THE MULTIGRID METHOD WITH CORRECTION PROCEDURE*

CAI ZHI-QIANG (蔡智强)
(Department of Mathematics, Huazhong University of Science and Technology, Wuhan, China)

§ 1. Introduction

The general form of the MGE method for solving the boundary value problem of elliptic partial differential equations suggested in [1] suits both the finite difference scheme and the finite element scheme resulting from elliptic differential equations. In order to decrease the number of multigrid iterations on each level, Cai et al. suggested a revised MGE method by using auxiliary grids in [2—3]. For the special equation $-\Delta u=f(x, u)$, we combine the correction procedure with multigrid method, so that the interpolation level number is 1. The computational work needed by this method is less than any revised MGE method[2-3].

§ 2. The Multigrid Method with Correction Procedure

For simplicity, we consider the model problem

$$\begin{cases}\Delta u=f(x, u), & \text{in } \Omega,\\ u=g, & \text{on } \partial\Omega,\end{cases} \tag{1}$$

where Ω is a one-, two- or three-dimensional domain, and $\partial\Omega$ is the boundary of the domain Ω. Suppose that Ω consists of some squares in the two-dimensional case or of some cubes in the three-dimensional case, and that the solution u is smooth enough and

$$f_1(u)=f'_u(x, u)\geqslant 0. \tag{2}$$

$\Omega_k\subset\Omega$ $(k=0, 1, \cdots, 1)$ are uniform discretized grids of the domain, whose width is h_k, and

$$\Omega_k\subset\Omega_{k+1}, \quad h_k=\xi h_{k+1}.$$

The ratio of step size ξ is usually 2.

Let Δ_k be the 5-point approximation of the Laplace operator Δ in the two-dimensional case and the 7-point approximation in the three-dimensional case on the grid Ω_k as usual.

Let Δ_k^* be the 5-point approximation defined by

$$\Delta_k^* u(x_1, x_2)=\left(\sum u(x_1\pm h_k, x_2\pm h_k)-4u(x_1, x_2)\right)/2h_k^2, \tag{3}$$

$$\sum u(x_1\pm h_k, x_2\pm h_k)=u(x_1+h_k, x_2-h_k)+u(x_1+h_k, x_2+h_k)+u(x_1-h_k, x_2-h_k)+u(x_1-h_k, x_2+h_k)$$

in the two-dimensional case and the 9-point approximation defined by

* Received January 2, 1986.

$$\Delta_k^* u(x_1, x_2, x_3) = \left(\sum u(x_1 \pm h_k, x_2 \pm h_k, x_3 \pm h_k) - 8u(x_1, x_2, x_3)\right)/4h_k^2 \tag{4}$$

in the three-dimensional case.

Consider the finite difference solution u_k defined by

$$\begin{cases} \Delta_k u_k = f(x, u_k), & x \in \Omega_k, \\ u_k = g - h_k^2 f(x, g)/12, & x \in \partial\Omega_k \end{cases} \tag{5}$$

and a correction solution φ_k defined by the linearized finite difference equation

$$\begin{cases} (\Delta_k^* - f_1(u_k))\varphi_k = f(x, u_k) - f_1(u_k)u_k + h_k^2 f_1(u_k) f(x, u_k)/4, & x \in \Omega_k, \\ \varphi_k = g - h_k^2 f(x, g)/12, & x \in \partial\Omega_k. \end{cases} \tag{6}$$

The finite difference equations (5) and (6) can be denoted by the abstract equations

$$L_k u_k = F_k, \tag{7}$$

$$L_k^* \varphi_k = F_k^*, \tag{8}$$

where L_k, L_k^* are discretized matrices and u_k, φ_k, F_k and F_k^* are grid functions.

In [4], Lin and Lu have proved

$$\frac{2}{3} u_k + \frac{1}{3} \varphi_k + \frac{1}{12} h_k^2 f(x, u_k) = u + O(h_k^4), \ x \in \overline{\Omega}_k \tag{9}$$

under reasonable conditions. It is obvious that if the following condition

$$\bar{u}_k = u_k + O(h_k^4), \quad \bar{\varphi}_k = \varphi_k + O(h_k^4), \text{ in } \overline{\Omega}_k \tag{10}$$

is valid, then one has also

$$\frac{2}{3} \bar{u}_k + \frac{1}{3} \bar{\varphi}_k + \frac{1}{12} h_k^2 f(x, \bar{u}_k) = u + O(h_k^4), \quad \text{in } \overline{\Omega}_k. \tag{11}$$

In order to avoid solving equations (7) and (8) directly, we wish to find the approximations of u_k and φ_k indirectly by using the solutions u_{k-1} and φ_{k-1} at the level $k-1$. We can prove the following proposition.

Proposition 1. Let $h_{k-1} = 2h_k$ $(k = 1, \cdots, l)$; then

$$\frac{3}{4} u_{k-1} + \frac{1}{4} \varphi_{k-1} + \frac{1}{16} h_{k-1}^2 f(x, u_{k-1}) = u_k + O(h_{k-1}^4), \quad \text{on } \overline{\Omega}_k. \tag{12}$$

Proof. Let u_k^* be the solution of the equation

$$\begin{cases} (\Delta_k^* - f_1(u_k))u_k^* = f(x, u_k) - f_1(u_k)u_k, & x \in \overline{\Omega}_k, \\ u_k^* = g - \dfrac{1}{12} h_k^2 f(x, g), & x \in \partial\Omega_k. \end{cases}$$

Then one may obtain

$$u_{k-1} - u + \frac{1}{12} h_{k-1}^2 v_1 = O(h_{k-1}^4), \quad \text{on } \overline{\Omega}_{k-1}, \tag{13}$$

$$u_{k-1}^* - u + \frac{1}{12} h_{k-1}^2 v_2 = O(h_{k-1}^4), \quad \text{on } \overline{\Omega}_{k-1} \tag{14}$$

from the proof of Proposition 1 in [4]. Hence

$$\frac{3}{4} u_{k-1} + \frac{1}{4} u_{k-1}^* - u_k + \frac{1}{16} h_{k-1}^2 \left(\frac{2}{3} v_1 + \frac{1}{3} v_2\right) = O(h_{k-1}^4). \tag{15}$$

Set

$$w = \frac{2}{3} v_1 + \frac{1}{3} v_2.$$

Then we have