HIERARCHICAL ELEMENTS, LOCAL MAPPINGS AND THE h-p VERSION OF THE FINITE ELEMENT METHOD (I)*

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Abstract

This is the first half of an article which develops a theory of the hierarchical elements for the h-p version of the finite element method in the two-dimentional case. The approximation properties of the hierarchical elements are discussed. The second part will address the convergence rate when geometric meshes are used.

§ 0. Introduction

In the finite element method, when conforming elements are used, the rate of convergence for an elliptic problem is determined by the approximability of the finite element space to the solution (Céa's Lemma). As a consequence, the select ion of a good finite element space is the key for achieving maximal rate of convergence.

There are three basic versions in the finite element method. In the h version, which is the traditional one, the mesh size h goes to zero while the degrees of elements are fixed. In contrast, the p version increases the degrees and fixes the mesh. The h-p version combines the two and obtains convergence by both refining the mesh and increasing the degrees.

Since the h-p version considers both mesh and degree distribution, it is more advantageous in the approximability of the finite element space. It was proved in [1]—[3] that in one-dimension if the solution has a singularity of x^α-type, the best order of convergence for the h-p version is $q_0^{\sqrt{(\alpha-\frac{1}{2})N}}$, where N is the number of degrees of freedom of the finite element space and $q_0=(\sqrt{2}-1)^2\approx 0.1715$. To achieve this rate of convergence a geometric mesh with the ratio q_0 and a linear distribution of degrees of elements with the slope $2\alpha-1$ were used. In this setting, the elements closer to the singularity have smaller sizes and lower degrees. If the degrees of elements are uniformly distributed, they should be increased as a multiple of the number of the elements with a factor $2\alpha-1$, which gives a rate of convergence of $q_0^{\sqrt{(\alpha-\frac{1}{2})N/2}}$. In 2-dimensions, when on a cornered domain, r^α-type singularities of the solution will arise at the corners. It was shown in [4] that when a geometric mesh is used and the degrees of elements are either optimally,

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or uniformly distributed, an order of convergence $e^{-\gamma\cdot\sqrt{N}}$ ($\gamma>0$) can be reached. However, the rules for selecting the ratio of the geometric mesh and for the increase of degrees of elements were not clear.

An important property in one-dimension is that the ratio of the geometric mesh for achieving the best convergence order is independent of the strength of the ingularity, i.e. the value α. This has been called "the 0.15 rule" (theoretically 0.1715). It has also been guessed that this rule could be valid also for two-dimensions. In this paper we will show that this guess is true for optimal mesh-degree combinations.

The basic tools for implementing the h-p version are the hierarchical elements and local mappings. In contrast with Lagrange elements and Hermit elements, when the degrees of elements need to be incresed, one can simply add new basis functions to the old basis set. The concept of hierarchical basis can go back to the original Galerkin's method. It was introduced by [5] in 1971 and suggested by [6], [7], [8] for p-version in the early 80's. The hierarchical basis is closely related with spectral expansions. And it has been used in the p-version finite element analysis program PROBE. In this paper we will give a more detailed analysis for the approximation properties of the finite element space based on the hierarchical elements.

The use of local mappings is an old technique in the finite element calculation. To handle the curved boundary the means of isoparametric elements were successfully used in the h version. In this paper a discussion will be given to it in cooperation with the hierarchical elements.

This part contains section 1, C^0 hierarchical elements in one-dimension and section 2, the $Q^0_?$ hierarchical elements in two-dimensions (the square elemets). The other two sections (section 3, C^0 compatible local mappings and geometric meshes and section 4, the h-p method and its error analysis) are left to Part II.

§ 1. C^0 Hierarchical Elements in One Dimension

1.1. Preliminaries

In the following, H^1, H^k, L_2 are used in the usual sense for Sobolev spaces. $P_n(x)$ ($n=0, 1, 2, \cdots$) are the standard Legendre polynomials:

$$P_n(x)=\frac{1}{(2n)!!}\frac{d^n}{dx^n}[(x^2-1)^n]. \tag{1.1.1}$$

For the properties of Legendre polynomials we refer to [7], [8] and [9]. The most important properties for our discussion are listed below:

1° $\{P_n(x)\}$ is an orthogonal basis of $L_2(-1, 1)$. And we have

$$\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{n,m} \tag{1.1.2}$$

where
$$\delta_{n,m}=\begin{cases}0, & \text{if } n\neq m,\\ 1, & \text{if } n=m.\end{cases}$$

2° If $u\in L_1(-1, 1)$, then $u(x)$ has a Fourier-Legendre expansion (or simply the Legendre expansion):

$$u(x)\sim\sum_{n=0}^{\infty}u_nP_n(x), \tag{1.1.3}$$