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NUMERICAL ANALYSIS OF BIFURCATION PROBLEMS OF NONLINEAR EQUATIONS*1)

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Abstract

The paper presents some essential results of branch solutions of nonlinear problems and their numerical approximation. The general theory is applied to the bifurcation problems of the Navier-Stokes equations.

§ 1. Introduction

The purpose of this paper is to study the bifurcation problems of the nonlinear equation

$$F(\lambda, u)=u+T(\lambda)G(\lambda, u)=0 \tag{1.1}$$

and its discretized form

$$F_h(\lambda, u)=u+T_h(\lambda)G(\lambda, u)=0, \tag{1.2}$$

where we assume that for some Banach spaces V and W, $\{T(\lambda);\ \lambda\in\Lambda\}$ and $\{T_h(\lambda);\ \lambda\in\Lambda\}$ are two families of linear bounded mappings from W into V, h is the discrete parameter which tends to 0, and $G(\lambda, u)$ is a nonlinear mapping from $\Lambda\times V$ into W, Λ being a subset of a Banach space.

We consider the bifurcation of the continuous problem (1.1) and the convergence of its numerical approximations. The outline of the paper is as follows.

Section 2 is devoted to general analysis of singular points of nonlinear mapping F and parameterization of its branch solutions. In Section 3 we discuss the approximation of simple limit points of F. Section 4 deals with the numerical prediction of a singular point of F. The bifurcation problem of the Navier-Stokes equations is considered in Section 5 and Section 6 provides a numerical method for computing its branch solutions.

§ 2. Simple Singular Points

Let V, W be Banach spaces, and Λ a subset of a Banach space. Suppose that

1) G: $\Lambda\times V\to W$ is a C^m $(m\geq 2)$ bounded mapping;

2) T, T_h: $\Lambda\times W\to V$ are C^m bounded mappings with respect to λ and for any fixed $\lambda\in\Lambda$, $T(\lambda)$, $T_h(\lambda)\in L(W, V)$.

Define the mappings F, F_h: $\Lambda\times V\to V$ at follows:

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$$F(\lambda, u) = u + T(\lambda)\cdot G(\lambda, u) + u^*, \qquad (2.1)$$
$$F_h(\lambda, u) = u + T_h(\lambda)\cdot G(\lambda, u) + u^*,$$

where u^* is a given point in V.

Theorem 2.1.[7] *Let Λ be a compact set and $u(\lambda)$: $\Lambda\to V$ be a nonsingular solution of F, i.e.*

1) $F(\lambda, u(\lambda)) = 0, \quad \forall\lambda\in\Lambda$;

2) $D_uF(\lambda, u(\lambda))$ *is an isomorphism on* V;

3) $u(\lambda)$ *is a* C^m *mapping.*

If in addition the following conditions are satisfied:

$$\text{i)} \lim_{h\to 0,\lambda\in\Lambda}\sup \|D^l_\lambda T_h(\lambda) - D^l T(\lambda)\| = 0, \quad 0\leqslant l\leqslant m, \qquad (2.2)$$

$$\text{ii)} \sup_{\lambda\in\Lambda}\|D^m_\lambda T_h(\lambda)\|\leqslant C, \quad C \text{ is independent of } h, \qquad (2.3)$$

then there exist constants a, h_0, $K>0$, such that if $h\leqslant h_0$, there is a unique C^m mapping $u_h(\lambda)$: $\Lambda\to V$ satisfying:

$$F_h(\lambda, u_h(\lambda)) = 0, \quad \forall\lambda\in\Lambda, \qquad (2.4)$$
$$\|u_h(\lambda) - u(\lambda)\|\leqslant a,$$

and

$$\|D^l_\lambda u_h(\lambda^*) - D^l u(\lambda)\|\leqslant K\Big\{|\lambda^*-\lambda| + \sum_{i=0}^{l}\Big\|\frac{d^i}{d\lambda^i}[(T_h(\lambda)-T(\lambda))\cdot G(\lambda, u(\lambda))]\Big\|\Big\},$$
$$\forall\lambda^*, \lambda\in\Lambda, \quad 0\leqslant l\leqslant m-1, \qquad (2.5)$$

where $|\cdot|$ stands for the norm of the Banach space that contains Λ.

Definition. *A pair of $(\lambda_0, u_0)\in\Lambda\times V$ is called a simple singular point of F if (λ_0, u_0) satisfies:*

1) $F^0 = F(\lambda_0, u_0) = 0, \qquad (2.6)$

2) $T(\lambda_0)D_uG(\lambda_0, u_0)$ *is a compact operator and -1 is one of its eigenvalues with algebraic multiplicity* 1.

Denote $D_uF^0 = D_uF(\lambda_0, u_0)$, and in the sequel V' stands for the dual space of V and $\langle\,,\rangle$ represents the dual pairing between them.

Lemma 2.1. *Let (λ_0, u_0) be a simple singular point of F. Then there are $\{\varphi_i\}_{i=1}^p\subset V$, $\{\varphi_i^*\}_{i=1}^p\subset V'$ ($p\geqslant 1$ integer) such that*

$$D_uF^0\varphi_i = 0, \quad \|\varphi_i\| = 1, \quad 1\leqslant i\leqslant p,$$
$$V_1\equiv \mathrm{Ker}(D_uF^0) = [\varphi_1, \varphi_2, \cdots, \varphi_p],$$
$$(D_uF)^*\varphi_i^* = 0, \quad \langle\varphi_i, \varphi_j^*\rangle = \delta_{ij},$$

and

$$V_2\equiv \mathrm{Range}(D_uF^0) = [\varphi_1^*, \varphi_2^*, \cdots, \varphi_p^*]^\perp,$$
$$V = V_1 \dotplus V_2,$$
$$D_uF^0 \text{ is an isomorphism from } V_2 \text{ onto } V_2,$$

where $[\varphi_1, \varphi_2, \cdots, \varphi_p]$ is a linear space spanned by $\varphi_1, \varphi_2, \cdots, \varphi_p$.

The proof can be found in [10].

For simplicity, we shall write $L = (D_uF^0/V_2)$ as the inverse isomorphism of D_uF^0 on V_2. Let us now define a projection Q: $V\to V_2$ by

$$Qv = v - \sum_{i=1}^{p}\langle v, \varphi_i^*\rangle\varphi_i.$$