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GAUSS–NEWTON–REGULARIZING METHOD FOR SOLVING COEFFICIENT INVERSE PROBLEM OF PARTIAL DIFFERENTIAL EQUATION AND ITS CONVERGENCE*

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Introduction

In this paper we define a new nonlinear operator of the coefficient inverse problem of the wave equation and heat equation such that the inverse problem will be reduced to a new nonlinear operator equation. This new nonlinear operator is most useful for extending to 2–D and 3–D inverse problem of the wave equation and heat equation. We used the Gauss–Newton method to solve the regularizing equation of the above nonlinear operator equation of the inverse problem of 2–D elastic wave equation. The iterative process is very stable and yields excellent numerical results ([3], [4]). Here, we study in detail the properties of this nonlinear operator and prove the convergence of this iterative solution to the regularizing solution of the inverse problem in Tikhonov's sense. In particular, we prove that the conditions of Theorem 5.1 are entirely satisfied.

§ 1. A New Statement of the Coefficient Inverse Problem of 1–D Wave Equation and Heat Equation

1.1. The inverse problem of 1–D wave equation and heat equation

It is well known that, after Laplace transformation ([3]), the coefficient inverse problem of 1–D wave equation and heat equation can be reduced to finding a coefficient function $k(x)$ in

$$\Sigma^* = \{k(x) \in C^1[0,1],\ 0<\gamma_1 \leqslant k(x) \leqslant \gamma_2,\ \|dk(x)/dx\| \leqslant \beta_1\},$$

such that a solution, $u(x, s)$, of the ordinary differential equation

$$d(k(x)du/dx)/dx + s^\theta u = 0, \quad s>0,\ 0<x<1, \tag{1.1}$$

$$u(0, s) = F(s),\ du/dx(1, s) = 0, \quad s>0, \tag{1.2}$$

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satisfies the additional condition:

$$du/dx(0, s) = G(s), \quad s>0, \tag{1.3}$$

where $\theta=2$ for the wave equation, $\theta=1$ for the heat equation. The $F(s)$ and $G(s)$ are known data, and satisfy certain compatible conditions such that the above inverse problem has a solution.

In what follows, we will consider the inverse problem of the weak form of (1.1)—(1.3).

1.2. A new nonlinear operator of the coefficient inverse problem

Definition.

$$\Sigma = \{k(x) \in C[0, 1],\ 0<k_0 \leqslant k(x) \leqslant k_1\}. \tag{1.4}$$

For fixed $k(x) \in \Sigma$ and $s>0$, let $u_1(x, s; k)$ be a solution of

$$\int_0^1 \{k(x) \cdot du_1/dx \cdot dv/dx + s^\theta \cdot u_1 \cdot v\} dx = 0, \quad s>0, \tag{1.5}$$

$$u_1(0, s) = F(s),\ s>0, \text{ for all } v \in H^1[0, 1],\ v(0)=0, \tag{1.6}$$

and let $u_2(x, s; k)$ be a solution of

$$\int_0^1 \{k(x) \cdot du_2/dx \cdot dv/dx + s^\theta \cdot u_2 \cdot v\} dx = -k(0) G(s) v(0), \tag{1.7}$$

for

$$\text{all} \quad v \in H^1[0, 1], \quad s>0. \tag{1.8}$$

Definition.

$$T(k) = \int_0^1 \{k(x) (d(u_1-u_2)/dx)^2 + s^\theta (u_1-u_2)^2\} dx. \tag{1.9}$$

$T(k)$ is a new nonlinear operator of the coefficient inverse problem of 1-D wave equation and heat equation, so that the coefficient inverse problem of weak form of the wave equation and heat equation will be reduced to the following nonlinear operator equation

$$T(k) = 0. \tag{1.10}$$

§ 2. Gauss-Newton-Regularizing Method for Solving the Coefficient Inverse Problem of the Wave Equation and Heat Equation

2.1. Hilbert space $L_w(0, \infty)$

In Section 1.2, we took Σ as a domain of the operator $T(k)$, here we have to define a range of $T(k)$.

Definition. *$L_w(0, \infty)$ is a Banach space consisting of all functions $f(s)$ which are defined in $(0, \infty)$ and satisfies*

$$\int_0^\infty f^2(s) w(s) ds < +\infty, \tag{2.1}$$

with a norm

$$\|f\|_w^2 = \int_0^\infty f(s)^2 \cdot w(s) ds, \tag{2.2}$$

where