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# ON THE MINIMUM PROPERTY OF THE PSEUDO $\varkappa$-CONDITION NUMBER FOR A LINEAR OPERATOR*

KUANG JIAO-XUN (匡蛟勋)

*(Shanghai Teachers University, Shanghai, China)*

## Abstract

It is well known that the $\varkappa$-condition number of a linear operator is a measure of ill condition with respect to its generalized inverses and a relative error bound with respect to the generalized inverses of operator $T$ with a small perturbation operator $E$, namely,

$$\frac{\|(T+E)^+-T^+\|}{\|T^+\|}\leqslant\frac{\varkappa(T)\frac{\|E\|}{\|T\|}}{1-\varkappa(T)\frac{\|E\|}{\|T\|}},$$

where $\varkappa(T)=\|T\|\cdot\|T^+\|$. The problem is whether there exists a positive number $\mu(T)$ independent of $E$ but dependent on $T$ such that the above relative error bound holds and $\mu(T)<\varkappa(T)$.

In this paper, an answer is given to this problem. The main result is

**Theorem.** *Let $X$, $Y$ be two Banach spaces, $T, E\in B[X, Y]$ and $\|E\|\cdot\|T^+\|<1$. Suppose*

$$\frac{\|(T+E)^+-T^+\|}{\|T^+\|}\leqslant\mu(T)\frac{\|E\|}{\|T\|}.$$

*Then $\varkappa(T)\leqslant\mu(T)$, where $\mu(T)$ is a positive number independent of $E$ but dependent on $T$ and $(I_Y+ET^+)^{-1}(T+E)$ maps $\mathscr{N}(T)$ into $\mathscr{R}(T)$. This theorem shows that $\varkappa(T)$ is minimum in the above sence.*

## § 1. Introduction

In [1], the author showed the minimum property of $\omega$-condition number for a linear operator, and extended the results of [2]. The results of [1] are only related to the relative error bound of an inverse linear operator with a small perturbation operator, or the relative error bound of the a regular solution of linear equations with small perturbation.

In this paper, we will discuss the relative error bound of a generalized inverse of a linear operator from a Banach space to another Banach space and a generalized solution of liear equations whose operator has a small perturbation. In addition, we will show the minimum property of the pseudo $\varkappa$-condition number. The results are very extensive and the results of [1] and [2] are the obvious corollaries.

## § 2. Generalized Inverses of a Linear Operator in a Banach Space

In general, the letters $X$, $Y$ denote the Banach space, $B[X, Y]$ is the Banach space consisting of all bounded linear operators from $X$ into $Y$, $\mathscr{D}(T)$ and $\mathscr{R}(T)$ denote the domain and range of $T$ respectively, and $\mathscr{N}(T)$ denotes the null of $T$.

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We assume that the closed subspace $\mathcal{N}(T)$ of $X$ has a topological complement $\mathcal{N}(T)^c$ and the closed subspace $\overline{\mathcal{R}(T)}$ of $Y$ has a topological complement $\overline{\mathcal{R}(T)}^c$, namely

$$X=\mathcal{N}(T)\oplus\mathcal{N}(T)^c;\quad Y=\overline{\mathcal{R}(T)}\oplus\overline{\mathcal{R}(T)}^c.$$

In this case, $\mathcal{N}(T)$ and $\overline{\mathcal{R}(T)}$ are closed, however a closed subspace does not necessarily have a topological complement. A subspace $\mathcal{N}(T)$ ($\overline{\mathcal{R}(T)}$) has a topological complement if and only if there exists a projector $P(Q)$ of $X(Y)$ onto $\mathcal{N}(T)$ ($\overline{\mathcal{R}(T)}$), i.e., $PX=\mathcal{N}(T)$ ($QY=\overline{\mathcal{R}(T)}$), see [7]. Nashed pointed out that if the decompositions

$$X=\mathcal{N}(T)\oplus\mathcal{N}(T)^c;\ Y=\overline{\mathcal{R}(T)}\oplus\overline{\mathcal{R}(T)}^c$$

exist, then there exists uniquely the generalized inverse $T^+\equiv T^+_{P,Q}$ ($T^+_{P,Q}$ implies that the operator $T^+$ depends on the projectors $P$ and $Q$) such that

$$\begin{cases}\mathcal{D}(T^+)=\mathcal{R}(T)\oplus\overline{\mathcal{R}(T)}^c;\ \mathcal{N}(T^+)=\overline{\mathcal{R}(T)}^c,\\ \mathcal{R}(T^+)=\mathcal{N}(T)^c;\ TT^+T=T;\ T^+TT^+=T^+ \text{ on } \mathcal{D}(T^+),\\ T^+T=I-P;\ TT^+=Q|_{\mathcal{D}(T^+)},\end{cases}\tag{1}$$

where $Q|_{\mathcal{D}(T^+)}$ is the restriction of $Q$ on $\mathcal{D}(T^+)$. $T^+$ is bounded if and only if $\mathcal{R}(T)$ is closed in $Y$. In this paper, we consider the case that $\mathcal{R}(T)$ is closed; then we have obviously

$$\begin{cases}X=\mathcal{N}(T)\oplus\mathcal{N}(T)^c;\ Y=\mathcal{R}(T)\oplus\mathcal{R}(T)^c,\\ \mathcal{D}(T^+)=Y;\ \mathcal{N}(T^+)=\mathcal{R}(T)^c,\\ \mathcal{R}(T^+)=\mathcal{N}(T)^c,\end{cases}\tag{2}$$

$$\begin{cases}TT^+T=T;\ T^+TT^+=T^+,\\ T^+T=P_{\mathcal{N}(T)^c};\ TT^+=P_{\mathcal{R}(T)}.\end{cases}\tag{3}$$

From (3) we can obtain easily

$$\begin{cases}T^+P_{\mathcal{R}(T)}=T^+;\ P_{\mathcal{N}(T)^c}T^+=T^+,\\ TP_{\mathcal{N}(T)^c}=T;\ P_{\mathcal{R}(T)}T=T.\end{cases}\tag{4}$$

In the following section, we consider the case that the perturbation $S=T+E$ of $T$ has a generalized inverse and estimate the error bound between $S^+$ and $T^+$. We suppose that $y_0\in Y$, $y_0=y_1+y_2$ and $\|y_0\|=1$ imply $\|y_1\|\leqslant 1$.

## § 3. The Minimum Property of the Pseudo $\varkappa$-Condition Number

**Lemma 1.** *Let $T\in B[X, Y]$ and suppose $X=\mathcal{N}(T)\oplus\mathcal{N}(T)^c$ and $Y=\mathcal{R}(T)\oplus\mathcal{R}(T)^c$. Let $T^+_{\mathcal{N}(T)^c,\mathcal{R}(T)^c}$ be the generalized inverses of $T$ with respect to these decompositions. Let $E\in B[X, Y]$ and $S=T+E$. Suppose*

$$\|ET^+\|<1\tag{5}$$

*and*

$$(I_y+ET^+)^{-1}S \text{ maps } \mathcal{N}(T) \text{ into } \mathcal{R}(T).\tag{6}$$

*Then*

$$X=\mathcal{N}(S)\oplus\mathcal{R}(T^+);\quad Y=\mathcal{R}(S)\oplus\mathcal{N}(T^+)$$

*and*