THE CALAHAN METHOD FOR PARABOLIC EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS*

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Abstract

A modified Calahan method for parabolic equations with time-dependent coefficients is presented. It is shown that the convergence order is $O(h^{r+1}+k^3)$ while the convergence order obtained in [1] for a standard Calahan method is only $O(h^{r+1}+k^2)$.

§ 1. Introduction

Sammon[1] considered the single step methods (including fully discrete finite element methods) for parabolic equations with time-dependent coefficients (PETC). The convergence order he obtained is $O(h^{r+1}+k^{\min(2,q)})$ under the restriction $k\leqslant Ch^2$ where h and k are respectively the step lengths of space and time, while $O(h^{r+1}+k^q)$ is the optimal order that we usually get for parabolic equations with time-independent coefficients. It was asserted in [1] that this estimate cannot be generally improved. It is then an interesting problem whether we could give some schemes for the PETC with convergence order better than $O(h^{r+1}+k^{\min(2,q)})$ in some special cases. As a trial in this respect, we consider in this paper a modified Calahan method ($q=3$ in this case, see [2]) for the PETC. The convergence order we shall show is optimal, $O(h^{r+1}+k^3)$, under the same restriction $k\leqslant Ch^2$ as in [1].

We shall describe in this section the problem to be dealt with and give some notations and definitions. In Sections 2 and 3 we shall do some preparative work and in the last section we shall give our main result, the convergence order estimate.

Let Ω be a bounded region in R^N with smooth boundary Γ and $t^*>0$ a constant, $T=[0, t^*]$. We consider the following PETC,

$$\frac{du}{dt}=Au, \quad (x, t)\in\Omega\times T, \tag{1.1}$$

$$u|_\Gamma=0, \quad t\in T, \tag{1.2}$$

$$u|_{t=0}=u^0, \quad x\in\Omega, \tag{1.3}$$

where
$$A=A(t)=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right), \quad a_{ij}=a_{ij}(x,t)$$

are prescribed smooth functions in $\overline{\Omega}\times T$, $\overline{\Omega}=\Omega+\Gamma$; $u^0=u^0(x)$ is a given smooth function in $\overline{\Omega}$.

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Let $H^s=H^s(\Omega)$ $(s\geq 0)$ be Sobolev spaces of degree s with norm $\|\cdot\|_s$ and H_0^1 the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_1$; $(\ ,\ )$ denotes the inner product in $L^2=H^0$. We also use $(\ ,\ )$ to denote the dual pair (cf. [5]) of H_0^1 and its dual space $(H_0^1)'=H^{-1}$ normed by

$$\|v\|_{-1}=\sup_{\substack{w\in H_0^1\\ \|w\|=1}}(v, w).$$

Let $S_h^r(h\geq 0)$ be a family of finite dimensional subspaces of H_0^1 with the approximation property

(H1) $$\inf_{v_h\in S_h}(\|v-v_h\|_0+h\|v-v_h\|_1)\leq Ch^{r+1}\|v\|_{r+1},\quad \forall v\in H^{r+1}\cap H_0^1,$$

where and bellow we always use C to denote generalized constants which are not necessarily the same at any two places.

Define

$$a(t;\ v,\ w)=\sum_{i,j=1}^{N}\left(a_{ij}\frac{\partial v}{\partial x_j}, \frac{\partial w}{\partial x_i}\right),\ \text{for}\ v,\ w\in H_0^1,\ t\in T.$$

Take $k=t^*/M$ as the time step length and $t^s=sk$ for $s\geq 0$. The modified Calahan method for (1.1)—(1.3) is to find $\{U^n\}_{n=0}^M\subset S_h^r$ such that for $n=0, 1, \cdots, M-1$ and all $v_h\in S_h^r$,

$$(W^{n+1},\ v_h)+bka(t^{n+\frac{2}{3}};\ W^{n+1},\ v_h)=-ka(t^{n+b};\ U^n,\ v_h),\tag{1.4}$$

$$(Z^{n+1},\ v_h)+bka(t^{n+\frac{2}{3}};\ Z^{n+1},\ v_h)=-ka(t^{n+b};\ U^n,\ v_h)$$
$$+cka(t^{n+1};\ U^n,\ v_h)-a(t^n;\ U^n,\ v_h)+a(t^{n+\frac{2}{3}};\ W^{n+1},\ v_h),\tag{1.5}$$

$$U^{n+1}=U^n+W^{n+1}+\frac{1}{4}Z^{n+1},\tag{1.6}$$

$$U^0=\tilde{u}^0,\tag{1.7}$$

where $b=\frac{1}{2}\left(1+\frac{1}{3}\sqrt{3}\right)$, $c=\frac{2}{3}\sqrt{3}$; $\{W^n\}_{n=1}^M$, $\{Z^n\}_{n=1}^M\subset S_h^r$; $\tilde{u}^0$ is an approximation of u^0 in S_h^r satisfying

(H2) $$\|u^0-\tilde{u}^0\|_0+h\|u^0-\tilde{u}^0\|_1\leq Ch^{r+1}\|u^0\|_{r+1}.$$

Throughout this paper we shall always assume that the conditions (H1), (H2) and the following hypotheses (H3) and (H4) are valid.

(H3)
1) $a_{ij}=a_{ji}$, for $(x,\ t)\in\bar{\Omega}\times T$;
2) $\sum_{i,j=1}^{N}a_{ij}r_ir_j\geq C\sum_{i=1}^{N}r_i^2$, for $(r_1, \cdots, r_N)\in R^N$, $(x,\ t)\in\bar{\Omega}\times T$;
3) $\frac{\partial^s a_{ij}}{\partial t}\in C^0(\bar{\Omega}\times T)$, for $i,\ j=1, 2, \cdots, N$; $s=0, 1, \cdots, 4$;
4) (1.1)—(1.3) possesses a unique solution u and
$$\|u\|_{p,\infty}=\sup_{t\in T}\|u(t)\|_p<\infty,\ p=\max(r+4,\ 8);$$
5) $\|v\|_{s+2}\leq C\|Av\|_s$, for $v\in H^{s+2}\cap H_0^1$, $t\in T$, $s\geq -1$.

(H4) $$k\leq Ch^2.$$

Remark 1. Condition 5) of (H3) is valid when Γ and a_{ij}'s are sufficiently