THE CONVERGENCE OF THE MULTIGRID ALGORITHM FOR NAVIER-STOKES EQUATIONS*

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Abstract

This paper deals with a multigrid algorithm for the numerical solution of Navier-Stokes problems. The convergence proof and the estimation of the contraction number of the multigrid algorithm are given.

§ 1. Introduction

The multigrid method is a new method for working out the numerical solutions of elliptic differential equations. Briefly, it consists of smoothing process and coarse-grid correction procedure such that the operational time can be saved and the convergence rate improved. The multigrid method for solving large systems of linear equations, which arise in the numerical solution of boundary value problems by finite elements, has been discussed by many authors, e.g. Astrachancev[1], Nicolaides[10], Bank & Dupont[2], Hackbusch[6-7], Wesseling[11] and Verfüth[12]. In this paper, we discuss the convergence properties of the multigrid algorithm for the nonlinear Navier-Stokes problem:

$$\begin{cases} -\mu\Delta \boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\operatorname{grad} p=\boldsymbol{f}, & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{u}=0, & \text{in } \Omega, \\ \boldsymbol{u}=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}=(u_1, u_2, \cdots, u_d)$ and p are the velocity and the pressure of fluid respectively, μ is its viscosity, $\Omega\subset R^d$ a sufficiently smooth domain, and

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u}=\Big(\sum_{j=1}^{d} u_j\,\partial u_i/\partial x_j\Big)_{i=1,\cdots,d}.$$

The general structure of our convergence analysis for the multigrid procedure is similar to that of Hackbusch[6-7], and the convergence result of the multigrid algorithm for Navier-Stokes equation is based on the convergence theorem of nonlinear multigrid methods[7]. It is known that the main sufficient conditions of the convergence of the multigrid method are the smoothing properties and the approximation properties. Therefore in this paper we first give the proof of the smoothing properties in some discrete norms, then prove the approximation property from the usual approximation assumptions in terms of Sobolev spaces and finally

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discuss the convergence of the multigrid algorithm and estimate the contraction number under general assumptions. We have to consider the nonlinearity of equation (1.1) and different orders of differentiability of $\boldsymbol{u}$ and p in (1.1). This will be compensated for by considering the nonlinear multigrid methods inside the neighbourhood of solution $[\boldsymbol{u}, p]$ and by introducing mesh-dependent norms. To simplify the analysis we present a smoothing procedure which is related to the Jacobi iteration for scalar problems[7].

§ 2. Preliminaries

Consider Navier-Stokes equation (1.1) in a smooth enough domain and its variational formulation:

Find $[\boldsymbol{u}, p] \in Z = X \times Y$ such that

$$\begin{cases} \mu a_0(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v})_0, & \forall \boldsymbol{v} \in X, \\ b(\boldsymbol{u}, q) = 0, & \forall q \in Y, \end{cases} \tag{2.1}$$

where

$$a_0 = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_0,$$

$$a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = ((\boldsymbol{w} \cdot \nabla) \boldsymbol{u}, \boldsymbol{v})_0,$$

$$b(\boldsymbol{u}, p) = -(\operatorname{div} \boldsymbol{u}, p)_0,$$

$$X = H_0^1(\Omega)^d, \quad X^0 = L^2(\Omega)^d,$$

$$Y = L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega p\, dx = 0 \right\},$$

and $(\cdot, \cdot)_0$ denotes the L^2-inner product.

Let Ω be a smooth enough domain such that a_0, a_1 and b satisfy the continuity, coercivity and Brezzi's conditions[6], respectively. In addition, we assume the following regularity assumptions of problem (1.1) and its duality problem: If $\boldsymbol{f} \in L^2(\Omega)^d$, then $[\boldsymbol{u}, p]$, $[\boldsymbol{w}, q] \in H^2(\Omega)^d \times H^1(\Omega)$ with

$$\| [\boldsymbol{u}, p] \|_{2.1} \leqslant c \| \boldsymbol{f} \|_0, \tag{2.2a}$$

$$\| [\boldsymbol{w}, q] \|_{2.1} \leqslant c \| \boldsymbol{f} \|_0, \tag{2.2b}$$

where c denotes the generic constant and $[\boldsymbol{w}, q]$ satisfies the duality problem of (1.1).

Introduce a Navier-Stokes' operator

$$\mathscr{L}' = \begin{bmatrix} -\mu\Delta + (I \cdot \nabla) & \operatorname{grad} \\ -\operatorname{div} & 0 \end{bmatrix}, \tag{2.3a}$$

where I denotes an identity operator; then (2.1) is equivalent to

$$\mathscr{L}[\boldsymbol{u}, p] = \mathscr{L}'[\boldsymbol{u}, p] - [\boldsymbol{f}, 0] = 0. \tag{2.3b}$$

Obviously, the linearization of $\mathscr{L}'$ equals

$$L[\boldsymbol{u}, p] = \begin{bmatrix} -\mu\Delta + \delta L_{11}(\boldsymbol{u}) & \operatorname{grad} \\ -\operatorname{div} & 0 \end{bmatrix} = \begin{bmatrix} \mu L_{11} + \delta L_{11}(\boldsymbol{u}) & L_{12} \\ L_{21} & 0 \end{bmatrix}$$

$$= L' + \begin{bmatrix} \delta L_{11}(\boldsymbol{u}) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.4}$$