ON A CLASS OF NON-LINEAR METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, a class of non-linear methods proposed in [1] is discussed. A new derivation of the methods is given. The analysis based on the new derivation shows that this class of methods is not suitable for stiff problems. The numerical tests support our argument.

§ 1. Introduction

In [1], Qin Zeng-fu put forward a class of non-linear methods for numerical integration of ordinary differential equations. The derivation of the formulas is based on the Frenet frame and the regular representation of curves.

Let the initial value problem be in the form

$$\begin{cases} \dfrac{dy_i}{dx}=f_i(x, y_1, \cdots, y_m), \\ y_i(x_0)=y_{i0}, \quad i=1, 2, \cdots, m. \end{cases} \tag{1.1}$$

By introducing

$$y_0=x,$$
$$f_0(y_0, y_1, \cdots, y_m)=1$$

and writing

$$Y=(y_0, y_1, \cdots, y_m)^T,$$
$$F=(f_0, f_1, \cdots, f_m)^T,$$

the initial value problem (1.1) can be rewritten in the form

$$\begin{cases} \dfrac{dY}{dx}=F(Y), \\ Y(x_0)=Y_0. \end{cases} \tag{1.2}$$

The solution of (1.2) is a curve in the space R^{m+1}. With the aid of the Frenet frame and the regular representation of the solution curve a class of non-linear formulas can be constructed. The derivation is rather complicated, for detail see [1]. Two of the non-linear formulas are as follows:

(I) $$Y_{n+1}=Y_n+\frac{h}{2}\left(\frac{1}{l_n}F_n+\frac{1}{l_n^*}F_n^*\right),$$

where

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$$l_n = \|F_n\|_2,$$
$$F_n^* = F\left(Y_n + h\frac{1}{l_n}F_n\right),$$
and
$$l_n^* = \|F_n^*\|_2,$$

(II)
$$Y_{n+1} = Y_n + \frac{h}{l_n}F_n + \frac{h^2}{6}\left(\left(\frac{1}{l_n}U_n - \frac{q}{l_n^4}F_n\right) + 2\left(\frac{1}{l_n^{*2}}U_n^* - \frac{q}{l_n^{*4}}F_n^*\right)\right),$$
where
$$l_n = \|F_n\|_2,$$
$$F_n^* = F\left(Y_n + \frac{h}{2l_n}F_n + \frac{h^2}{8}\left(\frac{1}{l_n^2}U_n - \frac{q}{l_n^4}F_n\right)\right),$$
$$U_n = \frac{\partial F(Y_n)}{\partial Y}F_n,$$
$$q_n = F_n^T \cdot U_n,$$
$$l_n^* = \|F_n^*\|_2,$$
$$U_n^* = U\left(Y_n + \frac{h}{2l_n}F_n + \frac{h^2}{8}\left(\frac{1}{l_n^2}U_n - \frac{q}{l_n^4}F_n\right)\right),$$
$$q_n^* = U_n^{*T} \cdot F_n^*.$$

It should be pointed out that the h in the formulas is a step-size in arc length along the solution curve rather than a common step-size in the independent variable. It has been shown that formula (I) is a two-stage method of order 2 and formula (II) a two-stage method of order 4. Formula (II) is recommended by Qin Zeng-fu for solving stiff ordinary differential equations. In addition the step-size criterion derived from (I) or (II) (for the standard test equation $y = \lambda y$) is

$$h < \frac{2l^3}{\alpha(l^2+1)}, \quad \alpha = \text{Re}\,\lambda, \tag{1.3}$$

$$h_{\text{perm}} = \frac{4(l^2-1)}{\varkappa l^2(l^2+1)} \tag{1.4}$$

where $\varkappa$ is the curvature.

In this paper, we give another derivation of the non-linear methods by which a wider class of non-linear formulas can be easily constructed. However it can be seen from the new derivation that the non-linear formulas of [1] are essentially the results obtained by applying certain "explicit linear methods" to the ordinary differential equations which have been transformed with an independent variable transformation. One can expect that such kind of methods will have the same restriction on step-size as a general explicit method. The analysis in § 3 verifies this expectation and the numerical tests in § 4 is identical with the analysis.

§ 2. A New Derivation of the Formulas

Consider the initial value problem

$$\begin{cases} \dfrac{dY}{dx} = F(Y), \\ Y(x_0) = Y_0. \end{cases} \tag{2.1}$$