THE COUPLING OF FINITE ELEMENT METHOD AND BOUNDARY ELEMENT METHOD FOR TWO-DIMENSIONAL HELMHOLTZ EQUATION IN AN EXTERIOR DOMAIN*1)

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Abstract

This paper presents a new coupling of the Finite Element Method (FEM) and the Boundary Element Method (BEM) to solve the two-dimensional exterior Helmholtz problems by using the asymptotic radiation conditions in [1], in which the coupling relations are the same as C. Johnson and J. C. Nedelec's[2]. The error estimates are derived and results of numerical calculation in comparison with analytic solution verify the theoretical estimates.

§ 0. Introduction

In practical engineering we often encounter boundary value problems of unbounded domain of PDE, such as the flow around a symmetric body, the acoustic scattering and diffusion or electromagnetic scattering by an arbitrarily shaped body, etc. The numerical computation of the above problems is very important in many areas of cappliation, e.g. the design of wave guides, the study of engine noise, the assessment of damage by an electromagnetic pulse and the biological effects of microwave radiation, etc. Mathematically, the problems have the form of an exterior boundary value problem of PDE, which gives rise to particular difficulties emerging from the facts that the domain is unbounded and the solution is oscillatory for large values of frequency. Specially, the acoustic of electromagnetic scattering by an arbitrary body can be formulated by Dirichlet's (or Neumann) boundary value problem of Helmholtz equation satisfying Sommerfield's radiation conditions at infinity (see [3]). There are varieties of numerical methods for the exterior Helmholtz problem in recent years, such as the BEM; we refer to [4—8]. In [2, 9, 10], the coupling of FEM and BEM is presented and successfully applied to many physical problems. P. Bettess gave an infinite element method in [11]. For more infinite element methods for treating problems of unbounded domain see [12] and [13]. C. I. Goldstein[14, 15] presented a method by which the three dimensional exterior Helmholtz equation is replaced by an approximate problem in a sphere with sufficiently large radius, and the boundary condition on the surface of the sphere is approximated by the Sommerfield condition at infinity (called the artificial boundary condition). This approximate problem is then solved using FEM with

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nonuniform mesh sizes. Error estimates are obtained, provided that certain relationships hold between the frequency mesh size and outer radius. If we, however, want to obtain a highly accurate approximate solution, we must take the diameter of the sphere sufficiently large. In this case high cost of computation is inevitable although nonuniform mesh sizes are used. In order to get a highly accurate numerical solution, the approximate errors of the artificial boundary conditions must be decreased. To this end, with the help of the asymptotic radiation conditions of high degree in [1] a new coupling of FEM is presented in this paper for the two-dimensional exterior Helmholtz problem with Dirichlet's boundary condition. The method does not artificially add any unknown variable as in [16, 17]. Furthermore, the existence, uniqueness and convergence of the solution of the discrete problem for arbitrary wave number k are proved and the error estimates are obtained. For references treating the exterior Helmholtz equation using the FEM, see [18] and [19].

We end this section by outlining the remainder of the paper. Sect. 1 is mainly contributed to the asymptotic radiation condition. In Sect. 2 we give the details of the coupling process of the FEM and BEM. Thus we establish the errors of the asymptotic radiation condition in Sect. 3. In Sect. 4 we prove the existence, uniqueness and convergence of the solution of the corresponding finite element approximate problems, and obtain the error estimates. A numerical example is presented for a given wave number k in Sect. 5; results of numerical computation verify the theoretical estimates. At last we make a simple discussion on results of numerical computation, and on an extension of the method of this paper to the corresponding three-dimensional problems.

§ 1. A Family of Asymptotic Radiation Conditions for the Helmholtz Equation

In this section, we will mainly introduce the forms of the asymptotic radiation conditions of arbitrary order given by Feng Kang in [1].

Let $k>0$ be the wave number, $\forall x=(x_1, x_2)\in R^2$, $r=|x|=(x_1^2+x_2^2)^{1/2}$. The domain $\Omega_R=\{(x_1, x_2)\,|\,r>R\}$ is the exterior to the circle $\Gamma_R=\{(x_1, x_2)\,|\,r=R\}$ of radius $R>0$. Based on the Fourier expansion of the Helmholtz equation in Ω_R, the asymptotic expansion and the properties of Hankel functions, and Laplace-Beltrami operator, we can obtain the asymptotic radiation conditions for the two-dimensional Helmholtz equation on Γ_R as follows:

$$\begin{cases} (\mathrm{F}_0) \quad -\dfrac{\partial u}{\partial r}=F_0u=iku, \\ (\mathrm{F}_1) \quad -\dfrac{\partial u}{\partial r}=F_1u=\left(ik+\dfrac{1}{2R}\right)u, \\ (\mathrm{F}_2) \quad -\dfrac{\partial u}{\partial r}=F_2u=\left(ik+\dfrac{1}{2R}+\dfrac{1}{8kR^2}\right)u+\dfrac{i}{2kR^2}\Lambda_1u, \\ (\mathrm{F}_3) \quad -\dfrac{\partial u}{\partial r}=F_3u=F_2u-\dfrac{1}{2k^2R^3}\left(\dfrac{u}{4}+\Lambda_1u\right), \\ \cdots\cdots \end{cases} \tag{1.1}$$