STABILITY OF IMPLICIT DIFFERENCE SCHEMES WITH SPACE AND TIME-DEPENDENT COEFFICIENTS*1)

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Abstract

A stability theorem is derived for implicit difference schemes approximating multidimensional initial-value problems for linear hyperbolic systems with variable coefficients, and lots of widely used difference schemes are proved to be stable under the conditions similar to those for the cases of constant coefficients. This theorem is an extension of the stability theorem due to Lax-Nirenberg[3]. The proof is quite simple.

§ 1. Introduction

In the 1960s, stability of difference schemes for initial-value problems for linear hyperbolic systems with variable coefficients was extensively and intensively studied, and some well-known and deep results were obtained, such as Kreiss dissipative theorem[3], Lax-Nirenberg's stability theorem[3].

However, most of the results are only suitable to explicit schemes and some of them are only applicable to the time-independent cases. Also the conditions ensuring the stability of schemes are very strong and hard to be checked. And the proofs are very complicated.

In this paper, combining a skill in [1] with Lax-Nirenberg's theorem for difference operators[2], we obtain a stability theorem. The schemes considered here could be both explicit and implicit, and their coefficients may depend on time variable as well as space ones. The conditions needed are natural and easy to be checked pointwise. The proof is quite simple. This theorem is an extension of Lax-Nirenberg's. As a consequence, we prove that lots of widely used schemes are stable under the conditions similar to those for the cases of constant coefficients. Dissipation and symmetry (conjugacy) are not mentioned.

§ 2. Results

For convenience, we first introduce some notation and state the Lax-Nirenberg theorem for difference operators[2].

Let $P_\alpha(x)$ be $N\times N$ complex matrices with elements depending on variables $x\in R^p$, and $u(x)$ be a complex vector function with N components $\in L^2(R^p)$. The difference operator P_h with a single parameter h(positive real) is defined in the following form:

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$$(P_h u)(x)=\sum_\alpha P_\alpha(x)T^\alpha u(x),$$

where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_p)$, α_i $(i=1, 2, \cdots, p)$ are integers and T^α are shift-operators:

$$T^\alpha u(x)=u(x_1+\alpha_1 h, x_2+\alpha_2 h, \cdots, x_p+\alpha_p h).$$

Define
$$P(x, \xi)=\sum_\alpha P_\alpha(x)e^{i\alpha\cdot\xi} \quad \text{for } \xi\in R^p,$$
$$|P|_{l,m}=\sum_\alpha \sup_{\substack{|\beta|=l\\ x\in R^p}} \|\partial_x^\beta P_\alpha(x)\|_2\cdot(\alpha)^m,$$

where $\|\cdot\|$ is the spectral norm of matrix, β are multi-indices, β_i $(i=1, 2, \cdots, p)$ are non-negative integers, $|\beta|=\sum_{i=1}^p|\beta_i|$, $(\alpha)^2=\sum_{i=1}^p\alpha_i^2$, $\partial_x^\beta P_\alpha(x)=\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\cdots\partial_{x_p}^{\beta_p}P_\alpha(x)$, and l, m are integers.

The Lax-Nirenberg Theorem. If $P(x, \xi)$ is a non-negative Hermitian matrix for all $x, \xi\in R^p$ and $|P|_{2,0}$, $|P|_{0,2}$ are bounded, then

$$\mathrm{Re}(u, P_h u)\geqslant-\frac{1}{2}h(C|P|_{0,2}+|P|_{2,0})\|u\|^2,$$

for all $u\in L^2(R^p)$, where $(\cdot, \cdot)$ is the scalar product in L^2, $\|\cdot\|$ is the corresponding norm, $\mathrm{Re}(u, P_h u)$ is the real part of $(u, P_h u)$ and C is an absolute positive constant.

In this paper, we will discuss the following schemes:

$$\sum_\mu R_\mu(x, t, \Delta)T^\mu u^{n+1}(x)=\sum_\mu S_\mu(x, t, \Delta)T^\mu u^n(x), \tag{$*$}$$

where the two sides of $(*)$ are similar to the definition of P_h. The difference between them is that the elements of R_μ, S_μ depend on x, as well as on t and Δ (Δ represents time and space meshsizes), and $|\mu|$ are not larger than some constant. Superscript n indicates that vector u depends on time variable $t=n\Delta t$, $n=0, 1, \cdots, t\leqslant T$ (constant).

In constant coefficient cases, $\Delta t/h$ is usually a constant. It is more reasonable to assume $\Delta t/h$ to satisfy

$$0<\mathrm{const}_1\leqslant\Delta t/h\leqslant\mathrm{const}_2<+\infty,$$

because the coefficients here are variable.

Theorem. *If the following condition* (A) *holds, then the schemes* $(*)$ *are stable with respect to initial-value in the sense of* Lax[3] *with* L^2 *norm, that is, there exists a positive constant* C *such that*

$$\|u^n\|\leqslant C\|u^0\|, \quad 0<n\Delta t\leqslant T, \quad n=1, 2, \cdots.$$

Condition (A). There exist two positive constants C_1, C_2 and two invertible matrices $M(x, t)$, $G(x, t)$ such that for all $x, \xi\in R^p$, $0<t\leqslant T$,

a) $(\sum_\mu MR_\mu(x, t, 0)Ge^{i\mu\xi})^*(\sum_\mu MR_\mu(x, t, 0)Ge^{i\mu\xi})-(\sum_\mu MS_\mu(x, t, 0)Ge^{i\mu\xi})^*(\sum_\mu MS_\mu(x, t, 0)Ge^{i\mu\xi})\geqslant 0$;

b) $(\sum_\mu MR_\mu(x, t, 0)Ge^{i\mu\xi})^*(\sum_\mu MR_\mu(x, t, 0)Ge^{i\mu\xi})-C_1I\geqslant 0$ (I is the unit matrix of order N);

c) $r(x, t, \Delta)$, $s(x, t, \Delta)$ (elements of $R(x, t, \Delta)$, $S(x, t, \Delta)$ respectively) and their main parts $r(x, t, 0)$, $s(x, t, 0)$ satisfy

$$|r(x, t, \Delta)-r(x, t, 0)|\leqslant C_2\Delta t, \quad |s(x, t, \Delta)-s(x, t, 0)|\leqslant C_2\Delta t,$$