THE CHEBYSHEV SPECTRAL METHOD FOR BURGERS-LIKE EQUATIONS*

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Abstract

The Chebyshev polynomials have good approximation properties which are not affected by boundary values. They have higher resolution near the boundary than in the interior and are suitable for problems in which the solution changes rapidly near the boundary. Also, they can be calculated by FFT. Thus they are used mostly for initial-boundary value problems for P. D. E. 's (see [1, 3—4, 6, 8—11]). Maday and Quarteroni[8] discussed the convergence of Legendre and Chebyshev spectral approximations to the steady Burgers equation. In this paper we consider Burgers-like equations

$$\begin{cases} \partial_t u+F(u)_x-\nu u_{xx}=0, & -1\leqslant x\leqslant 1,\ 0<t\leqslant T, \\ u(-1,t)=u(1,t)=0, & 0\leqslant t\leqslant T, \\ u(x,0)=u_0(x), & -1\leqslant x\leqslant 1, \end{cases} \tag{0.1}$$

where $F\in C(\mathbf{R})$ and there exists a positive function $A\in C(\mathbf{R})$ and a constant $p>1$ such that

$$|F(z+y)-F(z)|\leqslant A(z)(|y|+|y|^p).$$

We develop a Chebyshev spectral scheme and a pseudospectral scheme for solving (0.1) and establish their generalized stability and convergence.

§ 1. Notations and Lemmas

Let $I=(-1,1)$, and let $L^2(I)$ be equipped with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Suppose $\omega(x)=(1-x^2)^{-1/2}$; set

$$L^2_\omega(I)=\{v: I\to R\,|\,v \text{ is measurable and } (v,v)_\omega<\infty\},$$

where

$$(u,v)_\omega=\int_I u(x)v(x)\omega(x)dx,\quad \|u\|_\omega=(u,u)_\omega^{1/2}.$$

For any positive integer m, define

$$\|u\|^2_{m,\omega}=\sum_{j=0}^m\left\|\frac{d^ju}{dx^j}\right\|^2_\omega,\ H^m_\omega(I)=\{v\in L^2_\omega(I)\,|\,\|v\|_{m,\omega}<\infty\},$$

$$H^1_{0,\omega}(I)=\{v\in H^1_\omega(I)\,|\,v(-1)=v(1)=0\}.$$

For any positive integer N, let S_N be the space of algebraic polynomials of degree at most N. Set

$$V_N=\{\varphi\in S_N\,|\,\varphi(-1)=\varphi(1)=0\}=S_N\cap H^1_{0,\omega}(I).$$

Let $P_N: L^2_\omega(I)\to V_N$ be the L^2_ω-orthogonal projection operator, i.e.,

$$(P_Nv,\varphi)_\omega=(v,\varphi)_\omega,\quad \forall\varphi\in V_N,$$

and $P_{1,N}: H^1_{0,\omega}(I)\to V_N$ be as follows:

$$((P_{1,N}v-v)_x,(\varphi\omega)_x)=0,\quad \forall\varphi\in V_N.$$

Denote by $\{x_j,\omega_j\}$ the nodes and weights of the Gauss-Lobatto integration

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formula, where $x_j=\cos\frac{\pi}{N}j(0\leqslant j\leqslant N)$, $\omega_0=\omega_N=\frac{\pi}{2N}$ and $\omega_j=\frac{\pi}{N}$, $1\leqslant j\leqslant N-1$. Then

$$\int_I f(x)\omega(x)dx=\sum_{j=0}^{N}f(x_j)\omega_j,\quad \forall f\in S_{2N-1}. \tag{1.1}$$

Let P_c: $C(\bar{I})\to S_N$ be the interpolation operator: $P_c u(x_j)=u(x_j)$ $(0\leqslant j\leqslant N)$. Introduce the discrete inner product and norm

$$(u,v)_{N,\omega}=\sum_{j=0}^{N}u(x_j)v(x_j)\omega_j,\quad \|u\|_{N,\omega}=(u,u)_{N,\omega}^{1/2}.$$

The constants c in the following lemmas are independent of N and of the function v, which may be different in different cases.

Lemma 1. *If $v\in H^1_{0,\omega}(I)$, then*

$$\|v\omega^2\|_\omega\leqslant|v|_{1,\omega}. \tag{1.2}$$

Proof. Let $g(t)=t^{-1}\int_0^t f(s)ds$. We get from Theorem 4.1 in Chapter 3 of [7]

$$\int_0^\infty|g(t)|^2t^{-1/2}dt\leqslant\frac{16}{9}\int_0^\infty|f(t)|^2t^{-1/2}dt.$$

Set $t=1+x$; then

$$\int_{-1}^0|g(1+x)|^2(1+x)^{-1/2}dx\leqslant\frac{16}{9}\int_{-1}^\infty|f(1+x)|^2(1+x)^{-1/2}dx.$$

Now take

$$f(1+x)=\begin{cases}v_x(x), & -1\leqslant x\leqslant 0,\\ 0, & x>0.\end{cases}$$

Then $g(1+x)=(1+x)^{-1}\int_{-1}^x f(1+\xi)d\xi=(1+x)^{-1}v(x)$ and so

$$\int_{-1}^0|v(x)|^2\omega^5(x)dx\leqslant\frac{16}{9}\sqrt{2}\int_{-1}^0|v_x(x)|^2\omega(x)dx.$$

Similarly we get the result on the interval $[0,1]$. Therefore

$$\|v\omega^2\|_\omega^2\leqslant\frac{16}{9}\sqrt{2}\,|v|_{1,\omega}^2,$$

which leads to $\lim_{x\to\pm1}v^2(x)\omega^3(x)=0$, and we get from integration by parts

$$\int_I v_x(v\omega)_x\,dx=\int_I|v_x|^2\omega\,dx-\frac{1}{2}\int_I(1+2x^2)v^2\omega^5dx, \tag{1.3}$$

$$\int_I v_x(v\omega)_x dx=\int_I|(v\omega)_x|^2\omega^{-1}dx+\frac{1}{2}\int_I v^2\omega^5dx, \tag{1.4}$$

Subtracting (1.4) from (1.3) yields

$$\int_I|v_x|^2\omega dx-\int_I(1+x^2)v^2\omega^5\,dx=\int_I|(v\omega)_x|^2\omega^{-1}dx\geqslant0,$$

and so (1.2) follows.

Lemma 2. *If $v\in H^1_{0,\omega}(I)$, then*

$$\int_I v_x(v\omega)_x dx\geqslant\frac{1}{4}\|v\|_{1,\omega}^2. \tag{1.5}$$

Proof. We get from (1.3) and (1.4)

$$\int_I v_x(v\omega)_x dx=\frac{1}{4}\int_I|v_x|^2\omega dx+\frac{1}{4}\int_I v^2\omega^3dx+\frac{3}{4}\int_I|(v\omega)_x|^2\omega^{-1}dx\geqslant\frac{1}{4}\|v\|_{1,\omega}^2.$$