FINITE ELEMENT APPROXIMATION TO AXIAL SYMMETRIC STOKES FLOW*

YING LONG-AN (应隆安)
(*Peking University, Beijing, China*)

The finite element method for Stokes flow has been extensively and intensively studied, and the method for axial symmetric elliptic problems has also been touched, see e.g. [1]. The purpose of this paper is to discuss the finite element method for axial symmetric Stokes flow and prepare for the discussion of the infinite element approximation to axial symmetric Stokes flow, which will be published in another paper.

Let us give the classical statement of the three dimensional axial symmetric Stokes flow. Let $x=(x_1, x_2)\in\mathbb{R}^2$ and Ω be a bounded polygonal region on the half plane $x_1>0$. We consider the following problem: to find $u(x)=(u_1(x), u_2(x))$ and $p(x)$, satisfying

$$\nu(-\nabla(x_1\nabla u_1)/x_1+u_1/x_1^2)+\frac{\partial p}{\partial x_1}=f_1, \quad x\in\Omega,$$

$$-\nu\nabla(x_1\nabla u_2)/x_1+\frac{\partial p}{\partial x_2}=f_2, \quad x\in\Omega,$$

$$\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)=0, \quad x\in\Omega,$$

$$u=0, \quad x\in\partial\Omega\backslash\{x_1=0\},$$

$$u_1=0, \quad x\in\partial\Omega\cap\{x_1=0\}.$$

If Ω rotates around the x_2-axis, then a three-dimensional region $\tilde{\Omega}$ is formed. The above problem is a description of the incompressible viscous flow on $\tilde{\Omega}$ with low Reynold's number, where the constant $\nu>0$ is viscosity, u velocity, p pressure and $f=(f_1, f_2)$ body force.

We need some weighted Sobolev spaces for the above problem. First we define the seminorm and norm as

$$|f|_{m,\Omega}=\Big(\sum_{|\alpha|=m}\int_\Omega x_1|D^\alpha f|^2dx\Big)^{1/2},$$

$$\|f\|_{m,\Omega}=\Big(\sum_{l=0}^m|f|_{l,\Omega}^2\Big)^{1/2}.$$

The corresponding Hilbert spaces are donoted by $Z^m(\Omega)$, where $\alpha=(\alpha_1, \alpha_2)$, $D^\alpha=\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. Then we define the norm as

$$|f|_{1,*,\Omega}=(|f|_{1,\Omega}^2+\|f/x_1\|_{0,\Omega}^2)^{1/2},$$

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* Received September 3, 1984.

If f can be expressed as $f=f_1f_2$, where $f_1\in C^\infty(\bar\Omega)$, $f_2\in C_0^\infty(\mathbb{R}_+^2)$, $\mathbb{R}_+^2=\{x\in\mathbb{R}^2;\ x_1>0\}$, then we denote $f\in C_*^\infty(\bar\Omega)$. The completion of $C_*^\infty(\bar\Omega)$ with respect to the norm $\|\cdot\|_{1,*,\Omega}$ is denoted by $Z_*^1(\Omega)$. We define $f\in Z_+^2(\Omega)$ if and only if $f\in Z_*^1(\Omega)\cap Z^2(\Omega)$, and $\|D^{(0,2)}f/x_1\|_{0,\Omega}$ is bounded.

Let $H(\Omega)=Z_*^1(\Omega)\times Z^1(\Omega)$, $H_0(\Omega)=\{f\in H(\Omega);\ f|_{\partial\Omega\setminus(x_1=0)}=0\}$, $M_0(\Omega)=\left\{p\in Z^0(\Omega);\ \int_\Omega x_1p\,dx=0\right\}$. We consider the bilinear form on $H(\Omega)\times H(\Omega)$:

$$a(u,\ v)=\nu\int_\Omega x_1(\nabla u_1\cdot\nabla v_1+\nabla u_2\cdot\nabla v_2+u_1v_1/x_1^2)dx,\quad u,\ v\in H(\Omega), \tag{1}$$

and the bilinear form on $H(\Omega)\times Z^0(\Omega)$:

$$b(v,\ p)=-\int_\Omega p\left\{\frac{\partial}{\partial x_1}(x_1v_1)+\frac{\partial}{\partial x_2}(x_1v_2)\right\}dx,\quad v\in H(\Omega),\ p\in Z^0(\Omega). \tag{2}$$

Then a weak formulation of the original problem is: to find $(u,\ p)\in H_0(\Omega)\times M_0(\Omega)$, such that

$$a(u,\ v)+b(v,\ p)=F(v),\quad \forall v\in H_0(\Omega), \tag{3}$$

$$b(u,\ q)=0,\quad \forall q\in M_0(\Omega), \tag{4}$$

where

$$F(v)=\int_\Omega x_1(f_1v_1+f_2v_2)dx.$$

We see from definitions (1), (2) that a, b are bounded and

$$a(u,\ u)=\nu(|u_1|_{1,*,\Omega}^2+|u_2|_{1,\Omega}^2),$$

and we notice that the inequality of Poincaré–Friedrichs type

$$\|u_1\|_{0,\Omega}^2+\|u_2\|_{0,\Omega}^2\leqslant Ca(u,\ u)$$

holds on $H_0(\Omega)$. Throughout the paper C will always denote a positive constant. We have

$$a(u,\ u)>a_0\|u\|_{H(\Omega)}^2,\quad \forall u\in H_0(\Omega), \tag{5}$$

where $a_0>0$. Moreover, if f_1, f_2 are appropriately regular, then problem (3), (4) has a unique solution[2].

Now we consider the finite element approximation to problem (3), (4). The region Ω is divided into finite convex polygonal regions Ω_k, $k=1,\ 2,\ \cdots$, by finite broken lines. Then each subregion Ω_k is further divided into triangular elements, and it is assumed that Ω_k keep fixed in the further refinement process. It is also assumed that any two elements in Ω meet only in the entire common side, or at only a common vertex, or do not meet at all. The vertices and midpoints of the sides of all elements are taken as nodes. The element is denoted by e, and the side is denoted by s, where each e is an open set and the end points are not included in s. We make quadratic polynomial interpolation for u, and p is a constant on e. Then the subspaces $H_{0h}(\Omega)$, $M_{0h}(\Omega)$ of $H_0(\Omega)$, $M_0(\Omega)$ are obtained, and so are the subspaces $H_h(\Omega_k)$, $M_h(\Omega_k)$ of $H(\Omega_k)$, $Z^0(\Omega_k)$.

This kind of triangulation and interpolation causes loss of precision[3]. To overcome this shortcoming, there are several approaches, see e.g. [4], [5]. But for simplicity, we only consider this kind of element.

The finite element approximation to problem (3), (4) is: to find $(u_h,\ p_h)\in$