A SYSTEM OF PLANE ELASTICITY CANONICAL INTEGRAL EQUATIONS AND ITS APPLICATION*

YU DE-HAO (余德浩)

(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

In this paper, we obtain a new system of canonical integral equations for the plane elasticity problem over an exterior circular domain, and give its numerical solution. Coupling with the classical finite element method, it can be used for solving general plane elasticity exterior boundary value problems. This system of highly singular integral equations is also an exact boundary condition on the artificial boundary. It can be approximated by a series of nonsingular integral boundary conditions.

§ 1. Introduction

The canonical boundary reduction, suggested by Feng Kang[1], can be applied to the plane elasticity problem[2,4]. Let Ω be a domain with smooth boundary Γ. Taking displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in directions x_1 and x_2 as basic unknown functions, we have the plane elasticity equations with traction boundary conditions as follows

$$\begin{cases} (\lambda+2\mu) \text{ grad div } \boldsymbol{u}-\mu \text{ rot rot } \boldsymbol{u}=0, & \text{in } \Omega, \\ \sum_{j=1}^{2} \sigma_{ij}n_j=g_i, \quad i=1, 2, & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\boldsymbol{u}=u_1\boldsymbol{e}_1+u_2\boldsymbol{e}_2$, σ_{ij} $(i, j=1, 2)$ are components of stress, λ and μ are Lamè coefficients, $\lambda>0$, $\mu>0$, and (n_1, n_2) are the outward normal direction cosines on Γ.

Let

$$\mathscr{R}=\{\boldsymbol{v}\in H^1(\Omega)^2 \mid \boldsymbol{v}=(c_1-c_3x_2, c_2+c_3x_1)^T, c_1, c_2, c_3\in \mathbb{R}\};$$

then problem (1) has one and only one solution in $H^1(\Omega)^2/\mathscr{R}$ when $\boldsymbol{g}$ satisfies the consistency conditions

$$\int_\Gamma g_i ds=0, \; i=1, 2 \quad \text{and} \quad \int_\Gamma (x_1g_2-x_2g_1)ds=0.$$

From [4] we know that the boundary value problem (1) is equivalent to the canonical integral equation on Γ:

$$\boldsymbol{g}(s)=\int_\Gamma K(s, s')\boldsymbol{u}_0(s')ds'. \tag{2}$$

Particularly, by using the separation of variables, [4] has given a system of canonical integral equations with respect to exterior circular domain:

* Received October 17, 1984.

$$\begin{bmatrix} g_r(\theta) \\ g_\theta(\theta) \end{bmatrix} = \begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix} * \begin{bmatrix} u_r(R, \theta) \\ u_\theta(R, \theta) \end{bmatrix}, \tag{3}$$

where

$$K_{rr} = K_{\theta\theta} = -\frac{ab}{(a+b)2\pi R \sin^2\theta/2} + \frac{2b^2}{(a+b)R}\delta(\theta) + \frac{ab}{\pi R(a+b)},$$

$$K_{r\theta} = -K_{\theta r} = -\frac{ab}{(a+b)\pi R}\operatorname{ctg}\frac{\theta}{2} + \frac{2b^2}{(a+b)R}\delta'(\theta),$$

$$\boldsymbol{g} = g_r\boldsymbol{e}_r + g_\theta\boldsymbol{e}_\theta, \quad \boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta, \quad a = \lambda + 2\mu, \quad b = \mu,$$

$*$ denotes the convolution, and R is the radius of the circle.

When Ω is the exterior to an arbitrary smooth closed curve Γ, the solution of problem (1) must satisfy the condition at infinity, i.e. u_1 and u_2 are bounded at infinity. Then problem (1) has one and only one solution in $W_0^1(\Omega)^2/\mathscr{R}$, where

$$W_0^1(\Omega) = \left\{\frac{u}{\sqrt{1+r^2}\ln(2+r^2)} \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i=1, 2,\ r = \sqrt{x_1^2+x_2^2}\right\},$$

$$\mathscr{R} = \{\boldsymbol{v} \in W_0^1(\Omega)^2 \mid \boldsymbol{v} = (c_1, c_2)^T,\ c_1, c_2 \in \boldsymbol{R}\},$$

when $\boldsymbol{g}$ satisfies the consistency conditions:

$$\int_\Gamma g_i ds = 0, \quad i = 1, 2.$$

We can draw in Ω a circle Γ' with radius R. Then the original problem reduces to a new boundary value problem over a bounded domain Ω_1; its boundary condition on the artificial boundary Γ' is just the system of canonical integral equations (3).

Of course the boundary condition on Γ' is not unique. Recently [5] gives another integral boundary condition. Noticing that

$$\begin{cases} \sum\limits_{n=1}^{\infty}\frac{1}{n}\cos n\theta = -\ln\left|2\sin\frac{\theta}{2}\right|, \\ \sum\limits_{n=1}^{\infty} n\cos n\theta = -\frac{1}{4\sin^2\theta/2}, \\ \sum\limits_{n=1}^{\infty} n\sin n\theta = -\pi\delta'(\theta), \end{cases} \tag{4}$$

we can write that condition as

$$\begin{bmatrix} -\frac{\partial u_1}{\partial r} + p\cos\theta \\ -\frac{\partial u_2}{\partial r} + p\sin\theta \end{bmatrix}_{\Gamma'}$$

$$= \frac{1}{R}\begin{bmatrix} \frac{2a}{a+b}\left(-\frac{1}{4\pi\sin^2\theta/2}\right) & -\frac{a-b}{a+b}\delta'(\theta) \\ \frac{a-b}{a+b}\delta'(\theta) & \frac{2a}{a+b}\left(-\frac{1}{4\pi\sin^2\theta/2}\right) \end{bmatrix} * \begin{bmatrix} u_1(R, \theta) \\ u_2(R, \theta) \end{bmatrix}, \tag{5}$$

where $p = -\frac{\lambda+\mu}{\mu}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)$. Obviously, it is different from (3) in that the left side of (5) is not the tractions on the artificial boundary Γ'.

Below we shall apply the method of complex analysis and obtain a new integral