AN ESTIMATE OF THE DIFFERENCE BETWEEN A DIAGONAL ELEMENT AND THE CORRESPONDING EIGENVALUE OF A SYMMETRIC TRIDIAGONAL MATRIX*

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Abstract

A sharper upperbound of the difference between a diagonal element and the corresponding eigenvalue of a symmetric tridiagonal matrix is given. The bound can be used in the QL and QR algorithms and Rayleigh quotient approximation. The change of eigenvalues is estimated when the first off-diagonal element β_1 is replaced by zero and when two neighboring off-diagonal elements β_{i-1}, β_i are replaced by zeros.

§ 1. Introduction

Let

$$T=T_{1,n}=\begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & 0 \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

be an unreduced symmetric tridiagonal matrix. Let

$$\lambda_1<\lambda_2<\cdots<\lambda_n$$

denote its eigenvalues. Let $\widetilde{T}_{1,n}$ be a matrix obtained by replacing β_1 in $T_{1,n}$ with zero. So α_1 is an eigenvalue of $\widetilde{T}_{1,n}$. Let

$$\mu_1\leqslant\mu_2\leqslant\cdots\leqslant\mu_n$$

denote n eigenvalues of $\widetilde{T}_{1,n}$ and $\alpha_1=\mu_j$. Hence

$$\mu_1, \mu_2, \cdots, \mu_{j-1}, \mu_{j+1}, \cdots, \mu_n$$

are $n-1$ eigenvalues of

$$T_{2,n}=\begin{pmatrix} \alpha_2 & \beta_2 & & 0 \\ \beta_2 & \alpha_3 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ 0 & & \beta_{n-1} & \alpha_n \end{pmatrix}.$$

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How close is the diagonal element α_1 to an eigenvalue of T ? The question is important for the shifted QL algorithm. There are several results on this topic, see [1—5]. In [5], there is an eigenvalue λ of T, and $a=\min\limits_{\lambda_i\neq\lambda}|\alpha_1-\lambda_i|$ is the gap, then

$$|\alpha_1-\lambda|\leqslant\beta_1^2/a(1-\beta_1^2/a^2).$$

In [4], a result is

$$|\alpha_1-\lambda|\leqslant\beta_1^2/a.$$

In [1], for β_1^2 is sufficiently small, the estimation is

$$|\alpha_1-\lambda_j|\leqslant\beta_1^2/b,$$

where $b=\min\limits_{i\neq j}|\alpha_1-\mu_i|$.

In this paper a two-sided estimate of $\lambda_j-\alpha_1$ is given as follows:

$$\beta_1^2\sum_{k=j+1}^{n}\frac{s_{1k}^2}{\xi_2-\mu_k}\leqslant\lambda_j-\alpha_1\leqslant\beta_1^2\sum_{k=1}^{j-1}\frac{s_{1k}^2}{\xi_1-\mu_k},\tag{1}$$

where

$$\xi_1=\frac{1}{2}\{(\alpha_1+\mu_{j-1})+\sqrt{(\alpha_1-\mu_{j-1})^2+4\beta_1^2s_{1j-1}^2}\},$$

$$\xi_2=\frac{1}{2}\{(\alpha_1+\mu_{j+1})-\sqrt{(\alpha_1-\mu_{j+1})^2+4\beta_1^2s_{1j+1}^2}\},$$

s_k is a unit eigenvector of $T_{2,n}$ corresponding to the eigenvalue μ_k and s_{ik} is the first component of s_k. Because $\xi_1>\alpha_1$, $\xi_2<\alpha_1$ and $\sum\limits_{\substack{k=1\\k\neq j}}^{n}s_{1k}^2=1$, the result (1) of this paper is always sharper than the result in [1]. Furthermore, $s_{1,j-1}^2$ and $s_{1,j+1}^2$ are often small when n is big. They can offset the influence of a small gap such as $\xi_1-\mu_{j-1}$ and $\xi_2-\mu_{j+1}$. Even when $\alpha_1=\mu_j=\mu_{j-1}$ or $\alpha_1=\mu_{j+1}$, the result (1) is still available. This is different from the result in [1].

In Section 2, we also discuss the difference between $\mu_i(i\neq j)$, and the eigenvalue of T.

In Section 3, we consider the matrix

$$\hat{T}=\begin{pmatrix}T_{1,i-1} & & 0\\ & \alpha_i & \\ 0 & & T_{i+1,n}\end{pmatrix}.$$

If α_i is the j-th eigenvalue of $\hat{T}$, then a similar estimate of $\lambda_j-\alpha_i$ is given.

Define the vector $x=(x_1, x_2, \cdots, x_n)^T$ and $y=(y_1, y_2, \cdots, y_n)^T$. Hereafter th norm will be denoted by $\|x\|=(x_1^2+x_2^2+\cdots+x_n^2)^{\frac{1}{2}}$ and the inner-product by $(x, y)=x_1y_1+\cdots+x_ny_n$.

§ 2. Estimate for α_1 or α_n

Let