AN ECONOMICAL FINITE ELEMENT SCHEME FOR NAVIER-STOKES EQUATIONS*

HAN HOU-DE (韩厚德)
(*Tsinghua University, Beijing, China*)

Abstract

In this paper, a new finite element scheme for Navier-Stokes equations is proposed, in which three different partitions (in the two dimensional case) are used to construct finite element subspaces of the velocity field and the pressure. The error estimate of the finite element approximation is given. The precision of this new scheme has the same order as the scheme Q_2/P_0 (biquadratic rectangular element for the velocity field, and constant rectangular element for the pressure), but it is more economical than the scheme Q_2/P_0.

§ 1. Introduction

In this paper we consider the boundary value problem of Navier-Stokes equations

$$-\nu\Delta \boldsymbol{u}+\sum_{j=1}^{2} u_j \frac{\partial \boldsymbol{u}}{\partial x_j}+\operatorname{grad}\lambda=\boldsymbol{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u}=0, \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, $\boldsymbol{u}=(u_1, u_2)$ is the velocity, λ is the pressure, ν is a positive constant which stands for the coefficient of kinematic viscosity, and $\boldsymbol{f}=(f_1, f_2)$ is given.

Let $W^{m,q}(\Omega)$ denote the Sobolev space on Ω with norm $\|\cdot\|_{m,q,\Omega}$. As usual, when $q=2$, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$, and $W^{0,q}(\Omega)$ is denoted by $L^q(\Omega)$. Moreover, let $H_0^1(\Omega)=\{u\in H^1(\Omega),\ u=0 \text{ on } \partial\Omega\}$, $X=(H_0^1(\Omega))^2$ with norm $\|\cdot\|_X=|\cdot|_{1,\Omega}$ and $M=\left\{\lambda\in L^2(\Omega),\ \int_\Omega \lambda\, dx=0\right\}$ with norm $\|\cdot\|_M=\|\cdot\|_{0,\Omega}$. Then the boundary value problem (1.1)—(1.3) is equivalent to the following variational problem:

Find $(\boldsymbol{u}, \lambda)\in X\times M$, such that

$$a_0(\boldsymbol{u}, \boldsymbol{v})+a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, \lambda)=(\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v}\in X, \tag{1.4}$$

$$b(\boldsymbol{u}, \mu)=0, \quad \forall \mu\in M, \tag{1.5}$$

where

$$a_0(\boldsymbol{u}, \boldsymbol{v})=\nu\sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx, \tag{1.6}$$

$$a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})=\frac{1}{2}\sum_{i,j=1}^{2}\int_\Omega w_j\left(\frac{\partial u_i}{\partial x_j}v_i-\frac{\partial v_i}{\partial x_j}u_i\right)dx, \tag{1.7}$$

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$$b(\boldsymbol{v}, \lambda) = -\int_\Omega \lambda \operatorname{div} \boldsymbol{v}\, dx, \tag{1.8}$$

$$(\boldsymbol{f}, \boldsymbol{v}) = \sum_{i=1}^{2} \int_\Omega f_i v_i\, dx. \tag{1.9}$$

For the low Reynold's number, the variational problem (1.4)—(1.5) can be reduced to the Stokes problem:

Find $(\boldsymbol{u}, \lambda) \in X \times M$, such that

$$a_0(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \lambda) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in X, \tag{1.4*}$$

$$b(\boldsymbol{u}, \mu) = 0, \quad \forall \mu \in M. \tag{1.5}$$

Suppose X_h and M_h are two finite element subspaces of X and M. Consider the finite element approximation of (1.4)—(1.5) and (1.4)*—(1.5) respectively:

Find $(\boldsymbol{u}_h, \lambda_h) \in X_h \times M_h$, such that

$$a_0(\boldsymbol{u}_h, \boldsymbol{v}_h) + a_1(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \lambda_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in X_h, \tag{1.10}$$

$$b(\boldsymbol{u}_h, \mu_h) = 0, \quad \forall \mu_h \in M_h, \tag{1.11}$$

and

Find $(\boldsymbol{u}_h, \lambda_h) \in X_h \times M_h$, such that

$$a_0(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \lambda_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in X_h, \tag{1.10*}$$

$$b(\boldsymbol{u}_h, \mu_h) = 0, \quad \forall \mu_h \in M_h. \tag{1.11}$$

The question we shall discuss is how to choose the finite element subspaces X_h and M_h, such that the error estimate of the finite element approximation $\{\boldsymbol{u}_h, \lambda_h\}$ is optimal. We know that if X_h and M_h are the optimal choice, then X_h and M_h should satisfy the following conditions[1-2]:

(a) The errors $\inf_{\boldsymbol{v}_h \in X_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_X$ and $\inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M$ have the same order in h, where $(\boldsymbol{u}, \lambda)$ is the solution of (1.4)*—(1.5) or (1.4)—(1.5).

(b) There exists a constant β independent of h, such that

$$\sup_{\boldsymbol{v}_h \in X_h} \frac{b(\boldsymbol{v}_h, \mu_h)}{\|\boldsymbol{v}_h\|_X} \geqslant \beta \|\mu_h\|_M, \quad \forall \mu_h \in M_h. \tag{1.12}$$

Condition (1.12) is called Babuška–Brezzi condition.

In the case Ω is a rectangle, the domain Ω can be divided into some smaller rectangles. We shall denote by $\mathscr{T}_h$ this partition, and set P_k for the space of all polynomials of degree $\leqslant k$ in the variables x_1, x_2 and Q_k for the space of all polynomials of degree $\leqslant k$ with respect to each of the two variables x_1, x_2. J. T. Oden and O. Jacquotte[3] listed different choices of the subspaces X_h and M_h which satisfy the Babuška–Brezzi condition. For example, the Q_2/P_0 scheme (biquadratic rectangular element for the velocity field $\boldsymbol{u}$, and constant rectangular element for the pressure λ) is one of their choices. But in this scheme the error estimate of the finite element approximation $(\boldsymbol{u}_h, \lambda_h)$ is $\|\boldsymbol{u} - \boldsymbol{u}_h\|_X + \|\lambda - \lambda_h\|_M = O(h)$ only, even though they use the biquadratic rectangular element for velocity field $\boldsymbol{u}$. So it is interest to find a "one order precision scheme" with an optimal error estimate. O. A. Karakaskian[4] presented a scheme in which two different triangulations $\mathscr{T}_{\hat{h}}$ and $\mathscr{T}_h$ are used for approximating $\boldsymbol{u}$ and λ (linear triangular element for $\boldsymbol{u}$ and constant triangular element for λ to form subspaces X_h and M_h). He proved that if $\hat{h}/h$ is sufficiently small, subspaces X_h and M_h satisfy the Babuška–Brezzi condition