THE UNSOLVABILITY OF INVERSE ALGEBRAIC EIGENVALUE PROBLEMS ALMOST EVERYWHERE*

SUN JI-GUANG (孙继广) YE QIANG (叶 强)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

This paper revises the definition for the unsolvability of inverse algebraic eigenvalue problems almost everywhere (a.e.) given by Shapiro [5], and gives some sufficient and necessary conditions such that the inverse algebraic eigenvalue problems are unsolvable a.e.

§ 1. Introduction

The general inverse algebraic eigenvalue problems are the following problems:

Problem G-1. Given $m+1$ real $n\times n$ symmetric matrices $A, A_1, \cdots, A_m$, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n (k\geqslant 1)$, find a real m-dimensional vector $c=(c_1, \cdots, c_m)^T$ such that the matrix $A+\sum_{i=1}^m c_iA_i$ has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (see [2], [5], [7]).

Problem A-1. Given a real $n\times n$ symmetric matrix A, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real n-dimensional vector $c=(c_1, \cdots, c_n)^T$ such that the matrix $A+\mathrm{diag}(c_1, \cdots, c_n)$ has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (see [2], [3]).

Problem G-2. Given $m+1$ real $n\times n$ matrices $A_0, A_1, \cdots, A_m$, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n (k\geqslant 1)$, find a real m-dimensional vector $c=(c_1, \cdots, c_m)^T$ such that the matrix $A+\sum_{i=1}^m c_iA_i$ is diagonalizable and has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (ref. [2]).

Problem A-2. Given a real $n\times n$ matrix A, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real n-dimensional vector $c=(c_1, \cdots, c_n)^T$ such that the matrix $A+\mathrm{diag}(c_1, \cdots, c_n)$ is diagonalizable and has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (ref. [2], [3]).

Problems A-1 and A-2 are additive inverse eigenvalue problems. Problems G-1 and G-2 are general inverse eigenvalue problems. Some of these problems arise often

* Received January 15, 1985.

in applied mathematics, and are studied by many authors (see [2]—[5], [7] and the references contained therein).

Recently A. Shapiro [5] has defined the unsolvability of Problem G–1 almost everywhere (a.e.), and has given a sufficient condition such that Problem G–1 is unsolvable a.e. By [5] Problem G–1 is said to be unsolvable a.e. if the set of matrices A (in the vector space of $n\times n$ symmetric matrices) at which it is solvable has measure zero. Shapiro [5] has proved the following conclusion.

Theorem 1.1. Problem G–1 is unsolvable a.e., if

$$\sum_{t=0}^{k}\frac{r_t(r_t+1)}{2}>m+k. \tag{1.1}$$

Undoubtedly, the study of the unsolvability of inverse eigenvalue problems a.e. is important. But it seems that the above mentioned definition given by Shapiro [5] is not enough to clarify the concept of unsolvability a.e. and condition (1.1) looks too strong. In this paper we give a more reasonable definition for the unsolvability of inverse eigenvalue problems a.e., and give some sufficient and necessary conditions such that Problems G–1, A–1, G–2 and A–2 are unsolvable a.e. respectively.

This paper uses the following notation. The symbol $\mathbb{R}^{m\times n}$ denotes the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. $I^{(n)}$ is the $n\times n$ identity matrix, and O is the null matrix. The superscript T is for transpose, and

$$S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^T=A\},\quad \mathbb{O}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^TA=I\},$$

$$S\mathbb{R}_0^{n\times n}=\{A=(a_{ij})\in S\mathbb{R}^{n\times n}:\ a_{ii}=0,\ 1\leqslant i\leqslant n\},$$

$$\mathbb{R}_0^{n\times n}=\{A=(a_{ij})\in \mathbb{R}^{n\times n}:\ a_{ii}=0,\ 1\leqslant i\leqslant n\}.$$

Besides, for $A=(a_{ij})\in\mathbb{R}^{n\times n}$ we write

$$|A|=(|a_{ij}|),\quad k_1(A)=\max_{1\leqslant j\leqslant n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|\Big)$$

and

$$k_2(A)=\max_{1\leqslant j\leqslant n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}a_{ij}^2\Big)^{1/2}.$$

Now we define the unsolvability of inverse eigenvalue problems a.e.

Definition 1.1. *Problem G–1 is said to be unsolvable almost everywhere (u.s.a.e.) if the set of matrices $A, A_1, \cdots, A_m\in S\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space* $\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{m+1}\times\mathbb{R}^k$.

Definition 1.2. *Problem A–1 is said to be u.s.a.e. if the set of matrices $A\in S\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $S\mathbb{R}^{n\times n}\times\mathbb{R}^k$.*

Definition 1.3. *Problem G–2 is said to be u.s.a.e. if the set of matrices $A, A_1, \cdots, A_m\in\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space* $\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m+1}\times\mathbb{R}^k$.

Definition 1.4. *Problem A–2 is said to be u.s.a.e. if the set of matrices $A\in\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $\mathbb{R}^{n\times n}\times\mathbb{R}^k$.*