estimates:*

$$\|e^{\varepsilon,k}\| + \|Re^{\varepsilon,k}\| + \lambda\|\mathcal{A}^{-1}\delta_x^2 e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| + \lambda\|Rn^{\varepsilon,k}\| \lesssim h^4 + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \quad 0 \leq k \leq \frac{T}{\tau}, \quad (4.4)$$

$$\|e^{\varepsilon,k}\| + \|Re^{\varepsilon,k}\| + \lambda\|\mathcal{A}^{-1}\delta_x^2 e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| + \lambda\|Rn^{\varepsilon,k}\| \lesssim h^4 + \tau^2 + \varepsilon^{\alpha^\dagger}, \quad 0 \leq k \leq \frac{T}{\tau}. \quad (4.5)$$

Furthermore, by taking the minimum, we obtain the  $\varepsilon$ -independent convergence rate

$$\begin{aligned} & \|e^{\varepsilon,k}\| + \|Re^{\varepsilon,k}\| + \lambda\|\mathcal{A}^{-1}\delta_x^2 e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| + \lambda\|Rn^{\varepsilon,k}\| \\ & \lesssim h^4 + \tau^{\frac{2\alpha^\dagger}{3}}, \quad 0 \leq k \leq \frac{T}{\tau}. \end{aligned} \quad (4.6)$$

**Theorem 4.2.** *Under the assumptions (A), (B) and (C), there exist  $h_0 > 0$  and  $\tau_0 > 0$  independent of  $\varepsilon \in (0, 1]$  such that, when  $h \in (0, h_0]$  and  $\tau \in (0, \tau_0]$ , the difference scheme (2.9)-(2.12) with ill-prepared initial data ( $\alpha, \beta \in [0, 1)$ ) satisfy the following error estimate:*

$$\begin{aligned} & \|e^{\varepsilon,k}\| + \|Re^{\varepsilon,k}\| + \lambda\|\mathcal{A}^{-1}\delta_x^2 e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| + \lambda\|Rn^{\varepsilon,k}\| \\ & \lesssim \frac{h^4}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^*}}, \quad 0 \leq k \leq \frac{T}{\tau}. \end{aligned} \quad (4.7)$$

In the subsequent discussion, we will prove the convergence of the developed scheme for solving the QZS (2.1). Denote

$$([v]_j)^{k-\frac{1}{2}} = \frac{1}{2}(v(x_j, t_k) + v(x_j, t_{k-1})), \quad ([v]_j)^{\bar{k}} = \frac{1}{2}(v(x_j, t_{k+1}) + v(x_j, t_{k-1})).$$

Define the local truncation error  $\eta^{\varepsilon,k}, \zeta^{\varepsilon,k} \in \mathbb{V}_h$  of the scheme (2.9)-(2.12) for  $k \geq 1$  and  $1 < j < M$  as

$$\begin{aligned} \eta_j^{\varepsilon,k} & := i\delta_t^- E^\varepsilon(x_j, t_k) + \mathcal{A}^{-1}\delta_x^2 ([E^\varepsilon]_j)^{k-\frac{1}{2}} - \lambda^2 (\mathcal{A}^{-1}\delta_x^2)^2 ([E^\varepsilon]_j)^{k-\frac{1}{2}} - ([N^\varepsilon]_j)^{k-\frac{1}{2}} ([E^\varepsilon]_j)^{k-\frac{1}{2}}, \\ \zeta_j^{\varepsilon,k} & := \varepsilon^2 \delta_t^2 N^\varepsilon(x_j, t_k) - \mathcal{A}^{-1}\delta_x^2 ([N^\varepsilon]_j)^{\bar{k}} + \lambda^2 (\mathcal{A}^{-1}\delta_x^2)^2 ([N^\varepsilon]_j)^{\bar{k}} - \mathcal{A}^{-1}\delta_x^2 |E^\varepsilon(x_j, t_k)|^2. \end{aligned} \quad (4.8)$$

**Lemma 4.1 (Local truncation error).** *Under assumption (A), we have*

$$\begin{aligned} & \|\eta^{\varepsilon,k}\| + \|R\eta^{\varepsilon,k}\| + \|\mathcal{A}^{-1}\delta_x^2 \eta^{\varepsilon,k}\| + \|\zeta^{\varepsilon,k}\| + \|R\zeta^{\varepsilon,k}\| \lesssim h^4 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}, \quad 1 \leq k \leq \frac{T}{\tau}, \\ & \|\delta_t \zeta^{\varepsilon,k}\| \lesssim \frac{h^4}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \quad 2 \leq k \leq \frac{T}{\tau} - 1. \end{aligned} \quad (4.9)$$

*Proof.* Using Taylor expansion, we derive

$$\begin{aligned} \eta_j^{\varepsilon,k} & = \frac{i\tau^2}{8} \int_0^1 \int_0^\theta \int_{-s}^s \partial_t^3 E^\varepsilon \left( x_j, \frac{\sigma\tau}{2} + t_{k-\frac{1}{2}} \right) d\sigma ds d\theta \\ & \quad + \frac{\tau^2}{8} \int_0^1 \int_{-\theta}^\theta \partial_t^2 \partial_x^2 E^\varepsilon \left( x_j, \frac{s\tau}{2} + t_{k-\frac{1}{2}} \right) ds d\theta \end{aligned}$$

$$\begin{aligned}
 & + \mathcal{A}^{-1} \delta_x^2 ([E^\varepsilon]_j)^{k-\frac{1}{2}} - ([\partial_x^2 E^\varepsilon]_j)^{k-\frac{1}{2}} \\
 & - \frac{\tau^2 \lambda^2}{8} \int_0^1 \int_{-\theta}^\theta \partial_t^2 \partial_x^4 E^\varepsilon \left( x_j, \frac{s\tau}{2} + t_{k-\frac{1}{2}} \right) dsd\theta \\
 & - \lambda^2 \left( \mathcal{A}^{-2} \delta_x^4 ([E^\varepsilon]_j)^{k-\frac{1}{2}} - ([\partial_x^4 E^\varepsilon]_j)^{k-\frac{1}{2}} \right) \\
 & - \frac{\tau^2}{8} E^\varepsilon(x_j, t_{k-\frac{1}{2}}) \int_0^1 \int_{-\theta}^\theta N_{tt}^\varepsilon \left( x_j, \frac{s\tau}{2} + t_{k-\frac{1}{2}} \right) dsd\theta \\
 & - \frac{\tau^2}{16} (N^\varepsilon(x_j, t_{k-1}) + N^\varepsilon(x_j, t_k)) \int_0^1 \int_{-\theta}^\theta \partial_t^2 E^\varepsilon \left( x_j, \frac{s\tau}{2} + t_{k-\frac{1}{2}} \right) dsd\theta.
 \end{aligned}$$

Noticing that

$$\begin{aligned}
 & \mathcal{A} \left( \mathcal{A}^{-1} \delta_x^2 E^\varepsilon(x_j, t_k) - \partial_x^2 E^\varepsilon(x_j, t_k) \right) \\
 & = \delta_x^2 E^\varepsilon(x_j, t_k) - \mathcal{A} \partial_x^2 E^\varepsilon(x_j, t_k) = -\frac{h^4}{240} \partial_x^6 E^\varepsilon(\xi_j, t_k),
 \end{aligned}$$

for some  $\xi_j \in (x_{j-1}, x_{j+1})$ , one easily obtains

$$\begin{aligned}
 & \left| \mathcal{A}^{-1} \delta_x^2 ([E^\varepsilon]_j)^{k-\frac{1}{2}} - ([\partial_x^2 E^\varepsilon]_j)^{k-\frac{1}{2}} \right| \lesssim h^4 \|\partial_x^6 E^\varepsilon\|_{L^\infty(\Omega_T)}, \\
 & \left| \mathcal{A}^{-2} \delta_x^4 ([E^\varepsilon]_j)^{k-\frac{1}{2}} - ([\partial_x^4 E^\varepsilon]_j)^{k-\frac{1}{2}} \right| \lesssim h^4 \|\partial_x^8 E^\varepsilon\|_{L^\infty(\Omega_T)}.
 \end{aligned}$$

Using assumption (A) and Lemma 3.3, we derive that

$$\begin{aligned}
 \|\eta^{\varepsilon,k}\| & \lesssim h^4 \left( \|\partial_x^6 E^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_x^8 E^\varepsilon\|_{L^\infty} \right) \\
 & \quad + \tau^2 \left( \|\partial_t^3 E^\varepsilon\|_{L^\infty} + \|\partial_t^2 \partial_x^2 E^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_t^2 \partial_x^4 E^\varepsilon\|_{L^\infty} \right. \\
 & \quad \left. + \|N^\varepsilon\|_{L^\infty} \|\partial_t^2 E^\varepsilon\|_{L^\infty} + \|E^\varepsilon\|_{L^\infty} \|\partial_t^2 N^\varepsilon\|_{L^\infty} \right) \\
 & \lesssim h^4 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \zeta_j^{\varepsilon,k} & = \varepsilon^2 \tau^2 \int_0^1 \int_0^\theta \int_0^s \int_{-\sigma}^\sigma \partial_t^4 N^\varepsilon(x_j, t_k + z\tau) dz d\sigma dsd\theta \\
 & \quad - \frac{\tau^2}{2} \int_0^1 \int_{-\theta}^\theta \partial_t^2 \partial_x^2 N^\varepsilon(x_j, s\tau + t_k) dsd\theta \\
 & \quad - \left( \mathcal{A}^{-1} \delta_x^2 ([N^\varepsilon]_j)^{\bar{k}} - ([\partial_x^2 N^\varepsilon]_j)^{\bar{k}} \right) \\
 & \quad + \frac{\tau^2 \lambda^2}{2} \int_0^1 \int_{-\theta}^\theta \partial_t^2 \partial_x^4 N^\varepsilon(x_j, s\tau + t_k) dsd\theta \\
 & \quad + \lambda^2 \left( \mathcal{A}^{-2} \delta_x^4 ([N^\varepsilon]_j)^{\bar{k}} - ([\partial_x^4 N^\varepsilon]_j)^{\bar{k}} \right) \\
 & \quad - \left( \mathcal{A}^{-1} \delta_x^2 |E^\varepsilon(x_j, t_k)|^2 - \partial_x^2 |E^\varepsilon(x_j, t_k)|^2 \right),
 \end{aligned}$$

and

$$\|\zeta^{\varepsilon,k}\| \lesssim h^4 \left( \|\partial_x^6 N^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_x^8 N^\varepsilon\|_{L^\infty} + \left\| \partial_x^6 |E^\varepsilon|^2 \right\|_{L^\infty} \right)$$

$$\begin{aligned} & + \tau^2 \left( \varepsilon^2 \|\partial_t^4 N^\varepsilon\|_{L^\infty} + \|\partial_t^2 \partial_x^2 N^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_t^2 \partial_x^4 N^\varepsilon\|_{L^\infty} \right) \\ & \lesssim h^4 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}. \end{aligned}$$

Applying  $\delta_x^+$  and  $\delta_t$  to  $\eta_j^{\varepsilon,k}$  and  $\zeta^{\varepsilon,k}$  respectively, we get

$$\begin{aligned} \|\delta_x^+ \eta^{\varepsilon,k}\| & \lesssim h^4 \left( \|\partial_x^7 E^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_x^9 E^\varepsilon\|_{L^\infty} \right) \\ & + \tau^2 \left( \|\partial_t^3 E_x^\varepsilon\|_{L^\infty} + \|\partial_t^2 \partial_x^3 E^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_t^2 \partial_x^5 E^\varepsilon\|_{L^\infty} \right. \\ & \quad + \|N^\varepsilon\|_{L^\infty} \|E_{xtt}^\varepsilon\|_{L^\infty} + \|E^\varepsilon\|_{L^\infty} \|N_{xtt}^\varepsilon\|_{L^\infty} \\ & \quad \left. + \|N_x^\varepsilon\|_{L^\infty} \|E_{tt}^\varepsilon\|_{L^\infty} + \|E_x^\varepsilon\|_{L^\infty} \|N_{tt}^\varepsilon\|_{L^\infty} \right) \\ & \lesssim h^4 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}, \\ \|\delta_t \zeta^{\varepsilon,k}\| & \lesssim h^4 \left( \|\partial_x^6 N_t^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_x^8 N_t^\varepsilon\|_{L^\infty} + \|\partial_t \partial_x^6 |E^\varepsilon|^2\|_{L^\infty} \right) \\ & + \tau^2 \left( \varepsilon^2 \|\partial_t^5 N^\varepsilon\|_{L^\infty} + \|\partial_t^3 \partial_x^2 N^\varepsilon\|_{L^\infty} + \lambda^2 \|\partial_t^3 \partial_x^4 N^\varepsilon\|_{L^\infty} \right) \\ & \lesssim \frac{h^4}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \end{aligned}$$

hence we get

$$\|\delta_x^+ \eta^{\varepsilon,k}\| \lesssim h^4 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}$$

by using (3.1). Similarly, we obtain the estimates for  $\|\delta_x^+ \eta^{\varepsilon,k}\|$  and  $\|\delta_x^+ \zeta^{\varepsilon,k}\|$ , hence for  $\|\mathcal{A}^{-1} \delta_x^+ \eta^{\varepsilon,k}\|$  and  $\|R\zeta^{\varepsilon,k}\|$  by noticing (3.1).  $\square$

**Lemma 4.2.** *Under the assumptions (A) and (C), the first step errors of the scheme (2.9)-(2.12) satisfy*

$$\|n^{\varepsilon,1}\| \lesssim \frac{\tau^3}{\varepsilon^{3-\alpha^\dagger}}, \quad \|\delta_t^+ n^{\varepsilon,0}\| \lesssim \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}. \quad (4.10)$$

*Proof.* It follows from (2.11), (2.12) and Taylor expansion that

$$\begin{aligned} n_j^{\varepsilon,1} & = N_0^\varepsilon(x_j) + \tau \partial_t N^\varepsilon(x_j, 0) + \frac{\tau^2}{2} \partial_{tt} N^\varepsilon(x_j, 0) + \frac{\tau^3}{6} \partial_t^3 N^\varepsilon(x_j, s_j) \\ & - \left[ N_0^\varepsilon(x_j) + 2\tau I_\lambda \left( \text{Im} \left( \Delta_\lambda E_0 \overline{E_0} \right) \right) (x_j) + \varepsilon^\beta \sin\left(\frac{\tau}{\varepsilon}\right) w_1(x_j) + 2\varepsilon^\alpha \sin^2\left(\frac{\tau}{2\varepsilon}\right) \delta_\lambda w_0(x_j) \right] \\ & = \frac{\tau^3}{6} \partial_t^3 N^\varepsilon(x_j, s_j) + \varepsilon^\beta \left( \sin\left(\frac{\tau}{\varepsilon}\right) - \frac{\tau}{\varepsilon} \right) w_1(x_j) + \left( 2\varepsilon^\alpha \sin^2\left(\frac{\tau}{2\varepsilon}\right) - \frac{\tau^2}{2} \varepsilon^{\alpha-2} \right) \Delta_\lambda w_0(x_j) \\ & = \frac{\tau^3}{6} \partial_t^3 N^\varepsilon(x_j, s_j) - \frac{\tau^3 \cos(s)}{6} \varepsilon^{\beta-3} w_1(x_j) - \frac{\tau^3 \sin(\theta)}{6} \varepsilon^{\alpha-3} \Delta_\lambda w_0(x_j), \end{aligned}$$

where  $s_j \in (0, \tau)$  and  $s, \theta \in (0, \tau/\varepsilon)$ . This together with the assumptions (A) and (C) yields (4.10).  $\square$

As we know, the boundedness of the solution plays an essential role in deriving the error estimates. As was shown in (4.2),  $\|E^{\varepsilon,k}\|_\infty$  is uniformly bounded in 1D. However, for  $d = 2, 3$ , the Sobolev inequality involves  $\|\delta_x^2 E^{\varepsilon,k}\|$ , whose bound is inversely proportional to  $\lambda$ . Thus we can only get an a priori bound for  $\|E^{\varepsilon,k}\|_\infty$  which is decreasing with respect to  $\lambda$ . When  $\lambda$  is very small, this bound can be large. In order to give a uniform error estimate which is

independent of  $\lambda \ll 1$ , we apply the cut-off technique to the nonlinear terms in QZS (2.1) as was done in [1, 4]. Choose a smooth function  $\rho(s) \in C^\infty(\mathbb{R})$  such that

$$\rho(s) = \begin{cases} 1, & |s| \leq 1, \\ \in [0, 1], & |s| \leq 2, \\ 0, & |s| \geq 2. \end{cases} \quad (4.11)$$

For all  $\varepsilon \in (0, 1]$ , let  $M_0$  be a uniform upper bound of  $E^\varepsilon(x, t)$  and  $E(x, t)$  on  $\Omega_T = \Omega \times [0, T]$ , i.e.,

$$M_0 = \max \left\{ C_a, \sup_{\varepsilon \in (0, 1]} \|E^\varepsilon(x, t)\|_{L^\infty(\Omega_T)}, \|E(x, t)\|_{L^\infty(\Omega_T)} \right\},$$

where  $C_a$  is defined as (4.2) and is missing in this definition for  $d = 2, 3$ . For  $s \in \mathbb{R}$ , define

$$f_B(s) = s\rho(s/B), \quad B = (M_0 + 1)^2, \quad g(u, v) = \int_0^1 f'_B(\theta|u|^2 + (1-\theta)|v|^2) d\theta. \quad (4.12)$$

Set  $\widehat{E}^{\varepsilon, 0} = E^{\varepsilon, 0}$ ,  $\widehat{N}^{\varepsilon, 0} = N^{\varepsilon, 0}$ ,  $\widehat{N}^{\varepsilon, 1} = N^{\varepsilon, 1}$  and  $\widehat{E}^{\varepsilon, k}$ ,  $\widehat{N}^{\varepsilon, k+1}$  are the solutions to the following equation:

$$\begin{aligned} i\delta_t^- \widehat{E}_j^{\varepsilon, k} &= \left[ -\mathcal{A}^{-1}\delta_x^2 + \lambda^2(\mathcal{A}^{-1}\delta_x^2)^2 + \widehat{N}_j^{\varepsilon, k-\frac{1}{2}} g\left(\widehat{E}_j^{\varepsilon, k}, \widehat{E}_j^{\varepsilon, k-1}\right) \right] \widehat{E}_j^{\varepsilon, k-\frac{1}{2}}, \\ \varepsilon^2 \delta_t^2 \widehat{N}_j^{\varepsilon, k} &= \left( \mathcal{A}^{-1}\delta_x^2 - \lambda^2(\mathcal{A}^{-1}\delta_x^2)^2 \right) \widehat{N}_j^{\varepsilon, \bar{k}} + \mathcal{A}^{-1}\delta_x^2 f_B\left(|\widehat{E}_j^{\varepsilon, k}|^2\right), \quad k \geq 1. \end{aligned} \quad (4.13)$$

Actually  $(\widehat{E}^{\varepsilon, k}, \widehat{N}^{\varepsilon, k})$  is a pair of another approximation of  $(E^\varepsilon(x, t_k), N^\varepsilon(x, t_k))$ . Noticing that  $f'_B(\cdot)$  is bounded, this means  $f_B$  and  $g$  are Lipschitz functions. Thus, it can be clearly seen that (4.13) have unique solution for sufficiently small time step  $\tau$ .

Next, we turn to prove the error estimates in Theorems 4.1 and 4.2 for  $(\widehat{E}^{\varepsilon, k}, \widehat{N}^{\varepsilon, k})$ , respectively.

#### 4.2. Proof of (4.4) type estimates for $(\widehat{E}^{\varepsilon, k}, \widehat{N}^{\varepsilon, k})$

Define the error functions  $\widehat{e}^{\varepsilon, k}, \widehat{n}^{\varepsilon, k}$  as

$$\widehat{e}_j^{\varepsilon, k} = E^\varepsilon(x_j, t_k) - \widehat{E}_j^{\varepsilon, k}, \quad \widehat{n}_j^{\varepsilon, k} = N^\varepsilon(x_j, t_k) - \widehat{N}_j^{\varepsilon, k}, \quad 1 \leq j \leq M, \quad (4.14)$$

and the local truncation error  $\widehat{\eta}^{\varepsilon, k}, \widehat{\zeta}^{\varepsilon, k}$  reads as

$$\begin{aligned} \widehat{\eta}_j^{\varepsilon, k} &:= i\delta_t^- E^\varepsilon(x_j, t_k) \\ &\quad + \left[ \mathcal{A}^{-1}\delta_x^2 - \lambda^2(\mathcal{A}^{-1}\delta_x^2)^2 - ([N^\varepsilon]_j)^{k-\frac{1}{2}} g(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1})) \right] ([E^\varepsilon]_j)^{k-\frac{1}{2}}, \\ \widehat{\zeta}_j^{\varepsilon, k} &:= \varepsilon^2 \delta_t^2 N^\varepsilon(x_j, t_k) - \mathcal{A}^{-1}\delta_x^2 ([N^\varepsilon]_j)^{\bar{k}} + \lambda^2(\mathcal{A}^{-1}\delta_x^2)^2 ([N^\varepsilon]_j)^{\bar{k}} \\ &\quad - \mathcal{A}^{-1}\delta_x^2 f_B\left(|E^\varepsilon(x_j, t_k)|^2\right). \end{aligned} \quad (4.15)$$

In view of the definition of  $f_B$  and  $g$ , it is easy to check that

$$g(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1})) = 1, \quad f_B\left(|E^\varepsilon(x_j, t_k)|^2\right) = |E^\varepsilon(x_j, t_k)|^2,$$

which implies  $\widehat{\eta}_j^{\varepsilon, k} = \widehat{\eta}_j^{\varepsilon, k}$  and  $\widehat{\zeta}_j^{\varepsilon, k} = \widehat{\zeta}_j^{\varepsilon, k}$ . Thus (4.9) holds for  $\widehat{\eta}_j^{\varepsilon, k}$  and  $\widehat{\zeta}_j^{\varepsilon, k}$ . For analyzing the error of the scheme (2.9), let  $\widehat{u}^{\varepsilon, k-\frac{1}{2}}$  be the solution to the equation

$$-\mathcal{A}^{-1}\delta_x^2 \widehat{u}_j^{\varepsilon, k-\frac{1}{2}} = \delta_t^- \widehat{n}_j^{\varepsilon, k}, \quad 1 \leq j \leq M-1 \quad \text{with} \quad \widehat{u}_0^{\varepsilon, k-\frac{1}{2}} = \widehat{u}_M^{\varepsilon, k-\frac{1}{2}} = 0. \quad (4.16)$$

Subtracting (4.13) from (4.15), we get

$$i\delta_t^- \widehat{e}_j^{\varepsilon,k} = -\mathcal{A}^{-1} \delta_x^2 \widehat{e}_j^{\varepsilon,k-\frac{1}{2}} + \lambda^2 (\mathcal{A}^{-1} \delta_x^2)^2 \widehat{e}_j^{\varepsilon,k-\frac{1}{2}} + W_j^k + \widehat{\eta}_j^{\varepsilon,k}, \quad (4.17a)$$

$$\varepsilon^2 \delta_t^2 \widehat{n}_j^{\varepsilon,k} = \mathcal{A}^{-1} \delta_x^2 \widehat{n}_j^{\varepsilon,k} - \lambda^2 (\mathcal{A}^{-1} \delta_x^2)^2 \widehat{n}_j^{\varepsilon,k} + \mathcal{A}^{-1} \delta_x^2 P_j^k + \widehat{\zeta}_j^{\varepsilon,k} \quad (4.17b)$$

with

$$W_j^k = ([N^\varepsilon]_j)^{k-\frac{1}{2}} g(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1})) ([E^\varepsilon]_j)^{k-\frac{1}{2}} - N_j^{\varepsilon,k-\frac{1}{2}} g(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1}) \widehat{E}_j^{\varepsilon,k-\frac{1}{2}}, \quad (4.18)$$

$$P_j^k = f_B(|E^\varepsilon(x_j, t_k)|^2) - f_B(|\widehat{E}_j^{\varepsilon,k}|^2). \quad (4.19)$$

Then  $W_j^k$  can be rewritten as

$$\begin{aligned} W_j^k &= \widehat{n}_j^{\varepsilon,k-\frac{1}{2}} g(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1}) \widehat{E}_j^{\varepsilon,k-\frac{1}{2}} + ([N^\varepsilon]_j)^{k-\frac{1}{2}} g(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1}) \widehat{e}_j^{\varepsilon,k-\frac{1}{2}} \\ &\quad + ([N^\varepsilon]_j)^{k-\frac{1}{2}} \left( g(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1})) - g(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1}) \right) ([E^\varepsilon]_j)^{k-\frac{1}{2}}. \end{aligned} \quad (4.20)$$

Denote  $g_e(z_1, z_2) = g(z_1, z_2)(z_1 + z_2)$  for  $z_1, z_2 \in \mathbb{C}$ . Employing the nice property of  $g$  and  $f_B(\cdot)$ , it is easy to check that [1, 4]

$$\begin{aligned} |g_e(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1})| &\lesssim 1, \quad |P_j^k| \leq \sqrt{C_B} |\widehat{e}_j^{\varepsilon,k}|, \\ |g(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1})) - g(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1})| &\lesssim |\widehat{e}_j^k| + |\widehat{e}_j^{k-1}|, \\ |g_e(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1})) - g_e(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1})| &\lesssim |\widehat{e}_j^k| + |\widehat{e}_j^{k-1}|, \quad (4.21) \\ |\delta_x^+(g_e(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1}))) - g_e(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1}))| &\lesssim \sum_{l=k-1}^k (|\widehat{e}_j^l| + |\widehat{e}_{j+1}^l| + |\delta_x^+ \widehat{e}_j^l|), \\ |\delta_x^2(g_e(E^\varepsilon(x_j, t_k), E^\varepsilon(x_j, t_{k-1}))) - g_e(\widehat{E}_j^{\varepsilon,k}, \widehat{E}_j^{\varepsilon,k-1}))| &\lesssim \sum_{l=k-1}^k (|\widehat{e}_j^l| + |\widehat{e}_{j+1}^l| + |\delta_x^+ \widehat{e}_j^l| + |\delta_x^2 \widehat{e}_j^l|), \end{aligned}$$

where  $C_B > 0$  is a positive constant depending on  $M_0$  and  $\rho(\cdot)$ . It follows from (4.21) that

$$|W_j^k| \lesssim |\widehat{n}_j^{\varepsilon,k}| + |\widehat{n}_j^{\varepsilon,k-1}| + |\widehat{e}_j^{\varepsilon,k}| + |\widehat{e}_j^{\varepsilon,k-1}|, \quad 1 \leq j \leq M. \quad (4.22)$$

Multiplying  $2\tau h \overline{\widehat{e}_j^{\varepsilon,k-1/2}}$  on both sides of equality (4.17a), summing together for  $1 \leq j \leq M$  and taking imaginary parts yields

$$\|\widehat{e}^{\varepsilon,k}\|^2 - \|\widehat{e}^{\varepsilon,k-1}\|^2 = 2\tau \operatorname{Im} \langle W^k, \widehat{e}^{\varepsilon,k-\frac{1}{2}} \rangle + 2\tau \operatorname{Im} \langle \widehat{\eta}^{\varepsilon,k}, \widehat{e}^{\varepsilon,k-\frac{1}{2}} \rangle. \quad (4.23)$$

Multiplying  $h\tau \delta_t^- \overline{\widehat{e}_j^{\varepsilon,k}}$  on both sides of equality (4.17a), summing together for  $1 \leq j \leq M$ , using Lemmas 3.1 and 3.2 and taking the real parts, we have

$$\begin{aligned} \frac{1}{2} \left( \|R\widehat{e}^{\varepsilon,k}\|^2 - \|R\widehat{e}^{\varepsilon,k-1}\|^2 \right) + \frac{\lambda^2}{2} \left( \|\mathcal{A}^{-1} \delta_x^2 \widehat{e}^{\varepsilon,k}\|^2 - \|\mathcal{A}^{-1} \delta_x^2 \widehat{e}^{\varepsilon,k-1}\|^2 \right) \\ + \operatorname{Re} \langle W^k, \widehat{e}^{\varepsilon,k} - \widehat{e}^{\varepsilon,k-1} \rangle = -\operatorname{Re} \langle \widehat{\eta}^{\varepsilon,k}, \widehat{e}^{\varepsilon,k} - \widehat{e}^{\varepsilon,k-1} \rangle. \end{aligned} \quad (4.24)$$

Multiplying  $\tau h (\widehat{u}_j^{\varepsilon,k+\frac{1}{2}} + \widehat{u}_j^{\varepsilon,k-\frac{1}{2}})$  on both sides of equality (4.17b), summing together for  $1 \leq j \leq M$ , making use of (4.16), we obtain

$$\varepsilon^2 \|R\widehat{u}^{\varepsilon,k+\frac{1}{2}}\|^2 - \varepsilon^2 \|R\widehat{u}^{\varepsilon,k-\frac{1}{2}}\|^2 + \frac{1}{2} \left( \|\widehat{n}^{\varepsilon,k+1}\|^2 - \|\widehat{n}^{\varepsilon,k-1}\|^2 \right)$$

$$\begin{aligned}
& + \langle P^k, \widehat{n}^{\varepsilon, k+1} - \widehat{n}^{\varepsilon, k-1} \rangle + \frac{\lambda^2}{2} \left( \|R\widehat{n}^{\varepsilon, k+1}\|^2 - \|R\widehat{n}^{\varepsilon, k-1}\|^2 \right) \\
& = -\tau \left\langle \widehat{\zeta}^{\varepsilon, k}, \widehat{u}^{\varepsilon, k+\frac{1}{2}} + \widehat{u}^{\varepsilon, k-\frac{1}{2}} \right\rangle.
\end{aligned} \tag{4.25}$$

Combining  $3C_B \cdot (4.23) + 4 \cdot (4.24) + (4.25)$ , one can derive that

$$\begin{aligned}
\widehat{S}^k - \widehat{S}^{k-1} & = 6\tau C_B \operatorname{Im} \left\langle W^k, \widehat{e}^{\varepsilon, k-\frac{1}{2}} \right\rangle + 6\tau C_B \operatorname{Im} \left\langle \widehat{\eta}^{\varepsilon, k}, \widehat{e}^{\varepsilon, k-\frac{1}{2}} \right\rangle \\
& \quad - 4 \operatorname{Re} \left\langle \widehat{\eta}^{\varepsilon, k}, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \right\rangle - \tau \left\langle \widehat{\zeta}^{\varepsilon, k}, \widehat{u}^{\varepsilon, k+\frac{1}{2}} + \widehat{u}^{\varepsilon, k-\frac{1}{2}} \right\rangle \\
& \quad + 2 \left\langle P^k - P^{k-1}, \widehat{n}^{\varepsilon, k-\frac{1}{2}} \right\rangle - 4 \operatorname{Re} \left\langle W^k, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \right\rangle,
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
\widehat{S}^k & = 3C_B \|\widehat{e}^{\varepsilon, k}\|^2 + 2 \|R\widehat{e}^{\varepsilon, k}\|^2 + 2\lambda^2 \|\mathcal{A}^{-1} \delta_x^2 \widehat{e}^{\varepsilon, k}\|^2 + \varepsilon^2 \|R\widehat{u}^{\varepsilon, k+\frac{1}{2}}\|^2 \\
& \quad + \frac{1}{2} \left( \|\widehat{n}^{\varepsilon, k+1}\|^2 + \|\widehat{n}^{\varepsilon, k}\|^2 \right) + \frac{\lambda^2}{2} \left( \|R\widehat{n}^{\varepsilon, k+1}\|^2 + \|R\widehat{n}^{\varepsilon, k}\|^2 \right) + 2 \left\langle P^k, \widehat{n}^{\varepsilon, k+\frac{1}{2}} \right\rangle.
\end{aligned}$$

Next we estimate each term in (4.26) separately. Firstly by applying the Cauchy inequality, it is easily derived that

$$\begin{aligned}
\left| \operatorname{Im} \left\langle W^k, \widehat{e}^{\varepsilon, k-1/2} \right\rangle \right| & \lesssim \sum_{l=k-1}^k \left( \|\widehat{n}^{\varepsilon, l}\|^2 + \|\widehat{e}^{\varepsilon, l}\|^2 \right), \\
\left| \operatorname{Im} \left\langle \widehat{\eta}^{\varepsilon, k}, \widehat{e}^{\varepsilon, k-1/2} \right\rangle \right| & \leq \|\widehat{\eta}^{\varepsilon, k}\|^2 + \frac{1}{8} \left( \|\widehat{e}^{\varepsilon, k}\|^2 + \|\widehat{e}^{\varepsilon, k-1}\|^2 \right), \\
\left| \left\langle P^k, \widehat{n}^{\varepsilon, k-1/2} \right\rangle \right| & \leq \|P^k\|^2 + \frac{1}{8} \left( \|\widehat{n}^{\varepsilon, k}\|^2 + \|\widehat{n}^{\varepsilon, k-1}\|^2 \right).
\end{aligned} \tag{4.27}$$

Furthermore, using (4.17a), one gets

$$\begin{aligned}
& \left| \operatorname{Re} \left\langle \widehat{\eta}^{\varepsilon, k}, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \right\rangle \right| \\
& = \tau \left| \operatorname{Im} \left\langle \widehat{\eta}^{\varepsilon, k}, -\mathcal{A}^{-1} \delta_x^2 \widehat{e}_j^{\varepsilon, k-\frac{1}{2}} + \lambda^2 (\mathcal{A}^{-1} \delta_x^2)^2 \widehat{e}_j^{\varepsilon, k-\frac{1}{2}} + W_j^k + \widehat{\eta}_j^{\varepsilon, k} \right\rangle \right| \\
& \lesssim \tau \sum_{l=k-1}^k \left( \|\widehat{e}^{\varepsilon, l}\|^2 + \|R\widehat{e}^{\varepsilon, l}\|^2 + \|\widehat{n}^{\varepsilon, l}\|^2 + \lambda^2 \|\mathcal{A}^{-1} \delta_x^2 \widehat{e}^{\varepsilon, l}\|^2 \right) \\
& \quad + \tau \left( \|R\widehat{\eta}^{\varepsilon, k}\|^2 + \|\widehat{\eta}^{\varepsilon, k}\|^2 + \|\mathcal{A}^{-1} \delta_x^2 \widehat{\eta}^{\varepsilon, k}\|^2 \right).
\end{aligned} \tag{4.28}$$

To estimate the other terms appearing in (4.26), we need the following lemma.

**Lemma 4.3.** *Under assumption (A), for  $1 \leq k \leq \frac{T}{\tau}$ , we have*

$$\begin{aligned}
& \left| \left\langle P^k - P^{k-1}, \widehat{n}^{\varepsilon, k} + \widehat{n}^{\varepsilon, k-1} \right\rangle - 4 \operatorname{Re} \left\langle W^k, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \right\rangle \right| \\
& \lesssim \tau \left( \|\widehat{\eta}^{\varepsilon, l}\|^2 + \sum_{l=k-1}^k \left( \|\widehat{n}^{\varepsilon, l}\|^2 + \|\widehat{e}^{\varepsilon, l}\|^2 + \|R\widehat{e}^{\varepsilon, l}\|^2 + \lambda^2 \|\mathcal{A}^{-1} \delta_x^2 \widehat{e}^{\varepsilon, l}\|^2 \right) \right),
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
& -\tau \sum_{l=1}^k \left\langle \widehat{\zeta}^{\varepsilon, l}, \widehat{u}^{\varepsilon, l+\frac{1}{2}} + \widehat{u}^{\varepsilon, l-\frac{1}{2}} \right\rangle \\
& \leq \frac{\tau}{2} \sum_{l=1}^{k-1} \left( C \|\delta_t \widehat{\zeta}^{\varepsilon, l}\|^2 + \|\widehat{n}^{\varepsilon, l}\|^2 \right) + \sum_{l=0}^1 \left( C \|\widehat{\zeta}^{\varepsilon, l+1}\|^2 + \frac{1}{8} \|\widehat{n}^{\varepsilon, l}\|^2 \right)
\end{aligned}$$

$$+ \sum_{l=k}^{k+1} \left( C \|\widehat{\zeta}^{\varepsilon, l-1}\|^2 + \frac{1}{8} \|\widehat{n}^{\varepsilon, l}\|^2 \right), \quad (4.30)$$

where  $C$  is a number independent of  $h, \tau$  and  $\varepsilon$ .

*Proof.* For (4.29), in view of the definition of  $W_j^k, P_j^k$ , Lemma 4.1 and the Cauchy inequality, we find

$$\begin{aligned} & 2 \langle P^k - P^{k-1}, \widehat{n}^{\varepsilon, k-\frac{1}{2}} \rangle - 4 \operatorname{Re} \langle W^k, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \rangle \\ &= 2 \operatorname{Re} \langle \widehat{n}^{\varepsilon, k-\frac{1}{2}} G_e^k, E^\varepsilon(\cdot, t_k) - E^\varepsilon(\cdot, t_{k-1}) \rangle \\ & \quad - \operatorname{Re} \langle (N^\varepsilon(\cdot, t_k) + N^\varepsilon(\cdot, t_{k-1})) G_e^k, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \rangle, \end{aligned} \quad (4.31)$$

where

$$G_e^k = g_e(E^\varepsilon(\cdot, t_k), E^\varepsilon(\cdot, t_{k-1})) - g_e(\widehat{E}^{\varepsilon, k}, \widehat{E}^{\varepsilon, k-1}).$$

According to (4.21), we see that

$$\|G_e^k\| \lesssim \|\widehat{e}^{\varepsilon, k}\| + \|\widehat{e}^{\varepsilon, k-1}\|.$$

Using (4.17a), (4.21), (4.22) and assumption (A), we obtain

$$\begin{aligned} & \left| \operatorname{Re} \langle \widehat{n}^{\varepsilon, k-\frac{1}{2}} G_e^k, E^\varepsilon(\cdot, t_k) - E^\varepsilon(\cdot, t_{k-1}) \rangle \right| \\ & \lesssim \tau \|\partial_t E^\varepsilon(\cdot, t)\|_{L^\infty} \left( \|\widehat{n}^{\varepsilon, k}\|^2 + \|\widehat{n}^{\varepsilon, k-1}\|^2 + \|\widehat{e}^{\varepsilon, k}\|^2 + \|\widehat{e}^{\varepsilon, k-1}\|^2 \right) \\ & \lesssim \tau \left( \|\widehat{n}^{\varepsilon, k}\|^2 + \|\widehat{n}^{\varepsilon, k-1}\|^2 + \|\widehat{e}^{\varepsilon, k}\|^2 + \|\widehat{e}^{\varepsilon, k-1}\|^2 \right), \end{aligned}$$

and

$$\begin{aligned} & \left| \operatorname{Re} \langle (N^\varepsilon(\cdot, t_{k-1}) + N^\varepsilon(\cdot, t_k)) G_e^k, \widehat{e}^{\varepsilon, k} - \widehat{e}^{\varepsilon, k-1} \rangle \right| \\ &= \tau \left| \operatorname{Im} \langle (N^\varepsilon(\cdot, t_{k-1}) + N^\varepsilon(\cdot, t_k)) G_e^k, \left( -\mathcal{A}^{-1} \delta_x^2 + \lambda^2 (\mathcal{A}^{-1} \delta_x^2)^2 \right) \widehat{e}^{\varepsilon, k-\frac{1}{2}} + W^k + \widehat{\eta}^{\varepsilon, k} \rangle \right| \\ & \lesssim \tau \left( \|\widehat{\eta}^{\varepsilon, k}\|^2 + \sum_{l=k-1}^k \left( \|\widehat{n}^{\varepsilon, l}\|^2 + \|\widehat{e}^{\varepsilon, l}\|^2 + \|R\widehat{e}^{\varepsilon, l}\|^2 + \lambda^2 \|\mathcal{A}^{-1} \delta_x^2 \widehat{e}^{\varepsilon, l}\|^2 \right) \right), \end{aligned}$$

which immediately gives (4.29). For (4.30), based on the Cauchy inequality, (4.16) and (3.3), we derive

$$\begin{aligned} & -\tau \sum_{l=1}^k \langle \widehat{\zeta}^{\varepsilon, l}, \widehat{u}^{\varepsilon, l+\frac{1}{2}} + \widehat{u}^{\varepsilon, l-\frac{1}{2}} \rangle = -\sum_{l=1}^k \langle (-\delta_x^2)^{-1} \mathcal{A} \widehat{\zeta}^{\varepsilon, l}, \widehat{n}^{\varepsilon, l+1} - \widehat{n}^{\varepsilon, l-1} \rangle \\ &= 2\tau \sum_{l=2}^{k-1} \langle \delta_t (-\delta_x^2)^{-1} \mathcal{A} \widehat{\zeta}^{\varepsilon, l}, \widehat{n}^{\varepsilon, l} \rangle + \sum_{l=0}^1 \langle (-\delta_x^2)^{-1} \mathcal{A} \widehat{\zeta}^{\varepsilon, l+1}, \widehat{n}^{\varepsilon, l} \rangle - \sum_{l=k}^{k+1} \langle (-\delta_x^2)^{-1} \mathcal{A} \widehat{\zeta}^{\varepsilon, l-1}, \widehat{n}^{\varepsilon, l} \rangle \\ & \leq \frac{\tau}{2} \sum_{l=2}^{k-1} \left( C_1 \|\delta_t \widehat{\zeta}^{\varepsilon, l}\|^2 + \|\widehat{n}^{\varepsilon, l}\|^2 \right) + \sum_{l=0}^1 \left( C_1 \|\widehat{\zeta}^{\varepsilon, l+1}\|^2 + \frac{1}{8} \|\widehat{n}^{\varepsilon, l}\|^2 \right) \\ & \quad + \sum_{l=k}^{k+1} \left( C_1 \|\widehat{\zeta}^{\varepsilon, l-1}\|^2 + \frac{1}{8} \|\widehat{n}^{\varepsilon, l}\|^2 \right), \end{aligned}$$

where  $C_1$  is a constant independent of  $\varepsilon, h$  and  $\tau$ . This completes the proof of Lemma 4.3.  $\square$

**Proof of (4.4) for  $(\widehat{e}^{\varepsilon,k}, \widehat{n}^{\varepsilon,k})$ .** It follows from (4.21) and (4.27) that

$$\left| \left\langle P^k, \widehat{n}^{\varepsilon,k+1/2} \right\rangle \right| \leq C_B \|\widehat{e}^{\varepsilon,k}\|^2 + \frac{1}{8} \left( \|\widehat{n}^{\varepsilon,k}\|^2 + \|\widehat{n}^{\varepsilon,k+1}\|^2 \right),$$

which implies

$$\begin{aligned} \widehat{S}^k &\geq C_B \|\widehat{e}^{\varepsilon,k}\|^2 + 2 \|R\widehat{e}^{\varepsilon,k}\|^2 + 2\lambda^2 \|\mathcal{A}^{-1}\delta_x^2 \widehat{e}^{\varepsilon,k}\|^2 + \varepsilon^2 \|R\widehat{u}^{\varepsilon,k+1/2}\|^2 \\ &\quad + \frac{\lambda^2}{2} \left( \|R\widehat{n}^{\varepsilon,k+1}\|^2 + \|R\widehat{n}^{\varepsilon,k}\|^2 \right) + \frac{1}{4} \left( \|\widehat{n}^{\varepsilon,k+1}\|^2 + \|\widehat{n}^{\varepsilon,k}\|^2 \right). \end{aligned}$$

Combining (4.26)-(4.29), and applying (3.1), we are led to

$$\begin{aligned} &\widehat{S}^k - \widehat{S}^{k-1} + \tau \left\langle \widehat{\zeta}^{\varepsilon,k}, \widehat{u}^{\varepsilon,k+1/2} + \widehat{u}^{\varepsilon,k-1/2} \right\rangle \\ &\leq \tau (\widehat{S}^k + \widehat{S}^{k-1}) + \tau \left( \|\widehat{\eta}^{\varepsilon,k}\|^2 + \|R\widehat{\eta}^{\varepsilon,k}\|^2 + \|\mathcal{A}^{-1}\delta_x^2 \widehat{\eta}^{\varepsilon,k}\|^2 \right). \end{aligned}$$

Summing together for  $k = 1, \dots, m$ , applying (4.30), Lemmas 4.1 and 4.2, we yield

$$\begin{aligned} \frac{1}{2} \widehat{S}^m &\leq \widehat{S}^m - \frac{1}{8} \left( \|\widehat{n}^{\varepsilon,m}\|^2 + \|\widehat{n}^{\varepsilon,m+1}\|^2 \right) \\ &\lesssim \left( \frac{h^4}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}} \right)^2 + \tau \sum_{k=1}^m \widehat{S}^k, \quad 1 \leq m \leq \frac{T}{\tau}. \end{aligned}$$

Applying the discrete Gronwall inequality, for sufficiently small  $\tau > 0$ , we conclude that

$$\widehat{S}^k \lesssim \left( \frac{h^4}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}} \right)^2, \quad 1 \leq k \leq \frac{T}{\tau},$$

which immediately yields

$$\begin{aligned} &\|\widehat{e}^{\varepsilon,k}\| + \|R\widehat{e}^{\varepsilon,k}\| + \lambda \|\mathcal{A}^{-1}\delta_x^2 \widehat{e}^{\varepsilon,k}\| + \|\widehat{n}^{\varepsilon,k}\| + \lambda \|R\widehat{n}^{\varepsilon,k}\| \\ &\lesssim \frac{h^4}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \quad 0 \leq k \leq \frac{T}{\tau}, \end{aligned} \tag{4.32}$$

and the proof is completed.  $\square$

### 4.3. Proof of (4.5) type estimates for $(\widehat{e}^{\varepsilon,k}, \widehat{n}^{\varepsilon,k})$

*Proof.* Define the error functions

$$\widetilde{e}_j^{\varepsilon,k} = E(x_j, t_k) - \widehat{E}_j^{\varepsilon,k}, \quad \widetilde{n}_j^{\varepsilon,k} = N(x_j, t_k) - \widehat{N}_j^{\varepsilon,k}, \quad 1 \leq j \leq M, \tag{4.33}$$

where  $E(x, t)$  is the solution of the QM-NLSE (2.6) and  $N(x, t) = -I_\lambda |E(x, t)|^2$ . Let  $\widetilde{u}^{\varepsilon,k-1/2}$  satisfy

$$-\mathcal{A}^{-1}\delta_x^2 \widetilde{u}_j^{\varepsilon,k-1/2} = \delta_t^- \widetilde{n}_j^{\varepsilon,k}, \quad 1 \leq j \leq M-1 \tag{4.34}$$

with  $\widetilde{u}_0^{\varepsilon,k-1/2} = \widetilde{u}_M^{\varepsilon,k-1/2} = 0$ . Introduce the local truncation errors  $\widetilde{\eta}^{\varepsilon,k}, \widetilde{\zeta}^{\varepsilon,k}$  as

$$\begin{aligned} \widetilde{\eta}_j^{\varepsilon,k} &:= i\delta_t^- E(x_j, t_k) + \left( \mathcal{A}^{-1}\delta_x^2 - \lambda^2 (\mathcal{A}^{-1}\delta_x^2)^2 - ([N]_j^{k-1/2} g(E(x_j, t_k), E(x_j, t_{k-1}))) \right) (E)_j^{k-1/2}, \\ \widetilde{\zeta}_j^{\varepsilon,k} &:= \varepsilon^2 \delta_t^2 N(x_j, t_k) - \left( \mathcal{A}^{-1}\delta_x^2 - \lambda^2 (\mathcal{A}^{-1}\delta_x^2)^2 \right) ([N]_j^k - \mathcal{A}^{-1}\delta_x^2 f_B(|E(x_j, t_k)|^2)). \end{aligned} \tag{4.35}$$



Employing similar arguments in Lemmas 4.1 and 4.2, the local truncation errors and the initial errors satisfy

$$\begin{aligned} \|\tilde{\eta}^{\varepsilon,k}\| + \|\delta_x^+ \tilde{\eta}^{\varepsilon,k}\| + \|\delta_x^2 \tilde{\eta}^{\varepsilon,k}\| &\lesssim h^4 + \tau^2, \\ \|\tilde{\zeta}^{\varepsilon,k}\| + \|\delta_t \tilde{\zeta}^{\varepsilon,k}\| + \|\delta_x^+ \tilde{\zeta}^{\varepsilon,k}\| &\lesssim h^4 + \tau^2 + \varepsilon^2, \end{aligned} \quad (4.36)$$

$$\begin{aligned} \|\tilde{n}^{\varepsilon,0}\| &\lesssim \varepsilon^\alpha, \quad \|\tilde{n}^{\varepsilon,1}\| \lesssim \tau h^4 + \tau^2 + \varepsilon^\beta + \varepsilon^\alpha, \\ \|\tilde{u}^{\varepsilon,\frac{1}{2}}\| &\lesssim \|\delta_t^- \tilde{n}^{\varepsilon,1}\| \lesssim h^4 + \tau + \varepsilon^{\beta-1} + \varepsilon^{\alpha-1}. \end{aligned} \quad (4.37)$$

Subtracting (4.13) from (4.35), we get the error equations

$$i\delta_t^- \tilde{e}_j^{\varepsilon,k} = - \left( \mathcal{A}^{-1} \delta_x^2 - \lambda^2 (\mathcal{A}^{-1} \delta_x^2)^2 \right) \tilde{e}_j^{\varepsilon,k-\frac{1}{2}} + \tilde{W}_j^k + \tilde{\eta}_j^{\varepsilon,k}, \quad (4.38)$$

$$\varepsilon^2 \delta_t^2 \tilde{n}_j^{\varepsilon,k} = \left( \mathcal{A}^{-1} \delta_x^2 - \lambda^2 (\mathcal{A}^{-1} \delta_x^2)^2 \right) \tilde{n}_j^{\varepsilon,k} + \delta_x^2 \tilde{P}_j^k + \tilde{\zeta}_j^{\varepsilon,k} \quad (4.39)$$

with

$$\begin{aligned} \tilde{W}_j^k &= ([N]_j)^{k-\frac{1}{2}} g(E(x_j, t_k), E(x_j, t_{k-1})) ([E]_j)^{k-\frac{1}{2}} - N_j^{k-\frac{1}{2}} g(\hat{E}_j^{\varepsilon,k}, \hat{E}_j^{\varepsilon,k-1}) E_j^{k-\frac{1}{2}}, \\ \tilde{P}_j^k &= f_B(|E(x_j, t_k)|^2) - f_B(|\hat{E}_j^{\varepsilon,k}|^2). \end{aligned} \quad (4.40)$$

Denote

$$\begin{aligned} \tilde{S}^k &= 3C_B \|\tilde{e}^{\varepsilon,k}\|^2 + 2 \|R\tilde{e}^{\varepsilon,k}\|^2 + 2\lambda^2 \|\mathcal{A}^{-1} \delta_x^2 \tilde{e}^{\varepsilon,k}\|^2 + \varepsilon^2 \|R\tilde{u}^{\varepsilon,k+\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2} \left( \|\tilde{n}^{\varepsilon,k+1}\|^2 + \|\tilde{n}^{\varepsilon,k}\|^2 \right) + \frac{\lambda^2}{2} \left( \|R\tilde{n}^{\varepsilon,k+1}\|^2 + \|R\tilde{n}^{\varepsilon,k}\|^2 \right) + \langle P^k, \tilde{n}^{\varepsilon,k+1} - \tilde{n}^{\varepsilon,k} \rangle. \end{aligned}$$

Applying similar arguments as used in the above subsection, we obtain for sufficiently small  $\tau$ ,

$$\tilde{S}^k \lesssim (h^4 + \tau^2 + \varepsilon^\alpha + \varepsilon^\beta + \varepsilon^2)^2, \quad 1 \leq k \leq \frac{T}{\tau},$$

which implies

$$\begin{aligned} &\|\tilde{e}^{\varepsilon,k}\| + \|R\tilde{e}^{\varepsilon,k}\| + \lambda \|\mathcal{A}^{-1} \delta_x^2 \tilde{e}^{\varepsilon,k}\| + \|\tilde{n}^{\varepsilon,k}\| + \lambda \|R\tilde{n}^{\varepsilon,k}\| \\ &\lesssim h^2 + \tau^2 + \varepsilon^\beta + \varepsilon^\alpha + \varepsilon^2, \quad 0 \leq k \leq \frac{T}{\tau}. \end{aligned}$$

This together with assumption (B) yields

$$\begin{aligned} &\|\tilde{e}^{\varepsilon,k}\| + \|R\tilde{e}^{\varepsilon,k}\| + \lambda \|\mathcal{A}^{-1} \delta_x^2 \tilde{e}^{\varepsilon,k}\| \\ &\leq \|\tilde{e}^{\varepsilon,k}\| + \|R\tilde{e}^{\varepsilon,k}\| + \lambda \|\mathcal{A}^{-1} \delta_x^2 \tilde{e}^{\varepsilon,k}\| + 2 \|E^\varepsilon(\cdot, t_k) - E(\cdot, t_k)\|_{H^1} \\ &\quad + 2\lambda \|E^\varepsilon(\cdot, t_k) - E(\cdot, t_k)\|_{H^2} \\ &\lesssim h^2 + \tau^2 + \varepsilon^{\alpha^\dagger} + \varepsilon^{1+\alpha^*}, \\ &\|\tilde{n}^{\varepsilon,k}\| + \lambda \|R\tilde{n}^{\varepsilon,k}\| \\ &\leq \|\tilde{n}^{\varepsilon,k}\| + \lambda \|R\tilde{n}^{\varepsilon,k}\| + \|N^\varepsilon(\cdot, t_k) - N(\cdot, t_k)\|_{L^2} \\ &\quad + 2\lambda \|N^\varepsilon(\cdot, t_k) - N(\cdot, t_k)\|_{H^1} \\ &\lesssim h^2 + \tau^2 + \varepsilon^{\alpha^\dagger}. \end{aligned} \quad (4.41)$$

The estimate is established.  $\square$

#### 4.4. Proof of Theorems 4.1 and 4.2

*Proof.* In view of (4.32) and (4.41), it suffices to show that  $\|\widehat{E}^{\varepsilon,k}\|_{\infty} \leq M_0 + 1$ , in which case  $(\widehat{E}^{\varepsilon,k}, \widehat{N}^{\varepsilon,k})$  are identical to  $(E^{\varepsilon,k}, N^{\varepsilon,k})$  and Theorems 4.1-4.2 are established.

Following the proof of Theorem 3.1 (see also [32]), we can derive the a priori bound of  $(\widehat{E}^{\varepsilon,k}, \widehat{N}^{\varepsilon,k})$ , and the scheme (4.13) conserves the mass  $\|\widehat{E}^{\varepsilon,k}\|$  and energy

$$\begin{aligned} \widehat{\mathcal{E}}^k &= \left( \|R\widehat{E}^{\varepsilon,k+1}\|^2 + \|R\widehat{E}^{\varepsilon,k}\|^2 \right) + \lambda^2 \left( \|\mathcal{A}^{-1}\delta_x^2\widehat{E}^{\varepsilon,k+1}\|^2 + \|\mathcal{A}^{-1}\delta_x^2\widehat{E}^{\varepsilon,k}\|^2 \right) + \varepsilon^2 \|R\widehat{U}^{\varepsilon,k+\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2} \left( \|\widehat{N}^{\varepsilon,k+1}\|^2 + \|\widehat{N}^{\varepsilon,k}\|^2 \right) + \frac{\lambda^2}{2} \left( \|R\widehat{N}^{\varepsilon,k+1}\|^2 + \|R\widehat{N}^{\varepsilon,k}\|^2 \right) \\ &\quad + \left\langle \widehat{N}^{\varepsilon,k+\frac{1}{2}}, f_B(|\widehat{E}^{\varepsilon,k+1}|^2) + f_B(|\widehat{E}^{\varepsilon,k}|^2) \right\rangle \equiv \widehat{\mathcal{E}}^0, \end{aligned} \quad (4.42)$$

where  $\widehat{U}^{\varepsilon,k+\frac{1}{2}} = (-\delta_x^2)^{-1}\delta_t^+\widehat{N}^{\varepsilon,k}$ . Noticing  $f_B(s) \leq s(s \geq 0)$ , similar to (4.2), we could obtain the same bound for  $\widehat{E}^{\varepsilon,k}$ ,

$$\|\widehat{E}^{\varepsilon,k}\|_{\infty} \leq C_a \leq M_0 + 1, \quad k \geq 0.$$

This implies the modified scheme (4.13) reduces to the scheme (2.10),  $(\widehat{E}^{\varepsilon,k}, \widehat{N}^{\varepsilon,k})$  agree with  $(E^{\varepsilon,k}, N^{\varepsilon,k})$  and (4.4)-(4.5), (4.7) hold. Furthermore, (4.6) can be derived immediately by taking the minimum of (4.4) and (4.5) when  $\alpha, \beta \geq 1$ .  $\square$

**Remark 4.1.** For  $d = 2, 3$ , to get an error estimate independent of  $\lambda$ , instead of bounding the solution  $\|\widehat{E}^{\varepsilon,k}\|_{\infty}$  by the conservation laws and Sobolev inequality, we estimate the solution via the error function

$$\|\widehat{E}^{\varepsilon,k}\|_{\infty} \leq \|E^{\varepsilon}(x, t_k)\|_{L^{\infty}} + \|\widehat{e}^{\varepsilon,k}\|_{\infty} \leq M_0 + \frac{1}{C_d(h)} \|\widehat{e}^{\varepsilon,k}\|_{H^1},$$

and we use the discrete Sobolev inequality [4, 27, 32] to control the  $l^{\infty}$ -norm as

$$\|\psi_h\|_{\infty} \lesssim \frac{1}{C_d(h)} \|\psi_h\|_{H^1}, \quad C_d(h) = \begin{cases} 1/|\ln(h)|, & d = 2, \\ h^{\frac{1}{2}}, & d = 3, \end{cases}$$

where  $\psi_h$  is the mesh functions over  $\Omega$  with zero at the boundary. Therefore, by assuming the extra conditions with  $h = o(\varepsilon^{2/7})$ ,  $\tau = o(\varepsilon^{3/2}C_d(h)^{1/2})$  for ill-prepared initial data,  $\tau = o(C_d(h)^{3/2})$  for less-ill-prepared initial data, and  $\tau = o(C_d(h)^{3/4})$  for well-prepared initial data, one easily find the same error bounds as those in Theorems 4.1 and 4.2.

## 5. Numerical Examples

The purpose of this section is to test validity of the scheme (2.9)-(2.12) for solving the QZS (2.1), and the corresponding numerical results will be shown in Tables 5.1-5.7.

**Example 5.1.** Consider the QZS (2.1) with the periodic boundary condition and the initial data

$$E_0(x) = e^{-\frac{x^2}{2}}, \quad \omega_0(x) = e^{-\frac{x^2}{4}}, \quad \omega_1(x) = xe^{-\frac{x^2}{4}}, \quad (5.1)$$

and the parameters  $\alpha$  and  $\beta$  in (2.7) are chosen as:

**Case 1.** Well-prepared initial data,  $\alpha = 2$  and  $\beta = 2$ .

**Case 2.** Less-ill-prepared initial data,  $\alpha = 1$  and  $\beta = 1$ .

**Case 3.** Ill-prepared initial data,  $\alpha = 0$  and  $\beta = 0$ .

The problem is fixed to  $\lambda = 0.01$ ,  $\Omega = (-200, 200)$ , such that the error due to the truncation is negligible. Since the exact solution is not known, the ‘exact’ solution  $(E^\varepsilon(x, t), N^\varepsilon(x, t))$  obtained by the proposed compact difference scheme with mesh size  $h = 1/160$  and time step  $\tau = 2^{-18}$ . In order to measure the numerical error, we use the following error functions:

$$e^\varepsilon(t_k) := \|e^{\varepsilon,k}\| + \|Re^{\varepsilon,k}\| + \lambda\|\mathcal{A}^{-1}\delta_x^2 e^{\varepsilon,k}\|, \quad n^\varepsilon(t_k) := \|n^{\varepsilon,k}\| + \lambda\|Rn^{\varepsilon,k}\|$$

with  $e^{\varepsilon,k} = E^\varepsilon(\cdot, t_k) - E^{\varepsilon,k}$  and  $n^{\varepsilon,k} = N^\varepsilon(\cdot, t_k) - N^{\varepsilon,k}$ .

To testify the spatial accuracy, we take a tiny time step  $\tau = 2^{-18}$  such that the temporal error is negligible. Tables 5.1-5.3 list the spatial errors of the compact finite difference method

Table 5.1: Spatial error at time  $t = 1$  for well-prepared initial data Case I, i.e.,  $\alpha = 2$  and  $\beta = 2$ .

$e^\varepsilon$	$h = 0.4$	$h/2$	$h/2^2$	$h/2^3$
$\varepsilon = 1/2$	2.81e-3	1.64e-4	1.01e-5	6.27e-7
rate	-	4.10	4.03	4.01
$\varepsilon = 1/2^4$	2.80e-3	1.64e-4	1.01e-5	6.27e-7
rate	-	4.09	4.02	4.01
$\varepsilon = 1/2^6$	2.81e-3	1.64e-4	1.01e-5	6.29e-7
rate	-	4.10	4.02	4.01
$n^\varepsilon$	$h = 0.4$	$h/2$	$h/2^2$	$h/2^3$
$\varepsilon = 1/2$	2.64e-3	1.54e-4	9.48e-6	5.90e-7
rate	-	4.10	4.02	4.00
$\varepsilon = 1/2^4$	4.60e-4	3.22e-5	1.99e-6	1.24e-7
rate	-	3.84	4.02	4.00
$\varepsilon = 1/2^6$	3.88e-4	2.41e-5	1.50e-6	9.47e-8
rate	-	4.01	4.00	3.99

Table 5.2: Spatial error at time  $t = 1$  for less-ill-prepared initial data Case II, i.e.,  $\alpha = \beta = 1$ .

$e^\varepsilon$	$h = 0.4$	$h/2$	$h/2^2$	$h/2^3$
$\varepsilon = 1/2$	2.96e-3	1.73e-4	1.07e-5	6.65e-7
rate	-	4.09	4.02	4.01
$\varepsilon = 1/2^4$	2.80e-3	1.64e-4	1.01e-5	6.26e-7
rate	-	4.09	4.02	4.01
$\varepsilon = 1/2^6$	2.81e-3	1.64e-4	1.01e-5	6.29e-7
rate	-	4.10	4.02	4.01
$n^\varepsilon$	$h = 0.4$	$h/2$	$h/2^2$	$h/2^3$
$\varepsilon = 1/2$	2.63e-3	1.54e-4	9.45e-6	5.89e-7
rate	-	4.10	4.02	4.00
$\varepsilon = 1/2^4$	4.80e-4	3.32e-5	2.06e-6	1.28e-7
rate	-	3.85	4.01	4.00
$\varepsilon = 1/2^6$	4.11e-4	2.55e-5	1.59e-6	1.00e-7
rate	-	4.01	4.00	3.99

Table 5.3: Spatial error at time  $t = 1$  for ill-prepared initial data Case III, i.e.,  $\alpha = \beta = 0$ .

$e^\varepsilon$	$h = 0.4$	$h/2$	$h/2^2$	$h/2^3$
$\varepsilon = 1/2$	3.29e-3	1.91e-4	1.18e-5	7.32e-7
rate	-	4.11	4.02	4.01
$\varepsilon = 1/2^4$	2.80e-3	1.65e-4	1.01e-5	6.32e-7
rate	-	4.08	4.02	4.01
$\varepsilon = 1/2^6$	2.81e-3	1.64e-4	1.01e-5	6.29e-7
rate	-	4.10	4.02	4.01
$n^\varepsilon$	$h = 0.4$	$h/2$	$h/2^2$	$h/2^3$
$\varepsilon = 1/2$	2.71e-3	1.58e-4	9.76e-6	6.09e-7
rate	-	4.10	4.02	4.00
$\varepsilon = 1/2^2$	2.85e-3	1.69e-4	1.04e-5	6.49e-7
rate	-	4.07	4.02	4.00
$\varepsilon = 1/2^3$	1.50e-3	9.69e-5	6.01e-6	3.75e-7
rate	-	3.95	4.01	4.00
$\varepsilon = 1/2^4$	2.21e-3	1.38e-4	8.57e-6	5.35e-7
rate	-	4.01	4.00	4.00
$\varepsilon = 1/2^5$	4.27e-3	2.64e-4	1.65e-5	1.03e-6
rate	-	4.02	4.00	4.00
$\varepsilon = 1/2^6$	8.48e-3	5.24e-4	3.27e-5	2.04e-6
rate	-	4.02	4.00	4.00

(2.9)-(2.12) for the QZS with Cases I, II, III, respectively. It can be clearly observed that the scheme is uniformly fourth order accurate with respect to  $\varepsilon \in (0, 1]$  for well-prepared and less-ill-prepared initial data (i.e., Cases I and II), which agrees with the theoretical estimate in Theorem 4.1. For ill-prepared initial data (i.e., Case III), Table 5.3 shows the error of the scheme depends on  $\varepsilon$  as  $\mathcal{O}(h^4/\varepsilon)$  for  $N^\varepsilon$ , which confirms the result in Theorem 4.2.

For the temporal errors, we set the mesh size  $h = 1/160$  such that the spatial error can be ignorable. The temporal errors of the compact finite difference method (2.9)-(2.12) for the QZS with Cases I, II, III are shown in Tables 5.4-5.6, respectively. From Tables 5.4-5.6, we clearly observe that our numerical method is convergent at the second order in time for any fixed  $0 < \varepsilon \leq 1$  for all cases of initial data when the time step is small enough. Specifically, the upper triangle parts of Tables 5.4-5.6 (for  $n^\varepsilon$ ) suggest that for each fixed  $0 < \varepsilon \leq 1$ , the error of  $N^\varepsilon$  behaves like  $\mathcal{O}(\tau^2/\varepsilon)$ ,  $\mathcal{O}(\tau^2/\varepsilon^2)$  and  $\mathcal{O}(\tau^2/\varepsilon^3)$  for well-prepared Case I, less-ill-prepared Case II, and ill-prepared Case III initial data, respectively. While the lower triangle parts of Tables 5.4-5.6 show the error of  $N^\varepsilon$  at  $\mathcal{O}(\tau^2 + \varepsilon^2)$ ,  $\mathcal{O}(\tau^2 + \varepsilon)$  and  $\mathcal{O}(\tau^2 + \varepsilon^0)$  for well-prepared, less-ill-prepared and ill-prepared initial data, respectively. The uniform convergence order is attained when the two types of estimates are compatible, which is confirmed by the degeneracy of the error estimates listed in Table 5.7 as  $\tau^2 \sim \varepsilon^3$  for well-prepared and less-ill-prepared initial data. Furthermore, the corresponding temporal convergence rate degenerates as  $\mathcal{O}(\tau^{4/3})$  and  $\mathcal{O}(\tau^{2/3})$ , respectively. These numerical results verify the theoretical estimate in Theorems 4.1 and 4.2.

Table 5.4: Temporal errors at time  $t = 1$  for well-prepared initial data Case I, i.e.,  $\alpha = \beta = 2$ .

$e^\varepsilon$	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$
$\varepsilon = 1/2$	6.70e-3	1.99e-3	5.05e-4	1.27e-4	3.17e-5	7.91e-6	1.95e-6
rate	-	1.75	1.98	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^4$	6.56e-3	1.88e-3	4.74e-4	1.19e-4	2.96e-5	7.38e-6	1.82e-6
rate	-	1.81	1.99	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^6$	6.5138e-03	1.87e-3	4.74e-4	1.18e-4	2.96e-5	7.38e-6	1.82e-6
rate	-	1.80	1.98	2.00	2.00	2.00	2.02
$n^\varepsilon$	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$
$\varepsilon = 1/2$	7.30e-3	1.89e-3	4.76e-4	1.19e-4	2.98e-5	7.43e-6	1.83e-6
rate	-	1.95	1.99	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^2$	1.86e-2	6.05e-3	1.61e-3	4.06e-4	1.02e-4	2.54e-5	6.27e-6
rate	-	1.62	1.91	1.98	2.00	2.00	2.02
$\varepsilon = 1/2^3$	1.42e-2	4.65e-3	2.02e-3	6.07e-4	1.55e-4	3.86e-5	9.55e-6
rate	-	1.61	1.20	1.74	1.97	2.00	2.02
$\varepsilon = 1/2^4$	1.07e-2	5.34e-3	1.96e-3	6.60e-4	2.39e-4	6.32e-5	1.57e-5
rate	-	0.99	1.45	1.57	1.47	1.92	2.01
$\varepsilon = 1/2^5$	4.78e-3	3.04e-3	1.86e-3	8.36e-4	2.65e-4	1.01e-4	2.96e-5
rate	-	0.65	0.71	1.15	1.66	1.40	1.76
$\varepsilon = 1/2^6$	4.97e-3	1.26e-3	8.33e-4	5.90e-4	3.23e-4	1.21e-4	3.90e-5
rate	-	1.98	0.59	0.50	0.87	1.42	1.63

Table 5.5: Temporal errors at time  $t = 1$  for less-ill-prepared initial data Case II, i.e.,  $\alpha = \beta = 1$ .

$e^\varepsilon$	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$
$\varepsilon = 1/2$	7.54e-3	2.12e-3	5.37e-4	1.34e-4	3.36e-5	8.37e-6	2.07e-6
rate	-	1.83	1.98	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^4$	6.71e-3	1.90e-3	4.78e-4	1.19e-4	2.98e-5	7.42e-6	1.83e-6
rate	-	1.82	1.99	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^6$	6.82e-3	1.93e-3	4.81e-4	1.20e-4	2.98e-5	7.42e-6	1.83e-6
rate	-	1.82	2.00	2.01	2.01	2.01	2.02
$n^\varepsilon$	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$
$\varepsilon = 1/2$	8.61e-3	2.21e-3	5.56e-4	1.39e-4	3.48e-5	8.68e-6	2.14e-6
rate	-	1.96	1.99	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^2$	2.49e-2	7.39e-3	1.93e-3	4.86e-4	1.22e-4	3.03e-5	7.50e-6
rate	-	1.76	1.94	1.99	2.00	2.00	2.02
$\varepsilon = 1/2^3$	6.10e-2	1.72e-2	4.71e-3	1.23e-3	3.10e-4	7.73e-5	1.91e-5
rate	-	1.82	1.87	1.94	1.99	2.00	2.02
$\varepsilon = 1/2^4$	1.13e-1	5.23e-2	1.60e-2	4.15e-3	1.06e-3	2.65e-4	6.56e-5
rate	-	1.11	1.71	1.94	1.97	2.00	2.02
$\varepsilon = 1/2^5$	8.20e-2	6.88e-2	4.08e-2	1.49e-2	4.02e-3	1.02e-3	2.52e-4
rate	-	0.25	0.76	1.45	1.89	1.99	2.01
$\varepsilon = 1/2^6$	4.72e-2	4.26e-2	3.91e-2	2.78e-2	1.31e-2	3.92e-3	9.90e-4
rate	-	0.15	0.13	0.49	1.08	1.74	1.99

Table 5.6: Temporal errors at time  $t = 1$  for ill-prepared initial data Case III, i.e.,  $\alpha = \beta = 0$ .

$e^\varepsilon$	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$
$\varepsilon = 1/2$	9.16e-3	2.56e-3	6.47e-4	1.62e-4	4.04e-5	1.01e-5	2.49e-6
rate	-	1.84	1.99	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^2$	1.47e-2	3.86e-3	9.61e-4	2.40e-4	5.98e-5	1.49e-5	3.68e-6
rate	-	1.93	2.01	2.00	2.00	2.01	2.02
$\varepsilon = 1/2^3$	1.16e-2	3.30e-3	8.68e-4	2.13e-4	5.31e-5	1.32e-5	3.26e-6
rate	-	1.82	1.93	2.03	2.01	2.01	2.02
$\varepsilon = 1/2^4$	1.26e-2	3.22e-3	8.38e-4	2.14e-4	5.41e-5	1.36e-5	3.38e-6
rate	-	1.96	1.94	1.97	1.98	1.99	2.01
$\varepsilon = 1/2^5$	2.94e-2	4.83e-3	1.17e-3	3.05e-4	7.85e-5	1.99e-5	4.95e-6
rate	-	2.61	2.05	1.93	1.96	1.98	2.00
$\varepsilon = 1/2^6$	9.98e-2	1.45e-2	2.16e-3	5.11e-4	1.34e-4	3.46e-5	8.70e-6
rate	-	2.79	2.75	2.08	1.93	1.96	1.99
$n^\varepsilon$	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$
$\varepsilon = 1/2$	1.24e-2	3.14e-3	7.89e-4	1.98e-4	4.94e-5	1.23e-5	3.04e-6
rate	-	1.98	1.99	2.00	2.00	2.00	2.02
$\varepsilon = 1/2^2$	6.72e-2	1.76e-2	4.47e-3	1.12e-3	2.81e-4	7.00e-5	1.73e-5
rate	-	1.93	1.98	1.99	2.00	2.00	2.02
$\varepsilon = 1/2^3$	4.60e-1	1.28e-1	3.29e-2	8.27e-3	2.07e-3	5.16e-4	1.28e-4
rate	-	1.84	1.96	1.99	2.00	2.00	2.02
$\varepsilon = 1/2^4$	1.77e+0	8.17e-1	2.48e-1	6.40e-2	1.61e-2	4.02e-3	9.93e-4
rate	-	1.12	1.72	1.95	1.99	2.00	2.02
$\varepsilon = 1/2^5$	2.60e+0	2.19e+0	1.29e+0	4.72e-1	1.27e-1	3.19e-2	7.89e-3
rate	-	0.25	0.76	1.45	1.90	1.99	2.01
$\varepsilon = 1/2^6$	2.96e+0	2.72e+0	2.49e+0	1.77e+0	8.35e-1	2.49e-1	6.28e-2
rate	-	0.12	0.13	0.49	1.09	1.75	1.99

Table 5.7: Degeneracy of temporal error at time  $t = 1$  for  $n^\varepsilon$ . The convergence orders are calculated with respect to time step  $\tau$ .

$\alpha = 2, \beta = 2$	$\tau_0 = 1/8, \varepsilon_0 = 1/2$	$\tau_0/2^3, \varepsilon_0/2^2$	$\tau_0/2^6, \varepsilon_0/2^4$	$\tau_0/2^9, \varepsilon_0/2^6$
$n^\varepsilon$	2.51e-2	2.02e-3	1.01e-4	5.01e-6
rate	-	3.63/3	4.33/3	4.33/3
$\alpha = 1, \beta = 1$	$\tau_0 = 1/8, \varepsilon_0 = 1/2$	$\tau_0/2^3, \varepsilon_0/2^2$	$\tau_0/2^6, \varepsilon_0/2^4$	$\tau_0/2^9, \varepsilon_0/2^6$
$n^\varepsilon$	3.08e-2	4.71e-3	1.02e-3	1.88e-4
rate	-	2.71/3	2.21/3	2.43/3

## 6. Conclusions

We proposed and analyzed a conservative linearly-implicit compact finite difference scheme to solve the quantum Zakharov system (QZS) with a dimensionless parameter  $\varepsilon \in (0, 1]$  which is inversely proportional to the acoustic speed. This method is very efficient in implementation

since one only needs to solve two independent linear system at each time step. When  $0 < \varepsilon \ll 1$ , there exist highly oscillatory initial layers in the solution, and the error estimates of the conservative linearly-implicit compact scheme, especially the dependence of spatial and temporal errors on the mesh size  $h$  and the time step  $\tau$  as well as the parameter  $\varepsilon$  are analysed rigorously. Several numerical simulations are reported to test the error behavior from the theoretical analysis.

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