# PARTICLE APPROXIMATION OF FIRST ORDER SYSTEMS*

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## Introduction

By particle methods of approximation of time-dependent problems in partial differential equations, we mean numerical methods where, for each time $t$, the exact solution is approximated by a linear combination of Dirac measures in the space variables $x$. Although these methods have not yet a very large range of applications as that of classical methods (finite difference methods, finite element methods or even spectral methods), they provide an effective way of solving convection-dominated problems. In fact, particle methods are commonly used in some problems of Physics and Fluid Mechanics.

In Physics, these methods have been considered very early for the numerical solution of kinetic equations such as Boltzmann, Vlasov or Fokker-Planck equations and have been mainly based on a Monte Carlo methodology. More recently, particle methods have received a great deal of attention in Plasma Physics and are now currently used in a number of physical problems. In that direction, see the book of Hockney and Eastwood[9].

In Fluid Mechanics, vortex simulations of incompressible fluid flows at high Reynolds numbers have been first introduced by Rosenhead and subsequently developed by Chorin, Leonard and Rehbach among other contributors (see the survey of Leonard[10]). On the other hand, particle in cell (P.I.C.) methods have been introduced by Harlow[8] for the numerical computation of compressible multifluid flows. Recently Gingold and Monaghan[7] have proposed a new particle method which may be viewed as an improvement of the P.I.C. method.

The purpose of this paper is to review some recent results recently obtained by the author in joint works with S. Gallic and J. Ovadia concerning the particle approximation of hyperbolic and parabolic systems and which are related to the approach of Gingold and Monaghan. In Section 1, we describe a particle method of approximation of first-order linear symmetric systems. Convergence results are stated in Section 2: they generalize previous results of the author on the particle approximation of hyperbolic equations[12,13]. We show in Section 3 how to adapt the method to the nonlinear hyperbolic system of gas dynamics, hence generalizing the ideas of [7]. Finally, Section 4 is devoted to the extension of the method to the numerical treatment of convection-diffusion equations.

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For a mathematical study of the vortex method, we refer to Beale and Majda[1,2] and the thesis of Cottet[3]; see also [12]. For a proof of convergence of the particle method for the Vlasov–Poisson equations arising in Plasma Physics, see [4].

## § 1. Description of the Particle Method

Let us consider the Cauchy problem for first-order systems written in conservation form

$$\begin{cases} \dfrac{\partial u}{\partial t}+\sum\limits_{i=1}^{n}\dfrac{\partial}{\partial x_i}(A^i u)+A^0 u=f, & x\in\mathbb{R}^n,\ t>0, \\ u(x,\,0)=u_0(x). \end{cases} \tag{1.1}$$

Here $u=u(x,\,t)$, $f=f(x,\,t)$ are column vectors with $p$ components and $A^i=A^i(x,\,t)$, $0\leqslant i\leqslant n$, are $p\times p$ matrices. Setting $Q_T=\mathbb{R}^n\times]0,\,T[$, $T>0$, we assume that

$$\begin{cases} A^i\in L^\infty(Q_T;\ \mathfrak{L}(\mathbb{R}^p)), & 0\leqslant i\leqslant n, \\ \dfrac{\partial A^i}{\partial x_j}\in L^\infty(Q_T;\ \mathfrak{L}(\mathbb{R}^p)), & 1\leqslant i,\,j\leqslant n \end{cases} \tag{1.2}$$

and

$$A^i(x,\,t)=A^i(x,\,t)^T,\quad 1\leqslant i\leqslant n. \tag{1.3}$$

Then, given $u_0\in L^2(\mathbb{R}^n)^p$ and $f\in L^1(0,\,T;\,L^2(\mathbb{R}^n)^p)$, it is a classical result that Problem (1.1) has indeed a unique weak solution $u\in C^0(0,\,T;\,L^2(\mathbb{R}^n)^p)$.

Assume next that the data $A^i$, $0\leqslant i\leqslant n$, $u_0$ and $f$ are continuous functions. In order to approximate the solution $u$ of Problem (1.1) by a particle method, we begin by introducing a system of moving coordinates. We write

$$A^i=a^i I+B^i,\quad 1\leqslant i\leqslant n, \tag{1.4}$$

where $I$ is the $p\times p$ identity matrix and the functions $a^i$ are continuous and satisfy

$$a^i,\ \frac{\partial a^i}{\partial x_j}\in L^\infty(Q_T),\quad 1\leqslant i,\,j\leqslant n. \tag{1.5}$$

Then, we consider the differential system

$$\frac{dx}{dt}=a(x,\,t),\quad a=(a_1,\,\cdots,\,a_n), \tag{1.6}$$

whose solutions are the characteristic curves associated with the first order differential operator

$$\frac{\partial}{\partial t}+\sum_{i=1}^{n}a_i\frac{\partial}{\partial x_i}.$$

We denote by $t\rightarrow x(\xi,\,t)$ the unique solution of (1.6) which satisfies the initial condition

$$x(0)=\xi,\quad \xi\in\mathbb{R}^n \tag{1.7}$$

and we set

$$J(\xi,\,t)=\det\left(\frac{\partial x_i}{\partial \xi_j}(\xi,\,t)\right). \tag{1.8}$$

Then it is a simple and classical matter to check that

$$\frac{\partial J}{\partial t}(\xi,\,t)=J(\xi,\,t)(\operatorname{div}a)(x(\xi,\,t),t),\quad \operatorname{div}a=\sum_{i=1}^{n}\frac{\partial a^i}{\partial x_i}. \tag{1.9}$$