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VARIABLE-ELLIPTIC-VORTEX METHOD FOR INCOMPRESSIBLE FLOW SIMULATION*[1)]

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Abstract

A variable-elliptic-vortex method, which is a generalization of the elliptic-vortex method proposed by the author in [1], is presented for the numerical simulation of incompressible flows. The most attractive feature of the new method is that the numerical vortex blobs used in this model like actual vortex blobs can be translated, rotated and deformed in elliptic shape. The new method provides a more reasonable and more accurate approach for flow simulation than the fixed-vortex methods. Numerical examples are presented to demonstrate the performance of the new method.

§1. Introduction

Vortex methods have provided an attractive and successful approach for the numerical simulation of incompressible fluid flows at high Reynolds number. The features of these methods are as follows: (1) the interactions of the numerical vortices mimic the physical mechanisms in actual fluid flow; (2) vortex methods are automatically adaptive, since the vortex "blobs" concentrate in the regions of physical interest; and (3) there are no inherent errors with behavior like the numerical viscosity of Eulerian difference methods. Such numerical viscosity often obscures the effects of physical viscosity in high Reynolds number flow simulation.

The first attempts by Rosenhead [2] to simulate flows by a vortex method used point vortices. But the point vortex method introduces a singularity of the velocity field in its centre. Chorin [3] and Kuwahara and Takami [4] introduced a vortex method with finite cores or vortex blobs, to smooth out the singularity and to stabilize the method. There have been a large number of successful flow simulations by vortex blob methods (see Leonard [5]).

It is however noticed that up to now all of the numerical vortex blobs in use are assumed to retain fixed shape for all time while the actual flow can undergo substantial distortion. The "unphysical behavior" of vortex blobs reduces the accuracy of the vortex methods, even though it does not interfere with the convergence of vortex methods ([6], [7], [13]). Thus there is considerable interest in finding an appropriate approach to form a method with variable vortex blobs, which can follow the distortion of actual vortex blobs.

In this paper a variable-elliptic-vortex method is presented to meet this need, which is a generalization of the elliptic-vortex method proposed by the author in [1]. The most attractive feature of the new model is that the variable vortex blobs

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can be translated, rotated and deformed in the shape of elliptic type according to the decomposition theorem of velocity in a small neighborhood. The main merits of the new model are as follows: (1) it provides a more flexible and more reasonable approach to mimic physical flows; (2) it has higher order accuracy in space than the fixed shape vortex method.

§ 2. Approximate Motion of a Small Elliptic Blob

In this paper we are mainly concerned with incompressible inviscid flows in two dimensions satisfying the Euler equations in the vortex form

$$\begin{cases} \frac{\partial\xi}{\partial t}+u\frac{\partial\xi}{\partial x}+v\frac{\partial\xi}{\partial y}=0, \\ \Delta\psi=-\xi, \\ u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x}, \end{cases} \tag{1}$$

where $\boldsymbol{u}=(u, v)$ is the velocity vector, $\boldsymbol{r}=(x, y)$ is the position, t is the time, ψ is the stream function, $\xi=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ is the vorticity, $\Delta\equiv\nabla^2$ is the Laplace operator.

In the following we consider an approximate motion of a small blob Ω_0 in fluid. Let Ω_0 be an elliptic blob at $t=0$ centred at $\boldsymbol{a}_0=(\alpha_{10}, \alpha_{20})$ defined by

$$\Omega_0=\{\boldsymbol{a}\,|\,(\boldsymbol{a}-\boldsymbol{a}_0)A(\boldsymbol{a}-\boldsymbol{a}_0)^T\leqslant 1,\ \boldsymbol{a}\in\mathbb{R}^2\},$$

where $A=(a_{ij})$ is a 2×2 positive definite matrix. A small elliptic blob Ω_0 means that its major axis is small.

We will use $\boldsymbol{a}=(\alpha_1, \alpha_2)$ for the Lagrangian coordinates of a fluid particle. Thus a particle starting at the position $\boldsymbol{a}\in\Omega_0$ at $t=0$ follows a trajectory $\boldsymbol{r}(t;\ \boldsymbol{a})$ determined by the equation

$$\begin{cases} \frac{d\boldsymbol{r}}{dt}=\boldsymbol{u}(\boldsymbol{r}, t), \\ \boldsymbol{r}(0;\ \boldsymbol{a})=\boldsymbol{a}. \end{cases} \tag{2}$$

In writing equations to approximate (2), we expand $\boldsymbol{u}(\boldsymbol{r}, t)$ at $\boldsymbol{r}_0(t)=\boldsymbol{r}(t;\ \boldsymbol{a}_0)$, the trajectory of the center $\boldsymbol{a}_0$ of Ω_0, by Taylor's theorem

$$\boldsymbol{u}(\boldsymbol{r}, t)=\boldsymbol{u}(\boldsymbol{r}_0, t)+(\boldsymbol{r}-\boldsymbol{r}_0)\cdot\nabla\boldsymbol{u}(\boldsymbol{r}_0, t)^T+O(|\boldsymbol{r}-\boldsymbol{r}_0|^2),$$

where $\nabla\boldsymbol{u}=\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$ denotes the Jacobian matrix of $\boldsymbol{u}$ and $(\boldsymbol{r}-\boldsymbol{r}_0)\cdot\nabla\boldsymbol{u}(\boldsymbol{r}_0, t)^T$ is a matrix multiplication. Substituting the expression into (2) and neglecting the term of $O(|\boldsymbol{r}-\boldsymbol{r}_0|^2)$, we get an approximate system

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{u}(\boldsymbol{r}_0, t)+(\boldsymbol{r}-\boldsymbol{r}_0)\cdot\nabla\boldsymbol{u}(\boldsymbol{r}_0, t)^T \tag{3}$$

which is a linear ordinary differential system if $\boldsymbol{r}_0(t)$ is assumed to be a known trajectory. From above we know that (3) approximates (2) with second order accuracy in space. In using notations of $\boldsymbol{z}=\boldsymbol{r}-\boldsymbol{r}_0$ and $\boldsymbol{\beta}=\boldsymbol{a}-\boldsymbol{a}_0$, (3) becomes